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THE GENERAL EOQ MODEL WITH INCREASING DEMAND AND COSTS

Keisuke Matsuyama
Akita Prefectural University

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Abstract Assume that one day's demand for products, one day's holding cost per one unit of product and setup cost are expressed as increasing functions of time. Moreover assume that we are given a time interval under these circumstances. The aim of our study is to find the optimal inventory policy which minimizes the total inventory cost required during this interval. How often orders are placed during this interval and how much is ordered at each ordering time point are our concerns. The techniques of DP are introduced to solve this problem.

1. Introduction

The classical EOQ theories have been often concerned with the case in which one day's holding cost per unit of product, setup costs and one day's demand for the product are constant. Under this condition the inventory policy is planned so that the one day's average inventory cost may become minimal.

Many fruitful results have been obtained in this field of study. Moreover there are many trial studies which may modify this condition. The author have also been interested in this field (cf. [5, 6, 7]).

Setup costs are not always constant. For example, the study of joint replenishment presumes the realistic assumption that the total setup cost for multi-item products is not the simple sum of setup costs required for the orders of each items. Mathematically to say, the joint setup costs have the properties of monotoneity and convexity. With these joint setup costs DP techniques are applied to define the procedures with which inventory costs for multi-item products become minimal. See, for example, Queyranne [11], Rosenblatt *et al.* [12] and Matsuyama [6].

Recently it is reported that in some case setup costs decrease gradually as setups of orders are repeated. About this problem a sort of learning curve is introduced. With this kind of setup cost, the optimal inventory policy is planned. See, for example, Neves [9] and Pratsini [10].

About holding costs, the cost functions which depend on the inventory level are examined. Total holding costs increase as the inventory level becomes great. But the marginal increase of total holding cost with respect to inventory level may not be simply regarded as a constant. In other words, the holding cost is not defined as a linear function of inventory level. Taking account of economy of scale, this assumption seems to be plausible. Moreover the quality deterioration of products due to their storage can be described as the change of the holding costs. In order to generalize EOQ models, various functions which describe holding costs are introduced. See, Baker *et al.* [1], Goh [2, 3], Goswami [4] and Weiss [13].

Similar modifications are tried in respect of demand function so that EOQ models may be generalized. Various functions defined through the differential equation are introduced to explain actual and practical problems. They depend on the inventory level. (cf. Baker *et al.* [1], Goh [2, 3], Muhleman [8].)

Our study is concerned with the situations under which all of the demand of products, one day's holding cost per one unit of product, and setup cost are increasing functions of time. This case is often observed when the price index is not constant but increasing. In this meaning this paper is to generalize the problems which were studied in Matsuyama [6].

For example, assume that we are given a time interval during which inventory policy must be planned. In stead of considering minimum one day's average inventory cost, minimum total inventory cost required during this time interval is examined. Our aim is to find the procedures determining how often the order must be placed and how the ordering quantity is given for each ordering time point. It will be shown that DP techniques are very useful to this study. Mathematical properties of these procedures will be examined.

2. Formation of Problem

Assume T is given. We are to plan an inventory policy for a certain commodity. The following symbols are introduced;

t	variable expressing time.
$[0; T]$	time interval during which the inventory policy is to be planned.
$C(t)$	setup cost at t.
$p(t)$	one day's holding cost per one unit of the commodity at t.
$q(t)$	buying cost per unit of commodity at t.
$r(t)$	one day's demand for the commodity at t.

Moreover, we introduce the following assumptions.

Assumption 1

1. Demand for the commodity occurs continuously.
2. Shortage of the commodity is not allowed.
3. Lead time is regarded as zero.
4. The values of $C(t), p(t), q(t)$ and $r(t)$ are always positive.
5. $p(t), q(t), r(t)$ and $C(t)$ have their derivatives of the 2-nd order.
6. $q(t) > p(t)$.
7. $C(t) > \max[p(t), q(t), p(t)r(t), q(t)r(t)]$.

Inventory theory has been founded on certain tacit premises. 6 and 7 in Assumption 1 are such examples. Assume 6 is not valid. Then we have $p(t)r(t) > q(t)r(t)$. This means that one day's holding cost is greater than one day's buying cost. Assume $C(t) < p(t)r(t)$ is not valid. This means that one day's holding cost is greater than setup cost. Under these situations repeating ordering without maitaining inventory becomes more advantageous. The usual inventory theory does not deal with such cases.

In the classical theories of EOQ-Model, the assumption that $p(t)$ = constant, $q(t)$ = constant, $r(t)$ = constant and $C(t)$ = constant is introduced. Then, it is assumed that $\dot{q}(t) = 0$, $\dot{r}(t) = 0$ and $\dot{C}(t) = 0$. Instead of the assumption that variables are constant, the following assumptions are introduced. Under these assumptions variables may change moderately. In this meaning our analyses are concerned with more general cases than the cases assumed by EOQ-Model.

Assumption 2

1. $p(t) \gg \frac{d}{dt}p(t) \geq \frac{d^2}{dt^2}p(t) \geq 0.$
2. $q(t) \gg \frac{d}{dt}q(t) \geq \frac{d^2}{dt^2}q(t) \geq 0.$
3. $r(t) \gg \frac{d}{dt}r(t) \geq \frac{d^2}{dt^2}r(t) \geq 0.$
4. $C(t) \gg \frac{d}{dt}C(t) \geq \frac{d^2}{dt^2}C(t) \geq 0, \quad |\frac{d^2}{dt^2}C(t)| \ll 1.$

Assumption 3

1. $p(t) > \frac{d}{dt}q(t).$
2. $p(t) - \frac{d}{dt}q(t) > \frac{d}{dt}(q(t)r(t)), \quad \frac{d}{dt}p(t) - \frac{d^2}{dt^2}q(t) \geq 0.$
3. $q(t)r(t) \geq \frac{d}{dt}C(t).$

The right-sides of 1~3 of Assumption 3 are the derivative functions of time-dependent variables. Then, the meanings of the above assumptions are almost self-evident. Suppose that 3 is not valid. We have $q(t)r(t) < (d/dt) \cdot C(t)$. This brings the very abnormal results. Assume, for example, $q(0)r(0) = C(0)/200$. Then after one year (that is, $t = 365$), $C(t) > C(0) + (365/200)C(0) \approx 2.8$. Setup cost becomes almost three times as large as initial one after one year. This can not be observed ordinarily.

Let $R(t)$ and $Q(t_1, t_2)$ be defined by

$$R(t) = \int_0^t r(\tau)d\tau, \quad Q(t_1, t_2) = \int_{t_1}^{t_2} r(\tau)d\tau = R(t_2) - R(t_1),$$

when $t_1 < t_2$. $Q(t_1, t_2)$ denotes the total amount of demand which arises during the period $[t_1; t_2]$. We can easily verify that

$$z_1 < z_2 \quad \Longrightarrow \quad Q(z_1, z_2) > 0.$$

Assume that the order is placed once during $[t_1; t_2]$. The total buying cost is

$$q(t_1)Q(t_1, t_2) = q(t_1)(R(t_2) - R(t_1)).$$

Therefore, the total inventory cost during $[t_1; t_2]$ when the order is placed once is easily given.

Definition 1 For any positive numbers t_1 and t_2, the function $f(t_1, t_2)$ is given by

$$f(t_1, t_2) = C(t_1) + \int_{t_1}^{t_2} p(\tau)\{Q(t_1, t_2) - Q(t_1, \tau)\}d\tau + q(t_1)Q(t_1, t_2).$$

Definition 2 For any positive t, we define $F_n(t)$ recursively by

$$F_1(t) = f(0, t), \qquad F_2(t) = \min_{0 \leq \tau \leq t}[F_1(\tau) + f(\tau, t)]$$

and

$$F_{n+1}(t) = \min_{0 \leq \tau \leq t}[F_n(\tau) + f(\tau, t)].$$

In Definition 2, $F_n(t)$ denotes the minimal total inventory cost required when order is placed n times during $[0;t]$. It is easy to show

$$\begin{aligned} F_1(t) &= f(0,t) = C(0) + \int_0^t p(\tau)\{Q(0,t) - Q(0,\tau)\}d\tau + q(0)Q(0,t) \\ &= C(0) + R(t)\int_0^t p(\tau)d\tau - \int_0^t p(\tau)R(\tau)d\tau + q(0)R(t), \qquad (2.1) \\ F_2(0,t) &= \min_{0\le\tau\le t}[F_1(\tau) + f(\tau,t)] \\ &= \min_{0\le\tau\le t}[C(0) + \int_0^\tau p(\tau)\{Q(0,\tau) - Q(0,\xi)\}d\xi + q(0)Q(0,\tau) \\ &\qquad + C(\tau) + \int_\tau^t p(\xi)\{Q(\tau,t) - Q(\tau,\xi)\}d\xi + q(\tau)Q(\tau,t)]. \qquad (2.2) \end{aligned}$$

And

$$F_3(t) = \min_{0\le\tau\le t}[F_2(\tau) + f(\tau,t)] = \min_{0\le t_1\le t_2\le t}[F_1(t_1) + f(t_1,t_2) + f(t_2,t)]. \qquad (2.3)$$

It should be noted that in the above equation we have

$$\begin{aligned} f(t_1,t_2) &= C(t_1) + \int_{t_1}^{t_2} p(\tau)\{Q(t_1,t_2) - Q(t_1,\tau)\}d\tau + q(t_1)Q(t_1,t_2) \\ &= R(t_2)\int_0^{t_2} p(\tau)d\tau - \int_0^{t_2} p(\tau)R(\tau)d\tau - R(t_2)\int_0^{t_1} p(\tau)d\tau \\ &\quad + q(t_1)R(t_2) + C(t_1) + \int_0^{t_1} p(\tau)R(\tau)d\tau - q(t_1)R(t_1). \qquad (2.4) \end{aligned}$$

The equation (2.4) will be applied in many cases. Considering (2.1) $\sim$ (2.3), the following definitions are obtained by DP;

Definition 3 We define $F_1(T), t_{1.1}$ and $t_{1.2}$ by

$$F_1(T) = C(0) + R(T)\int_0^T p(\tau)d\tau - \int_0^T p(\tau)R(\tau)d\tau + q(0)R(T), \quad t_{1.1} = 0, \quad t_{1.2} = T.$$

And we define $F_2(T)$ and $t_{2.2}$ by

$$F_2(T) = \min_t G_2(t,T) = G_2(t_{2.2},T),$$

where $G_2(t,T)$ is given by

$$\begin{aligned} G_2(t,T) &= F_1(t) + f(t,T) \\ &= C(0) + R(T)\int_0^T p(\tau)d\tau - \int_0^T p(\tau)R(\tau)d\tau + q(0)R(T) \\ &\quad + C(t) + R(T)\int_0^t p(\tau)d\tau + q(0)R(t) - q(0)R(T) \\ &\quad - R(T)\int_0^t p(\tau)d\tau + q(t)R(T) - q(t)R(t). \qquad (2.5) \end{aligned}$$

We define $t_{2.1}$ and $t_{2.3}$ by

$$t_{2.1} = 0, \quad t_{2.3} = T.$$

When $F_n(T)$ is defined, $F_{n+1}(T)$ and $t_{n+1.n+1}$ are defined by

$$F_{n+1}(T) = \min_t G_{n+1}(t,T) = G_{n+1}(t_{n+1.n+1},T),$$

where

$$G_{n+1}(t,T) = F_n(t) + f(t,T). \tag{2.6}$$

Moreover the followings are defined;

$$t_{n+1.1} = 0, \quad t_{n+1.n+2} = T.$$

It is easy to show that

$$|F_n(T)| \geq n \cdot \min_{0\leq t\leq T} C(t) + \min_{0\leq t\leq T} q(t) \cdot R(0)$$

when T is given. Therefore, when T is given, we have

$$\lim_{n\to\infty} F_n(T) = \infty.$$

Definition 4 For the given T, the total inventory cost $F(T)$ required during $[0;T]$ is given by

$$F(T) = \min[F_1(T), F_2(T), F_3(T), \cdots].$$

In Definition 4, $F_i(T)$; $i = 1,2,3,\cdots$ denote the (imarginary) total costs which are expected when the inventory policy is planned at $t = 0$. It should be noted that only finite series of $F_1(T), F_2(T), F_3(T), \cdots, F_n(T)$ may be examined. As $\lim_{n\to\infty} F_n(T) = \infty$, a proper n_0 ($n_0 \ll [T] + 1$ in ordinary case) times of ordering must be considered. And $F_{n_0}(T)$ is actually realized.

Definition 5 Assume T and n are given. Moreover assume that $F_i(T)$ is defined for any integer i satisfying $2 \leq i \leq n$. Once $t_{n.n}$ is defined, $t_{n.n-1}$ is defined so that it may satisfy

$$F_{n-1}(t_{n.n}) = \min_t[F_{n-2}(t) + f(t,t_{n.n})] = F_{n-2}(t_{n.n-1}) + f(t_{n.n-1}, t_{n.n}).$$

Moreover, when $t_{n.n-i}$ ($0 \leq i \leq n$) is defined, $t_{n.n-i-1}$ ($n - i - 1 \geq 2$) is defined by

$$F_{n-i-1}(t_{n.n-i}) = \min_t[F_{n-i-2}(t) + f(t,t_{n.n-i})] = F_{n-i-2}(t_{n.n-i-1}) + f(t_{n.n-i-1}, t_{n.n-i}).$$

It is needless to say that $t_{n.i}$ ($1 \leq i \leq n$) is the i-th ordering time point when T and n are given. We must regard $t_{n.i}$ ($0 \leq i \leq n$) as the function of T. But in order to simplify our description, the notation $t_{n.i}$ is used instead of $t_{n.i}(T)$.

3. Properties of Inventory Cost

In this section we consider the relationships between the interval $[0;T]$ and $F_n(T)$. The results of our consideration will show that our definitions about the inventory cost are reasonable.

Theorem 1 $\dfrac{d}{dT}F_n(T) > 0.$

Proof Assume $n = 1$. From (2.5),

$$\frac{d}{dT}F_1(T) = r(T)\int_0^T p(\tau)d\tau + q(0)r(T) > 0. \tag{3.1}$$

Assume $n \geq 2$. $F_n(T)$ is defined as

$$F_n(T) = \min_t G_n(t,T) = G_n(t_{n.n},T). \tag{3.2}$$

where

$$G_n(t,T) = F_{n-1}(t) + f(t,T).$$

In other words, $t_{n.n}$ is defined so that it may satisfy

$$\frac{\partial}{\partial t_{n.n}} G_n(T_{n.n},T) = 0 \quad \Rightarrow \quad \frac{\partial}{\partial t_{n.n}} F_{n-1}(t_{n.n}) + \frac{\partial}{\partial t_{n.n}} f(t_{n.n},T) = 0. \tag{3.3}$$

It should be noted that $t_{n.n}$ is determined by T. But in stead of denoting $t_{n.n}(T)$, $t_{n.n}$ is used hereafter.

From (2.4) it is shown that

$$\frac{\partial}{\partial T} f(t_{n.n},T) = r(T)(\int_{t_{n.n}}^{T} p(\tau)d\tau + q(t_{n.n})). \tag{3.4}$$

The equation (3.2) defines $F_n(T)$ by

$$F_n(T) = F_{n-1}(t_{n.n}) + f(t_{n.n},T).$$

Differentiate the both sides of the above equation with respect to T. Then

$$\frac{d}{dT}F_n(T) = \frac{\partial}{\partial t_{n.n}}(F_{n-1}(t_{n.n}) + f(t_{n.n},T))\frac{dt_{n.n}}{dT} + \frac{\partial}{\partial T} f(t_{n.n},T). \tag{3.5}$$

Substituting (3.3) and (3.4) into (3.5),

$$\frac{d}{dT}F_n(T) = r(T)(\int_{t_{n.n}}^{T} p(\tau)d\tau + q(t_{n.n})) > 0.$$

Therefore the theorem is proven.

Theorem 2 $\quad \frac{d}{dt}r(t) > 0 \quad \Longrightarrow \quad \frac{d^2}{dT^2}F_n(T) > 0.$

Proof Assume $n = 1$. From (2.1) and (3.1) we have

$$\frac{d^2}{dT^2} f(0,T) = r'(T)(\int_0^T p(\tau)d\tau + q(0)) + r(T)p(T) > 0.$$

Assume $n \geq 2$. Differentiate the both sides of (3.5),

$$\begin{aligned} \frac{d^2}{dT^2}F_n(0,T) &= (\frac{\partial^2}{\partial t_{n.n}^2}(F_{n-1}(t_{n.n}) + f(t_{n.n},T)) \cdot \frac{dt_{n.n}}{dT} \\ &\quad + \frac{\partial^2}{\partial T \partial t_{n.n}} f(t_{n.n},T) \cdot \frac{dt_{n.n}}{dT} + \frac{\partial^2}{\partial T^2} f(t_{n.n},T). \end{aligned} \tag{3.6}$$

Differentiate both side of (3.3) with respect to T, and we have

$$\frac{\partial^2}{\partial t_{n.n}^2}(F_{n-1}(t_{n.n}) + f(t_{n.n},T))\frac{dt_{n.n}}{dT} + \frac{\partial^2}{\partial T \partial t_{n.n}} f(t_{n.n},T) = 0. \tag{3.7}$$

By substituting (3.7) into (3.6), we have

$$\frac{d^2}{dT^2}F_n(T) = \frac{\partial^2}{\partial T^2} f(t_{n.n},T).$$

The equation (3.4) shows

$$\frac{\partial^2}{\partial T^2} f(t_{n.n}, T) = r'(T)\left(\int_{t_{n.n}}^{T} p(\tau)d\tau + q(t_{n.n})\right) + r(T)p(T) > 0.$$

Therfore the theorem is proven.

Theorem 3 $T < T' \implies F(T) < F(T')$.

Proof From Theorem 1,

$$T < T' \Rightarrow F_n(T) < F_n(T')\ ;\ n = 1, 2, 3, \cdots.$$

Let $F(T)$ and $F(T')$ be given by

$$\begin{aligned} F(T) &= \min[F_1(T), F_2(T), F_3(T), \cdots] = F_n(T), \\ F(T') &= \min[F_1(T'), F_2(T'), F_3(T'), \cdots] = F_{n'}(T'). \end{aligned}$$

In the general case, there is no reason why $n = n'$ is supposed. As $T < T'$,

$$F_n(T) < F_n(T'), \quad F_{n'}(T) < F_{n'}(T').$$

Therefore,

$$F(T) = F_n(T) \le F_{n'}(T) < F_{n'}(T') = F(T').$$

Now consider the function defined by

$$G_n(t, T) = F_{n-1}(t) + f(t, T).$$

The variable t, which appears in the above equation, is not $t_{n.n}$ defined in Theorem 1~2. The variable t can vary freely in the domain $0 \le t \le T$. Assume that $t_{n.n}$ is defined under the condition that T is fixed and t varies freely. Then

$$\begin{aligned} \frac{d}{dt} G_n(t, T)|_{t=t_{n.n}} &= \frac{d}{dt}(F_{n-1}(t) + f(t, T))|_{t=t_{n.n}} = 0, \\ \frac{d^2}{dt^2} G_n(t, T)|_{t=t_{n.n}} &= \frac{d^2}{dt^2}(F_{n-1}(t) + f(t, T))|_{t=t_{n.n}} > 0. \end{aligned}$$

Assume that $d/dt\ r(t) > 0$. Then from Theorem 2 we have

$$\frac{d^2}{dt^2} F_{n-1}(t) > 0.$$

The possibility that $d^2/dt^2\ G_n(t, T) < 0$ is valid depends on the sign of $d^2/dt^2\ f(t, T)$. For example, if

$$\frac{d^2}{dt^2} f(t, T) > 0 \quad \text{and} \quad \frac{d}{dt} G_n(t, T)|_{t=0} > 0,$$

we can not define $t_{n.n}$. Under Assumption 1~3, (2.5) results in

$$\frac{d}{dt} f(t, T) = -\{R(T) - R(t)\}(p(t) - \frac{d}{dt}q(t)) + \frac{d}{dt}C(t) - q(t)r(t) < 0, \tag{3.8}$$

$$\begin{aligned} \frac{d^2}{dt^2} f(t, T) &= r(t)(p(t) - \frac{d}{dt}q(t)) - \frac{d}{dt}q(t) \cdot r(t) - q(t)\frac{d}{dt}r(t) \\ &\quad -\{R(T) - r(t)\}(\frac{d}{dt}p(t) - \frac{d^2}{dt^2}q(t)) + \frac{d^2}{dt^2}C(t). \end{aligned} \tag{3.9}$$

It should noted that $-\{R(T)-r(t)\}$ is an increasing function of t.

Lemma 1 Assume that

$$\frac{d}{dt}p(t)-\frac{d^2}{dt^2}q(t)$$

is increasing with respect to t. Then for $n \geq 2$ there exists at the very most one t^* satisfying

$$\frac{d}{dt}G_n(t,T)|_{t=t^*}=0 \quad \text{and} \quad \frac{d^2}{dt^2}G_n(t,T)|_{t=t^*}>0.$$

Proof

$$\{r(t)-R(T)\}(\frac{d}{dt}p(t)-\frac{d^2}{dt^2}q(t))+\frac{d^2}{dt^2}C(t)$$

is an increasing (non-decreasing) function of t. From Theorem 1 and 2,

$$\frac{d}{dt}F_{n-1}(t)>0 \quad \text{and} \quad \frac{d^2}{dt^2}F_{n-1}(t)>0.$$

The equation (3.8) shows that $d/dt\ f(t,T)<0$. Assume that at $t=t^*$ it is valid that

$$\frac{d}{dt}G_n(t,T)|_{t=t^*}=\frac{d}{dt}(F_{n-1}(t)+f(t,T))|_{t=t^*}=0$$

and

$$\frac{d^2}{dt^2}G_n(t,T)|_{t=t^*}=\frac{d^2}{dt^2}(F_{n-1}(t)+f(t,T))|_{t=t^*}>0.$$

Applying (3.9) and the assumption of this lemma, we have $d^2/dt^2\ G_n(t,T)>0$ for any t satisfying $t>t^*$. Moreover, if $t>t^*$ it is easy to show that

$$\frac{d^2}{dt^2}F_{n-1}(t)>0 \Rightarrow \frac{d}{dt}F_{n-1}(t)|_{t=t}>\frac{d}{dt}F_{n-1}(t)|_{t=t^*},$$
$$\frac{d^2}{dt^2}f(t,T)>0 \Rightarrow \frac{d}{dt}f(t,T)|_{t=t}>\frac{d}{dt}F_{n-1}(t)|_{t=t^*}.$$

In other words, if $t>t^*$,

$$\frac{d^2}{dt^2}G_n(t,T)>0 \quad \text{and} \quad \frac{d}{dt}G_n(t,T)>0.$$

Corollary 1 Assume that $t_{n.n}$ is defined for n. Then

$$\frac{d}{dt}G_n(t,T)|_{t=t^*}<0 \Rightarrow t^*<t_{n.n}.$$

4. Ordering Time Points

In this section the fundamental properties of the ordering time points will be examined. Analyses on these properties will be presented in some theorems. As a result the meanings of the general EOQ model will be clarified.

Theorem 4 Assume that $t_{n.n}$ is defined under Assumption 2. Then

$$\frac{d}{dt}(p(t)-\frac{d}{dt}q(t))|_{t=t_{n.n}}>0 \quad \Rightarrow \quad \frac{d}{dT}t_{n.n}>0.$$

Proof Applying (2.4) and (2.5), $G_n(t,T)$ is expressed by

$$G_n(t,T) = H_n(t) + \Gamma_T(t),$$

where

$$\begin{aligned} H_n(t) &= F_{n-1}(t) + C(t) + \int_0^t p(\tau)R(\tau)d\tau - q(t)R(t), & (4.1)\\ \Gamma_T(t) &= R(T)\int_0^T p(\tau)d\tau - \int_0^T p(\tau)R(\tau)d\tau - R(T)\int_0^t p(\tau)d\tau + q(t)R(T). & (4.2) \end{aligned}$$

It should be noted that t does not appear in $H_n(t)$. At $t = t_{n.n}$,

$$\frac{d}{dt}G_n(t,T)|_{t=t_{n.n}} = 0 \quad \text{and} \quad \frac{d^2}{dt^2}G_n(t,T)|_{t=t_{n.n}} > 0.$$

In the above, $t_{n.n}$ satisfying these conditions is unique. Therefore,

$$\begin{aligned} &\frac{d}{dt}(H_n(t)+\Gamma_T(t))|_{t=t_{n.n}} = 0, \quad \frac{d^2}{dt^2}(H_n(t)+\Gamma_T(t))|_{t=t_{n.n}} > 0,\\ &\frac{d}{dt}\Gamma_T(t)|_{t=t_{n.n}} = -R(T)(p(t) - \frac{d}{dt}q(t))|_{t=t_{n.n}} < 0, & (4.3)\\ &\frac{d^2}{dt^2}\Gamma_T(t)|_{t=t_{n.n}} = -R(T)\frac{d}{dt}(p(t) - \frac{d}{dt}q(t))|_{t=t_{n.n}} < 0, & (4.4) \end{aligned}$$

and

$$\frac{d}{dt}H_n(t)|_{t=t_{n.n}} > 0, \quad \frac{d^2}{dt^2}H_n(t)|_{t=t_{n.n}} > 0.$$

In other words at $t = t_{n.n}$, $d/dt\ H_n(t)$ is increasing and $d/dt\ \Gamma_T(t)$ is decreasing. We have $R(t) < R(T^*)$ for any T^* satisfying $T < T^*$. From (4.4),

$$0 > \frac{d}{dt}\Gamma_T(t)|_{t=t_{n.n}} > \frac{d}{dt}\Gamma_{T^*}(t)|_{t=t_{n.n}}.$$

Assume that for a proper $t^*_{n.n}$

$$\frac{d}{dt}H_n(t)|_{t=t^*_{n.n}} + \frac{d}{dt}\Gamma_{T^*}(t)|_{t=t^*_{n.n}} = 0.$$

Then, it is easily shown that $t^*_{n.n} > t_{n.n}$. As $(p(t) - q'(t))'$ is an increasing function,

$$\frac{d^2}{dt^2}G_n(t,T^*)|_{t=t^*_{n.n}} > 0.$$

According to Lemma 1, $t^*_{n.n}$, that satisfies these properties, is unique, even if $t^*_{n.n}$ exists and is defined. Moreover, $t^*_{n.n}$ always exists whenever $t_{n.n}$ exists. Therefore we have $T < T^* \Rightarrow t_{n.n} < t^*_{n.n}$, if $t_{n.n}$ can be defined. In other words we have proven that $d/dT\ t_{n.n} > 0$.

Lemma 2

1. $p(t) - q'(t) > 0 \Rightarrow \int_0^t p(\tau)d\tau + p(0) - q(t) > 0$,
2. $\frac{d}{dT}f(t,T)|_{t=0} > \frac{d}{dT}f(t,T)|_{t>0}$.

Proof The proof is self-evident.

Lemma 3 Under Assumption 1~3, we have

$$r(T)(p(T)-q'(T)) < 2r(0)(p(0)-q'(0)) \quad \Rightarrow \quad d/dT\ t_{n.n} < 1.$$

Proof $t_{n.n}$ is a function of T. And the relationship between $t_{n.n}$ and T is given by

$$\begin{aligned}&F'_{n-1}(t_{n.n}) + C'(t_{n.n}) + p(t_{n.n})R(t_{n.n}) - q'(t_{n.n})R(t_{n.n}) - q(t_{n.n})r(t_{n.n})\\ &\quad = r(T)\{p(t_{n.n}) - q'(t_{n.n})\}.\end{aligned} \tag{4.5}$$

Differentiating the both sides with respect to T,

$$\begin{aligned}&\Big[F''_{n-1}(t_{n.n}) + C''(t_{n.n}) + p'(t_{n.n})R(t_{n.n}) + p(t_{n.n})r(t_{n.n}) - q''(t_{n.n})R(t_{n.n})\\ &-q'(t_{n.n})r(t_{n.n}) - q'(t_{n.n})r(t_{n.n}) - q(t_{n.n})r'(t_{n.n}) - R(T)\{p'(t_{n.n}) - q''(t_{n.n})\}\Big]\frac{dt_{n.n}}{dT}\\ &= r'(T)\{p(t_{n.n}) - q'(t_{n.n})\}.\end{aligned} \tag{4.6}$$

Moreover, we have

$$\frac{d^2}{dt^2_{n.n}}F_{n-1}(t_{n.n}) = r'(t_{n.n})\int_{t_{n-1}}^{t_{n.n}} p(\tau)d\tau + q(t_{n.n})r'(t_{n.n}) + p(t_{n.n})r(t_{n.n}),$$

where t_{n-1} signifies $t_{n-1.n-1}$ which is obtained through assuming $T = t_{n.n}$.

$$\begin{aligned}&R(t_{n.n})\{p'(t_{n.n}) - q''(t_{n.n})\} - R(T)\{p'(t_{n.n}) - q''(t_{n.n})\}\\ &\quad \geq r(0)\cdot t_{n.n}\cdot\{p'(t_{n.n}) - q''(t_{n.n})\} - r(T)\cdot T\cdot\{p'(T) - q''(T)\}\\ &\quad \geq r(0)(p(0)-q'(0)) - r(T)(p(T)-q'(T)) \geq -r(0)(p(0)-q'(0)),\\ &r(t_{n.n})(p(t_{n.n}) - q'(t_{n.n})) > r(0)(p(0)-q'(0)),\\ &r'(t_{n.n})\int_{t_{n-1}}^{t_{n.n}} p(\tau)d\tau \geq r'(t_{n.n})\cdot(t_{n.n}-t_{n-1})\cdot p(0)\\ &\quad \geq (t_{n.n}-t_{n-1})\cdot r'(t_{n.n}-t_{n-1})p(0) \geq r(0)p(0) > r(0)(p(0)-q'(0)).\end{aligned}$$

Substituting these into (4.6),

$$\begin{aligned}&\{2r(0)(p(0)-q'(0))\}\frac{d}{dT}t_{n.n} < r(T)(p(t_{n.n}) - q'(t_{n.n}))\\ &\quad < r(T)(p(T)-q'(T)) < 2r(0)(p(0)-q'(0)).\end{aligned}$$

Therefore the theorem is proven.

Theorem 5 Assume $t_{2.2}, t_{3.3}, t_{4.4}, \cdots$ are defined under the assumptions introduced in Lemma 3. Then,

$$t_{n.n} < t_{n+1.n+1};\ n = 1, 2, 3, \cdots.$$

Proof When $n = 1$, the theorem is self-evident, for $t_{1.1} = 0$ from the definition.

Assume that the theorem is proven when $n = m$. From definition,

$$\frac{d}{dt}(F_m(t) + f(t,T))|_{t=t_{m+1.m+1}} = 0, \quad \frac{d}{dt}(F_{m-1}(t) + f(t,T))|_{t=t_{m.m}} = 0.$$

Let $F_m(t_{m+1.m+1})$ be given by

$$F_m(t_{m+1.m+1}) = F_{m-1}(t') + f(t', t_{m+1.m+1}),$$

where t' is defined so that it may satisfy

$$\frac{d}{d\tau}(F_{m-1}(\tau)+f(\tau,t_{m+1.m+1}))|_{\tau=t'}=0.$$
$$\frac{d}{dt}F_m(t)|_{t=t_{m+1.m+1}}=\{\frac{\partial}{\partial\tau}(F_{m-1}(\tau)+f(\tau,t))\frac{d\tau}{dt}+\frac{\partial}{\partial t}f(\tau,t)\}|_{t=t_{m+1.m+1},\ \tau=t'}.$$

In other words, we have

$$\frac{d}{dt}F_m(t)|_{t=t_{m+1.m+1}}>\frac{\partial}{\partial\tau}(F_{m-1}(\tau)+f(\tau,t))|_{t=t_{m+1.m+1},\ \tau=t'}.$$

Therefore

$$\begin{aligned} 0 &= \frac{d}{dt}(F_m(t)+f(t,T))|_{t=t_{m+1.m+1}} \\ &> \frac{\partial}{\partial\tau}(F_{m-1}(\tau)+f(\tau,t))\frac{d\tau}{dt}|_{t=t_{m+1.m+1},\tau=t'}+\frac{d}{dt}f(t,T)|_{t=t_{m+1.m+1}} \\ &= \frac{d}{dt}(F_{m-1}(t)+f(t,T))|_{t=t_{m+1.m+1}}. \end{aligned}$$

With the results of Corollary 1, theorem is proven when $n=m+1$.

Corollary 2 Assume that $t_{2.2},t_{3.3},t_{4.4},\cdots$ are defined. Then
1. $t_{2.2}>t_{3.2}>t_{4.2}>t_{5.2}>\cdots$
2. $t_{3.3}>t_{4.3}>t_{5.3}>t_{6.3}>\cdots$
3. $t_{n.3}>t_{n-1.2}\ ;\ n\geq 3$.

Proof Express the function, which determines $t_{n.n}$ from T, by $t_{n.n}=\varphi_{n.n}(T)$. With this function, corollary is proven.

1 It is shown that $T>t_{3.3}$. From Definition 3 and Theorem 4,

$$t_{2.2}=\varphi_{2.2}(T)\geq\varphi_{2.2}(t_{3.3})=t_{3.2}.$$

More generally

$$\begin{aligned} &T>t_{n+1.n+1} \quad\Rightarrow\quad t_{n.n}=\varphi_{n.n}(T)\geq\varphi_{n.n}(t_{n+1.n+1})=t_{n+1.n}. \\ &t_{n.2}=\varphi_{2.2}\varphi_{3.3}\cdots\varphi_{n-1.n-1}(t_{n.n})\geq\varphi_{2.2}\varphi_{3.3}\cdots\varphi_{n-1.n-1}(t_{n+1.n})=t_{n+1.2}. \end{aligned}$$

2 is proven in the similar way.

3 $t_{3.3}>t_{2.2}$ is self-evident from Theorem 5. Moreover Theorem 5 shows $t_{4.4}>t_{3.3}$. From Theorem 4 and Theorem 5,

$$t_{4.3}=\varphi_{3.3}(t_{4.4})>\varphi_{3.3}(t_{3.3})>\varphi_{2.2}(t_{3.3})=t_{3.2}.$$

In the same ways,

$$\begin{aligned} &t_{5.5}>t_{4.4} \quad\Rightarrow\quad t_{5.4}=\varphi_{4.4}(t_{5.5})>\varphi_{3.3}(t_{4.4})=t_{4.3}, \\ &t_{5.3}=\varphi_{3.3}(t_{5.4})>\varphi_{3.3}(t_{4.3})>\varphi_{2.2}(t_{4.3})=t_{4.2}. \end{aligned}$$

5. Time Interval and Frequency of Orderings

In this section, the relationship between the length of time interval and the number of times of ordering will be examined. Let $\phi(\tau_1, \tau_2)$ be defined by

$$\begin{aligned}\phi(\tau_1,\tau_2) &= F_1(\tau_1) - F_1(\tau_2) + f(\tau_1,\tau_2) \\ &= C(\tau_1) + R(\tau_1)\int_0^{\tau_1} p(\tau)d\tau + q(0)R(\tau_1) - q(0)R(\tau_2) \\ &\quad -R(\tau_2)\int_0^{\tau_1} p(\tau)d\tau + q(\tau_1)R(\tau_2) - q(\tau_1)R(\tau_1)\end{aligned} \tag{5.1}$$

for any τ_1 and τ_2 satisfying $0 \le \tau_1 \le \tau_2 \le T$. Then, for any t $(0 \le t \le T)$,

$$F_1(t) + f(t,T) = F_1(T) + \phi(t,T), \tag{5.2}$$

and for any $t_1, t_2, \cdots, t_{n-1}$ satisfying $0 \le t_1 \le t_2 \le \cdots \le t_{n-1} \le T$,

$$\begin{aligned}&F_1(t_1) + f(t_1,t_2) + f(t_2,t_3) + \cdots + f(t_{n-1},T) \\ &\quad = F_1(t_2) + f(t_2,t_3) + \cdots + f(t_{n-1},T) + \phi(t_1,t_2).\end{aligned} \tag{5.3}$$

Suppose $t_{n.2}, t_{n.3}, \cdots, t_{n.n-1}$ and $t_{n.n}$ are those which are defined by Definition 5, then

$$\begin{aligned}F_n(T) &= \min_{0\le t_1\le t_2\le\cdots\le t_{n-1}\le T}[F_1(t_1) + f(t_1,t_2) + \cdots + f(t_{n-2},t_{n-1}) + f(t_{n-1},T)] \\ &\le F_1(t_{n-1.2}) + f(t_{n-1.2},t_{n-1.3}) + \cdots + f(t_{n-1.n-1},T) + \min_{0\le t_1\le t_{n-1.2}}\phi(t_1,t_{n-1.2}) \\ &= F_{n-1}(T) + \min_{0\le t_1\le t_{n-1.2}}\phi(t_1,t_{n-1.2}).\end{aligned} \tag{5.4}$$

We have

$$\begin{aligned}F_n(T) &= f(0,t_{n.2}) + f(t_{n.2},t_{n.3}) + \cdots + f(t_{n.n},T) \\ &= f(0,t_{n.3}) + f(t_{n.3},t_{n.4}) + \cdots + f(t_{n.n},T) + \phi(t_{n.2},t_{n.3}).\end{aligned}$$

From definition,

$$F_1(t_{n.3}) + f(t_{n.3},t_{n.4}) + \cdots + f(t_{n.n},T) \ge F_{n-1}(T).$$

Therefore

$$F_n(T) \ge F_{n-1}(T) + \phi(t_{n.2},t_{n.3}) \ge F_{n-1}(T) + \min_{0\le t\le t_{n.3}}\phi(t,t_{n.3}). \tag{5.5}$$

Moreover from (5.1) it is easily shown that

$$t_{n.3} > t_{n-1.2} \quad \Rightarrow \quad \min_{0\le t\le t_{n+1.2}}\phi(t,t_{n+1.2}) \ge \min_{0\le t\le t_{n.3}}\phi(t,t_{n.3}). \tag{5.6}$$

Using these results, the following theorem is proven;

Theorem 6 For proper $L_1, L_2, L_3, \cdots$, which satisfy $0 < L_1 < L_2 < L_3, \cdots$, we have

$$\begin{aligned}0 < T \le L_1 &\quad\Rightarrow\quad F(T) = F_1(T), \\ L_1 < T \le L_2 &\quad\Rightarrow\quad F(T) = F_2(T), \\ L_2 < T \le L_3 &\quad\Rightarrow\quad F(T) = F_3(T).\end{aligned}$$

Proof

$$F_2(T) = F_1(T) + \min_{0\le t\le T}\phi(t,T)$$

$$F_3(T) \leq F_2(T) + \min_{0 \leq t \leq t_{2.2}} \phi(t, t_{2.2}) \tag{5.7}$$

$$F_3(T) \geq F_2(T) + \min_{0 \leq t \leq t_{3.3}} \phi(t, t_{3.3}) \tag{5.7$'$}$$

$$F_4(T) \leq F_3(T) + \min_{0 \leq t \leq t_{3.2}} \phi(t, t_{3.2}) \tag{5.8}$$

$$F_4(T) \geq F_3(T) + \min_{0 \leq t \leq t_{4.3}} \phi(t, t_{4.3}) \tag{5.8$'$}$$

$$\cdots$$

It is shown that

$$\phi(t,T) = C(t) + (R(t) - R(T))\{\int_0^t p(\tau)d\tau + q(0) - q(t)\}.$$

When $T = 0, t = 0$. As $\phi(0,0) = C(0) > 0, \phi(t,T) > 0$ for relatively small T. In this case $\min \phi(t,T) > 0$. As T becomes large enough, $\phi(t,T) < 0$, in other words it is shown $\min \phi(t,T) < 0$.

Let ξ_1 and ξ_2 be defined by

$$\xi_1 = \{T \mid \min_{0 \leq t \leq T} \phi(t,T) \geq 0,\ T > 0\}, \quad \xi_2 = \{T \mid t > 0,\ T \notin \xi_1\}.$$

Then, it is easily shown that

$$T_1 \in \xi_1, T_2 \in \xi_2 \ \Rightarrow\ T_1 < T_2;\ T_1 \in \xi_1, T_2 < T_1 \ \Rightarrow\ T_2 \in \xi_1;\ T_1 \in \xi_2, T_1 < T_2 \ \Rightarrow\ T_2 \in \xi_2.$$

According to the continuity of real numbers (Dedekind's axiom about the cut of real number), there exists a proper number L_1 satisfying

$$T < L_1 \quad \Rightarrow \quad F_1(T) < F_2(T), \tag{5.9}$$

$$T = L_1 \quad \Rightarrow \quad F_1(T) = F_2(T), \tag{5.9$'$}$$

$$T > L_1 \quad \Rightarrow \quad F_1(T) > F_2(T). \tag{5.9$''$}$$

Assume $T < L_1$. Then $F_1(T) < F_2(T) \ \Rightarrow\ \min_{0 \leq t \leq T} \ \phi(t,T) > 0$.

As $t_{3.3} < T, 0 \leq \min_{0 \leq t \leq T} \ \phi(t,T) \leq \min_{0 \leq t \leq t_{3.3}} \ \phi(t, t_{3.3})$. Therefore, from (5.7)$'$, $T < L_1 \ \Rightarrow\ F_2(T) < F_3(T)$. From Corollary 2, $t_{3.3} > t_{4.3} > t_{5.3} > \cdots$. So, from (5.8)$'$,

$$T < L_1 \ \Rightarrow\ F_2(T) < F_3(T) < F_4(T) < \cdots.$$

In other words,

$$T < L_1 \ \Rightarrow\ F(T) = \min[\ F_1(T), F_2(T), F_3(T), \cdots] = F_1(T).$$

Now, consider (5.7) and (5.7)$'$. When T is relatively small, $t_{3.3}$ is small. Then $\min_{0 \leq t \leq t_{3.3}}$ $\phi(t, t_{3.3}) > 0 \ \Rightarrow\ F_3(T) > F_2(T)$. But when T becomes large enough, $t_{2.2}$ becomes large. In this case, $\min_{0 \leq t \leq t_{2.2}} \ \phi(t, t_{2.2}) < 0$.

From (5.7), $F_3(T) < F_2(T)$. Speaking more exactly, almost similar procedures which are applied to $F_1(T)$ and $F_2(T)$ show that there exists a proper number L_2 satisfying

$$\begin{aligned} T < L_2 &\Rightarrow F_2(T) < F_3(T), \\ T = L_2 &\Rightarrow F_2(T) = F_3(T), \\ T > L_2 &\Rightarrow F_2(T) > F_3(T). \end{aligned} \tag{5.10}$$

As $t_{3.3} > t_{4.3} > t_{5.3} > \cdots$,

$$T < L_2 \Rightarrow F_3(T) \leq F_4(T) \leq F_5(T) \leq \cdots. \tag{5.11}$$

Moreover, it can be proven that $L_1 < L_2$. Assume, for example, $L_2 < L_1$. Then from (5.9) for T^* satisfying $L_2 < T^* < L_1, L_2 < T^* \Rightarrow F_2(T^*) < F_3(T^*)$, $T^* < L_1 \Rightarrow F_3(T^*) < F_2(T^*)$. This ia a contradiction. So, we must assume that $L_1 < L_2$.

For T satisfying $L_1 < T \leq L_2$,

$$F_2(T) \leq F_1(T). \tag{5.12}$$

From (5.10), (5.11) and (5.12), for any T satisfying $L_1 < T \leq L_2$,

$$F(T) = \min[F_1(T), F_2(T), F_3(T), \cdots] = F_2(T).$$

In order to complete the proof, the entirely same procedure is applied.

Moreover, we can easily show

$$\begin{aligned} L_1 < T < L_2 &\Rightarrow 0 < t_{2.2} < L_1, \\ L_2 < T < L_3 &\Rightarrow 0 < t_{3.2} < L_1 < t_{3.3} < L_2, \\ L_3 < T < L_4 &\Rightarrow 0 < t_{4.2} < L_1 < t_{4.3} < L_2 < t_{4.4}, \\ L_4 < T < L_5 &\Rightarrow 0 < t_{5.2} < L_1 < t_{5.3} < L_2 < t_{5.4} < L_4 < t_{5.5} < L_5. \end{aligned}$$

Corollary 4 $T < T'$, $F(T) = F_n(T)$, $F(T') = F_{n'}(T') \Rightarrow n < n'$.

6. Conclusions

The classical EOQ theories have successfully defined ordering cycle which minimizes one day's average inventory cost, when one day's demand for products, one day's holding cost per unit of product and setup cost are constant. But when they are not constant, it is not easy to define ordering cycle with classical EOQ theory. This is because under such a condition one day's average inventory cost depends on the time point from when the average is calculated.

Assume that one day's demand for the products, one day's holding cost per a product and setup costs are expressed as increasing functions of time respectively. Moreover assume we are given a time interval during which the optimal inventory policy is planned. Instead of considering the minimum one day's average inventory cost, we investigated the minimum total inventory cost required during this interval.

For the given time interval the total inventory cost was defined recursively in respect of the frequency of orderings. The frequency of orderings, which minimizes the total inventory cost, was selected to define the minimum inventory cost required during this interval.

The procedures, with which frequency of orderings and total inventory cost are defined, could be given recursively with the techniques of DP. Exact analytical forms of functions were not necessary to apply DP. Only the values of functions are sufficiently calculated in some ways.

For the given time interval, ordering time points and ordering quantities were determined recursively. Moreover a few mathematical properties of our procedures were examined finely. As the interval becomes longer, the total inventory cost required becomes greater. As the interval becomes longer, the number of time of ordering becomes greater. If DP is applied, the optimal inventory policy is defined effectively with the electronic computer.

In this paper, exact analytical expressions of functions were not assumed. If these are given, more concrete and fruitful results will be obtained. We assumed that the one day's demand for products and one day's holding cost per unit of product are increasing functions of time. Similar procedures will be easily defined even if they are not increasing functions. But rather different conclusions will be obtained under this condition.

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Keisuke Matsuyama
Akita Prefectural University
84-4 Ebino-kuchi, Tutiya-aza, Honjo-shi
Akita 015-0055, Japan