# PERFORMANCE ANALYSIS OF A TWO-QUEUE MODEL WITH AN ( $M, N$ )-THRESHOLD SERVICE SCHEDULE 

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#### Abstract

In this paper, we consider a polling system consisting of two-parallel queues and a single server under an ( $M, N$ )-threshold nonpreemptive priority service schedule. Two thresholds $M$ and $N(0 \leq M<N)$ are set up in one of two queues, say, the second queue. At each epoch of service completion, the server decides which queue is to be served next according to the control level reached by the number of customers in the second queue. For the queueing model, we carry out the performance analysis by using the transform method and propose an algorithm to compute the generating functions of the stationary joint queue length distributions at service completion instants. We also determine the Laplace-Stieltjes transforms of the waiting time distributions for both queues, and obtain their mean waiting times.


## 1. Introduction

Polling systems used for modeling distributed multiqueue systems sharing a single server such as a communication channel or a processor, have received a considerable amount of attention in the recent literature. Examples of such systems are local area networks (LAN), high-speed Asynchronous Transfer Mode (ATM) networks, multiprocessor systems, distributed computation, and so forth. An excellent survey may be found in Levy and Sidi [23]. See also the detailed discussion and references in Takagi[31, 32] on this subject. The polling system, in particular, consisting of two queues and a single server has an important application for modeling communication network systems with two different types of traffic: real-time traffic (such as voice and video) and non-real-time traffic (such as data), for example, hybrid switching voice/data transmission systems, and packetswitched voice/data transmission systems[5]. In order to be able to meet the quality of service requirements for different types of traffic, the model has been extensively studied by many researchers under various service schedules, such as the exhaustive, semi-exhaustive, gated, K-limited, Bernoulli service schedules or mixtures of these service schedules (see [1,2,3,6,7,10,11,12,14,15,18,20,21,22,24,26,28,30]).

Threshold-based service policies have been applied by many researchers to queueing systems with a single queue as policies to control service rate, number of servers or vacation, and proved to be optimal to some queueing systems ([17,19,25,27]). Especially in Nishimura and Jiang[27], two thresholds $n_{H}$ and $n_{R}$ have been used to control the service rate in an $\mathrm{M} / \mathrm{G} / 1$ vacation model. Recently, such threshold-based service policies have been applied by some researchers to the polling system consisting of two queues and a single server. Lee and Sengupta[20] have analyzed a threshold-based polling system, where a single threshold was set up in the high priority queue. If the queue length of the high priority queue exceeds the threshold, the server switches the service to the high priority queue, otherwise, the server serves two queues with a 1-limited service schedule. Boxma and Down[4] have considered
a similar threshold-based service policy which is different from the one in [20] only when the queue length of the high priority does not exceed the threshold, the server serves two queues with a exhaustive service schedule.

In the present paper, we analyze a polling system consisting of two-parallel queues and a single server controlled by two threshold levels $M$ and $N(0 \leq M<N)$. The service schedule is described in detail in Section 2. In modern communication network systems which employ the fixed packet sizes of ATM technology, different types of traffic have different requirements for quality of service. Sometimes these requirements vary according to the system state, and not anyone of these traffics has absolute priority. For example, in a voice/data multiplexing system, or a hybrid switching voice/data transmission system, the voice traffic and the data traffic should have their own priority segment. That is, according to variety of the system state, sometimes the voice traffic has a higher priority over the data traffic, and sometimes contrary. As will be seen, the ( $M, N$ )-threshold service schedule provides such a priority scheduling strategy. $Q_{1}$ has a priority over $Q_{2}$ in the segment $[0, M), Q_{2}$ has a priority over $Q_{1}$ in the segment $(N, \infty)$, and the segment $[M, N]$ is a non-priority part in the sense that server does not switch its service to another queue when the queue-length of $Q_{2}$ is in this segment. Furthermore, if we choose $N=\infty$, then $Q_{1}$ has an absolute priority over $Q_{2}$ and if we choose $N=1$, then $Q_{2}$ has an absolute priority over $Q_{1}$. These considerations motivate us to consider such an $(M, N)$-threshold service schedule for the polling system. To the best of our knowledge, analysis of such a model has not been studied before.

The organization of the paper is as follows. In Section 2 the model is described in detail, and the system equations of the generating functions of the stationary joint queue-length distributions are established. In Section 3, utilizing singularities of the coefficient function and the coefficient matrix, these equations are transformed into a matrix equation in the boundary probabilities. An algorithm to compute the generating functions is proposed. The Laplace-Stieltjes transforms of waiting time distributions, and the mean waiting times are given in Section 4. Finally in Section 5, some numerical results are included.

## 2. The Model and the Generating Function Equations

We consider a polling system consisting of two-parallel queues $Q_{1}$ and $Q_{2}$ with infinite buffer capacities, and a single server. The arrival processes of customers at $Q_{1}$ (corresponding to the real-time traffic) and $Q_{2}$ (corresponding to the non-real-time traffic) are Poisson processes with rates $\lambda_{1}$ and $\lambda_{2}$, respectively. The service times at $Q_{i}$ are independent, identically distributed sequences with general distribution $B_{i}(\cdot)$. Their first moment, second moment and $L S T$ (Laplace-Stieltjes Transform) are denoted by $b_{i}, b_{i}^{(2)}$, and $\hat{B}_{i}(\cdot)$, and assumed to be finite. Two thresholds $M$ and $N(0 \leq M<N)$ are set up in $Q_{2}$. The server serves two queues in accordance with an ( $M, N$ )-threshold nonpreemptive priority service schedule described as follows:
(1) At each epoch of service completion in $Q_{1}$ at which the queue is not empty, if the queue-length in $Q_{2}$ exceeds threshold $N$, the server switches the service to $Q_{2}$; otherwise it continues to serve the customers in $Q_{1}$.
(2) At each epoch of service completion in $Q_{2}$, if the queue-length in $Q_{2}$ is less than or equal to threshold $M$, and $Q_{1}$ is not empty, the server switches the service to $Q_{1}$; otherwise, it continues to serve the customers in $Q_{2}$.
(3) Whenever the queue being served becomes empty at an epoch of service completion, if the other queue is not empty, the server switches the service to that queue; otherwise, the server remains idle at the present queue until the arrival of the next customer in either $Q_{1}$ or $Q_{2}$.

The service is first-come-first-served within each queue and nonpreemptive. The server experiences a switching time in its transition from one queue to another. The successive switching times from $Q_{i}$ form independent, identically distributed sequences with the general distribution $S_{i}(\cdot)$. Their first moment, second moment and $L S T$ are denoted by $s_{i}, s_{i}^{(2)}$, and $\hat{S}_{i}(\cdot)$, and assumed to be finite. The service and switching times and the arrival process are all assumed to be mutually independent.

We introduce the following notations.

$$
\begin{array}{ll}
\lambda \equiv \lambda_{1}+\lambda_{2} ; & r_{i}=\lambda_{i} / \lambda, \quad i=1,2 \\
\rho_{i} \equiv \lambda_{i} b_{i}, i=1,2 ; & \rho \equiv \rho_{1}+\rho_{2}  \tag{2.1}\\
s \equiv s_{1}+s_{2} ; & s^{(2)} \equiv s_{1}^{(2)}+2 s_{1} s_{2}+s_{2}^{(2)}
\end{array}
$$

Then $\rho_{i}$ is the utilization at $Q_{i}$ and $\rho$ is the total utilization of the server. $s$ and $s^{(2)}$ are respectively the first moment, second moment of the total switching time during one cycle. For periodic polling systems with a mixture of various service schedules, Fricker and Jaibi[13] have presented a necessary and sufficient condition for stability. Especially, the condition can be written as follows for a polling system consisting of two queues.

$$
\begin{equation*}
\rho+\max _{1 \leq i \leq 2}\left(\lambda_{i} / L_{i}^{*}\right) s<1 \tag{2.2}
\end{equation*}
$$

where $L_{i}^{*}$ is the maximum expected number of customers served in $Q_{i}$ during one visit cycle. Appealing to this result we give a sufficient condition for the stability of the model considered here. Since $M$ is finite, the service schedule in $Q_{2}$, in fact, is a exhaustive-type one. Especially, when $M=0$, it becomes a pure exhaustive service schedule. Hence, we have that $L_{2}^{*}=\infty$. On the other hand, the server can serve a maximum number of the customers in $Q_{1}$ if the number of the customers left in $Q_{2}$ when the server switches the service from $Q_{2}$ to $Q_{1}$ is zero. That is, let $S_{2}$ be a generic switching time from $Q_{2}$ to $Q_{1}, A_{2}^{j}$ and $B_{1}^{j}$, for each $j$, generic interarrival times and service times in $Q_{2}$ and $Q_{1}$ respectively. Let

$$
\begin{equation*}
\tau=\min \left\{n, \quad S_{2}+\sum_{j=1}^{n} B_{1}^{j}>\sum_{j=1}^{N} A_{2}^{j}\right\} . \tag{2.3}
\end{equation*}
$$

We have that $L_{1}^{*}=E[\tau]$. Then we obtain a sufficient condition of the stability

$$
\begin{equation*}
\rho+\frac{\lambda_{1} s}{E[\tau]}<1 . \tag{2.4}
\end{equation*}
$$

This is consistent with the condition given by Boxma and Down[4]. Throughout the paper, we assume that the condition (2.4) holds.

Let $\left\{t_{k}, k \geq 1\right\}$ be the successive moments of service completion, $X_{k}^{(i)}, i=1,2, k \geq 1$, the number of customers at $Q_{i}$ at instant immediately after $t_{k}$, and $J_{k}, k \geq 1$, the type of the departing customer at $t_{k}$, i.e., $J_{k}=i$ if the $k$ th departing customer is from $Q_{i}$. Then $\left\{\left(X_{k}^{(1)}, X_{k}^{(2)}, J_{k}\right)\right\}_{k \geq 1}$ forms an imbedded vector-valued Markov chain. Let $V_{k+1}^{(i)}$ denote the number of the arriving customers in $Q_{i}$ during $\left(t_{k}, t_{k+1}\right]$. Note that when the both queues are not empty, $t_{k+1}-t_{k}=B_{j}$ if $J_{k}=J_{k+1}=j$, and $t_{k+1}-t_{k}=S_{i}+B_{j}$ if $J_{k}=i, J_{k+1}=j$, and $i \neq j$. According to the service discipline, the relations between $\left(X_{k}^{(1)}, X_{k}^{(2)}, J_{k}\right)$ and $\left(X_{k+1}^{(1)}, X_{k+1}^{(2)}, J_{k+1}\right)$ may be described as follows: for $i, j=1,2$, if $J_{k+1}=i$, then $X_{k+1}^{(i)}=\left[X_{k}^{(i)}-1\right]^{+}+V_{k+1}^{(i)}$ and $X_{k+1}^{(j)}=X_{k}^{(j)}+V_{k+1}^{(j)} ; j \neq i$, where $x^{+} \equiv \max \{0, x\}$. Let $\left\{\pi_{n, m, i} ; \quad n, m \geq 0, i=1,2\right\}$ denote the equilibrium probabilities of $\left\{\left(X_{k}^{(1)}, X_{k}^{(2)}, J_{k}\right)\right\}_{k \geq 1}$, namely,

$$
\begin{equation*}
\pi_{n, m, i} \equiv \lim _{k \rightarrow \infty} P\left(\left(X_{k}^{(1)}, X_{k}^{(2)}, J_{k}\right)=(n, m, i)\right) \tag{2.5}
\end{equation*}
$$

For $\left|z_{1}\right| \leq 1 ; \quad\left|z_{2}\right| \leq 1$, define the two-dimensional generating functions

$$
\begin{equation*}
\Phi^{(i)}\left(z_{1}, z_{2}\right) \equiv \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \pi_{n, m, i} z_{1}^{n} z_{2}^{m}, \quad i=1,2 \tag{2.6}
\end{equation*}
$$

Considering the transition probabilities of the Markov chain during two successive service completion epochs and using the above relations, we derive the following equations for $\Phi^{(1)}\left(z_{1}, z_{2}\right)$ and $\Phi^{(2)}\left(z_{1}, z_{2}\right)$
$\Phi^{(1)}\left(z_{1}, z_{2}\right)=r_{1} \hat{B}_{1}\left(\lambda_{1}\left(1-z_{1}\right)+\lambda_{2}\left(1-z_{2}\right)\right)\left(\Phi^{(1)}(0,0)+\hat{S}_{2}\left(\lambda_{1}\left(1-z_{1}\right)+\lambda_{2}\left(1-z_{2}\right)\right) \Phi^{(2)}(0,0)\right)$ $+z_{1}^{-1} \hat{B}_{1}\left(\lambda_{1}\left(1-z_{1}\right)+\lambda_{2}\left(1-z_{2}\right)\right)\left\{\sum_{m=0}^{N} \sum_{n=1}^{\infty} \pi_{n, m, 1} z_{1}^{n} z_{2}^{m}+\hat{S}_{2}\left(\lambda_{1}\left(1-z_{1}\right)+\lambda_{2}\left(1-z_{2}\right)\right)\right.$ $\left.\times \sum_{m=0}^{M} \sum_{n=1}^{\infty} \pi_{n, m, 2} z_{1}^{n} z_{2}^{m}\right\}$,
$\Phi^{(2)}\left(z_{1}, z_{2}\right)=r_{2} \hat{B}_{2}\left(\lambda_{1}\left(1-z_{1}\right)+\lambda_{2}\left(1-z_{2}\right)\right)\left(\hat{S}_{1}\left(\lambda_{1}\left(1-z_{1}\right)+\lambda_{2}\left(1-z_{2}\right)\right) \Phi^{(1)}(0,0)+\Phi^{(2)}(0,0)\right)$
$+z_{2}^{-1} \hat{B}_{2}\left(\lambda_{1}\left(1-z_{1}\right)+\lambda_{2}\left(1-z_{2}\right)\right)\left\{\hat{S}_{1}\left(\lambda_{1}\left(1-z_{1}\right)+\lambda_{2}\left(1-z_{2}\right)\right)\left(\Phi^{(1)}\left(0, z_{2}\right)-\Phi^{(1)}(0,0)\right)\right.$ $+\left(\Phi^{(2)}\left(0, z_{2}\right)-\Phi^{(2)}(0,0)\right)+\hat{S}_{1}\left(\lambda_{1}\left(1-z_{1}\right)+\lambda_{2}\left(1-z_{2}\right)\right) \sum_{m=N+1}^{\infty} \sum_{n=1}^{\infty} \pi_{n, m, 1} z_{1}^{n} z_{2}^{m}$ $\left.+\sum_{m=M+1}^{\infty} \sum_{n=1}^{\infty} \pi_{n, m, 2} z_{1}^{n} z_{2}^{m}\right\}$.
For clarity, define for $i=1,2$,

$$
B_{i}^{*}\left(z_{1}, z_{2}\right)=\hat{B}_{i}\left(\lambda_{1}\left(1-z_{1}\right)+\lambda_{2}\left(1-z_{2}\right)\right), \quad S_{i}^{*}\left(z_{1}, z_{2}\right)=\hat{S}_{i}\left(\lambda_{1}\left(1-z_{1}\right)+\lambda_{2}\left(1-z_{2}\right)\right)
$$

Furthermore, for $\left|z_{1}\right| \leq 1$, define the one-dimensional generating functions of the joint equilibrium probabilities $\left\{\pi_{n, m, 1} ; n \geq 0\right\}, 0 \leq m \leq N$ and $\left\{\pi_{n, m, 2} ; n \geq 0\right\}, 0 \leq m \leq M$,

$$
\begin{array}{ll}
\varphi_{m}\left(z_{1}\right) \equiv \sum_{n=0}^{\infty} \pi_{n, m, 1} z_{1}^{n}, & 0 \leq m \leq N \\
\psi_{m}\left(z_{1}\right) \equiv \sum_{n=0}^{\infty} \pi_{n, m, 2} z_{1}^{n}, & 0 \leq m \leq M \tag{2.10}
\end{array}
$$

We have

$$
\begin{align*}
& \sum_{m=N+1}^{\infty} \sum_{n=1}^{\infty} \pi_{n, m, 1} z_{1}^{n} z_{2}^{m}=\Phi^{(1)}\left(z_{1}, z_{2}\right)-\Phi^{(1)}\left(0, z_{2}\right)-\sum_{m=0}^{N}\left(\varphi_{m}\left(z_{1}\right)-\varphi_{m}(0)\right) z_{2}^{m}  \tag{2.11}\\
& \sum_{m=M+1}^{\infty} \sum_{n=1}^{\infty} \pi_{n, m, 2} z_{1}^{n} z_{2}^{m}=\Phi^{(2)}\left(z_{1}, z_{2}\right)-\Phi^{(2)}\left(0, z_{2}\right)-\sum_{m=0}^{M}\left(\psi_{m}\left(z_{1}\right)-\psi_{m}(0)\right) z_{2}^{m} \tag{2.12}
\end{align*}
$$

In particular,

$$
\begin{equation*}
\varphi_{0}(0)=\Phi^{(1)}(0,0)=\pi_{0,0,1}, \quad \psi_{0}(0)=\Phi^{(2)}(0,0)=\pi_{0,0,2} \tag{2.13}
\end{equation*}
$$

Using these notations and relations, the equations (2.7) and (2.8) can be rewritten as

$$
\begin{align*}
\Phi^{(1)}\left(z_{1}, z_{2}\right)= & r_{1} B_{1}^{*}\left(z_{1}, z_{2}\right)\left(\varphi_{0}(0)+\psi_{0}(0)\right)+z_{1}^{-1} B_{1}^{*}\left(z_{1}, z_{2}\right)\left\{\sum_{m=0}^{N}\left(\varphi_{m}\left(z_{1}\right)-\varphi_{m}(0)\right) z_{2}^{m}+S_{2}^{*}\left(z_{1}, z_{2}\right)\right. \\
& \left.\times \sum_{m=0}^{M}\left(\psi_{m}\left(z_{1}\right)-\psi_{m}(0)\right) z_{2}^{m}\right\}  \tag{2.14}\\
\Phi^{(2)}\left(z_{1}, z_{2}\right)= & \frac{B_{2}^{*}\left(z_{1}, z_{2}\right)}{z_{2}-B_{2}^{*}\left(z_{1}, z_{2}\right)}\left\{\left(r_{2} z_{2}+r_{1} B_{1}^{*}\left(z_{1}, z_{2}\right)-1\right) S_{1}^{*}\left(z_{1}, z_{2}\right) \varphi_{0}(0)+\left(r_{2} z_{2}+r_{1} B_{1}^{*}\left(z_{1}, z_{2}\right)\right.\right. \\
& \left.\times S_{1}^{*}\left(z_{1}, z_{2}\right) S_{2}^{*}\left(z_{1}, z_{2}\right)-1\right) \psi_{0}(0)+\frac{\left(B_{1}^{*}\left(z_{1}, z_{2}\right)-z_{1}\right) S_{1}^{*}\left(z_{1}, z_{2}\right)}{z_{1}} \sum_{m=0}^{N}\left(\varphi_{m}\left(z_{1}\right)-\varphi_{m}(0)\right) \\
& \left.\times z_{2}^{m}+\frac{B_{1}^{*}\left(z_{1}, z_{2}\right) S_{1}^{*}\left(z_{1}, z_{2}\right)-z_{1}}{z_{1}} \sum_{m=0}^{M}\left(\psi_{m}\left(z_{1}\right)-\psi_{m}(0)\right) z_{2}^{m}\right\} \tag{2.15}
\end{align*}
$$

As has been seen, $\Phi^{(1)}\left(z_{1}, z_{2}\right)$ and $\Phi^{(2)}\left(z_{1}, z_{2}\right)$ are completely determined by the onedimensional generating functions $\varphi_{m}\left(z_{1}\right), 0 \leq m \leq N$ and $\psi_{m}\left(z_{1}\right), 0 \leq m \leq M$. In order to solve $\varphi_{m}\left(z_{1}\right), 0 \leq m \leq N$ and $\psi_{m}\left(z_{1}\right), 0 \leq m \leq M$, we need to derive more equations about these unknown functions. This can be done by considering the equilibrium balance equations for $\left\{\pi_{n, m, 1} ; n \geq 0\right\}, 0 \leq m \leq N$ and $\left\{\pi_{n, m, 2} ; n \geq 0\right\}, 0 \leq m \leq M$. First, for every $m, 0 \leq m \leq N$, we have

$$
\begin{align*}
\pi_{n, m, 1}= & \sum_{i=1}^{n+1} \sum_{j=0}^{\min (m, M)} \pi_{i, j, 2} \int_{0}^{\infty} \frac{\left(\lambda_{1} t\right)^{n-i+1}}{(n-i+1)!} e^{-\lambda_{1} t} \frac{\left(\lambda_{2} t\right)^{m-j}}{(m-j)!} e^{-\lambda_{2} t} d F_{S_{2}+B_{1}}(t)+\pi_{0,0,2} r_{1} \int_{0}^{\infty} \frac{\left(\lambda_{1} t\right)^{n}}{n!} \\
& \times e^{-\lambda_{1} t} \frac{\left(\lambda_{2} t\right)^{m}}{m!} e^{-\lambda_{2} t} d F_{S_{2}+B_{1}}(t)+\sum_{i=1}^{n+1} \sum_{j=0}^{m} \pi_{i, j, 1} \int_{0}^{\infty} \frac{\left(\lambda_{1} t\right)^{n-i+1}}{(n-i+1)!} e^{-\lambda_{1} t} \frac{\left(\lambda_{2} t\right)^{m-j}}{(m-j)!} e^{-\lambda_{2} t} d B_{1}(t) \\
& +\pi_{0,0,1} r_{1} \int_{0}^{\infty} \frac{\left(\lambda_{1} t\right)^{n}}{n!} e^{-\lambda_{1} t} \frac{\left(\lambda_{2} t\right)^{m}}{m!} e^{-\lambda_{2} t} d B_{1}(t), \quad 0 \leq n<\infty \tag{2.16}
\end{align*}
$$

where $F_{S_{2}+B_{1}}(\cdot)$ denotes the distribution of the sum $S_{2}+B_{1}$. From (2.16), multiplying the $n$th equation by $z_{1}^{n}$ and summing yields

$$
\begin{align*}
\varphi_{m}\left(z_{1}\right)= & \frac{1}{z_{1}}\left\{\sum_{j=0}^{\min \{m, M\}} H_{1, m-j}\left(z_{1}\right)\left(\psi_{j}\left(z_{1}\right)-\psi_{j}(0)\right)+\sum_{j=0}^{m} G_{1, m-j}\left(z_{1}\right)\left(\varphi_{j}\left(z_{1}\right)-\varphi_{j}(0)\right)\right. \\
& \left.+r_{1} z_{1} H_{1, m}\left(z_{1}\right) \psi_{0}(0)+r_{1} z_{1} G_{1, m}\left(z_{1}\right) \varphi_{0}(0)\right\}, \quad 0 \leq m \leq N \tag{2.17}
\end{align*}
$$

where

$$
\begin{align*}
& H_{1, j}\left(z_{1}\right) \equiv \int_{0}^{\infty} \frac{\left(\lambda_{2} t\right)^{j}}{j!} e^{-\lambda_{2} t} e^{-\lambda_{1}\left(1-z_{1}\right) t} d F_{S_{2}+B_{1}}(t), \quad 0 \leq j \leq M,  \tag{2.18}\\
& G_{1, j}\left(z_{1}\right) \equiv \int_{0}^{\infty} \frac{\left(\lambda_{2} t\right)^{j}}{j!} e^{-\lambda_{2} t} e^{-\lambda_{1}\left(1-z_{1}\right) t} d B_{1}(t), \quad 0 \leq j \leq N . \tag{2.19}
\end{align*}
$$

Next, for every $m, 0 \leq m \leq M-1$, we have
$\pi_{n, m, 2}=\sum_{j=1}^{m+1} \pi_{0, j, 1} \int_{0}^{\infty} \frac{\left(\lambda_{1} t\right)^{n}}{n!} e^{-\lambda_{1} t} \frac{\left(\lambda_{2} t\right)^{m-j+1}}{(m-j+1)!} e^{-\lambda_{2} t} d F_{S_{1}+B_{2}}(t)+\pi_{0,0,1} r_{2} \int_{0}^{\infty} \frac{\left(\lambda_{1} t\right)^{n}}{n!} e^{-\lambda_{1} t}$

$$
\begin{align*}
& \times \frac{\left(\lambda_{2} t\right)^{m}}{m!} e^{-\lambda_{2} t} d F_{S_{1}+B_{2}}(t)+\sum_{j=1}^{m+1} \pi_{0, j, 2} \int_{0}^{\infty} \frac{\left(\lambda_{1} t\right)^{n}}{n!} e^{-\lambda_{1} t} \frac{\left(\lambda_{2} t\right)^{m-j+1}}{(m-j+1)!} e^{-\lambda_{2} t} d B_{2}(t) \\
& +\pi_{0,0,2} r_{2} \int_{0}^{\infty} \frac{\left(\lambda_{1} t\right)^{n}}{n!} e^{-\lambda_{1} t} \frac{\left(\lambda_{2} t\right)^{m}}{m!} e^{-\lambda_{2} t} d B_{2}(t), \quad 0 \leq n<\infty \tag{2.20}
\end{align*}
$$

where $F_{S_{1}+B_{2}}(\cdot)$ denotes the distribution of the sum $S_{1}+B_{2}$. In particular,

$$
\begin{align*}
& \pi_{0,0,2}=\pi_{0,1,1} \int_{0}^{\infty} e^{-\left(\lambda_{1}+\lambda_{2}\right) t} d F_{S_{1}+B_{2}}(t)+\pi_{0,0,1} r_{2} \int_{0}^{\infty} e^{-\left(\lambda_{1}+\lambda_{2}\right) t} d F_{S_{1}+B_{2}}(t) \\
&+\pi_{0,1,2} \int_{0}^{\infty} e^{\left(-\lambda_{1}+\lambda_{2}\right) t} d B_{2}(t)+\pi_{0,0,2} r_{2} \int_{0}^{\infty} e^{\left(-\lambda_{1}+\lambda_{2}\right) t} d B_{2}(t) \tag{2.21}
\end{align*}
$$

From (2.20), multiplying the $n$th equation by $z_{1}^{n}$ and summing yield

$$
\begin{gather*}
\psi_{m}\left(z_{1}\right)=\sum_{j=1}^{m+1} H_{2, m-j+1}\left(z_{1}\right) \varphi_{j}(0)+\sum_{j=1}^{m+1} G_{2, m-j+1}\left(z_{1}\right) \psi_{j}(0)+r_{2} H_{2, m}\left(z_{1}\right) \varphi_{0}(0)+r_{2} G_{2, m}\left(z_{1}\right) \psi_{0}(0) \\
0 \leq m \leq M-1 \tag{2.22}
\end{gather*}
$$

where

$$
\begin{align*}
& H_{2, j}\left(z_{1}\right) \equiv \int_{0}^{\infty} \frac{\left(\lambda_{2} t\right)^{j}}{j!} e^{-\lambda_{2} t} e^{-\lambda_{1}\left(1-z_{1}\right) t} d F_{S_{1}+B_{2}}(t), \quad 0 \leq j \leq M  \tag{2.23}\\
& G_{2, j}\left(z_{1}\right) \equiv \int_{0}^{\infty} \frac{\left(\lambda_{2} t\right)^{j}}{j!} e^{-\lambda_{2} t} e^{-\lambda_{1}\left(1-z_{1}\right) t} d B_{2}(t), \quad 0 \leq j \leq M \tag{2.24}
\end{align*}
$$

and from (2.21) we have

$$
\begin{equation*}
\psi_{0}(0)=\left(r_{2} \varphi_{0}(0)+\varphi_{1}(0)\right) H_{2,0}(0)+\left(r_{2} \psi_{0}(0)+\psi_{1}(0)\right) G_{2,0}(0) \tag{2.25}
\end{equation*}
$$

## 3. Determination of the Generating Functions

In this section, we derive the generating functions $\Phi^{(1)}\left(z_{1}, z_{2}\right)$ and $\Phi^{(2)}\left(z_{1}, z_{2}\right)$. The equations (2.14) and (2.15) show that $\Phi^{(1)}\left(z_{1}, z_{2}\right)$ and $\Phi^{(2)}\left(z_{1}, z_{2}\right)$ can be obtained after $\varphi_{m}\left(z_{1}\right) ; \quad 0 \leq$ $m \leq N$, and $\psi_{m}\left(z_{1}\right) ; 0 \leq m \leq M$ are determined. Therefore, the main aim here is to deduce system equations about these one-dimensional generating functions by using (2.15), (2.17) and (2.22), and obtain their solutions. First, we consider the equation (2.15). According to Takács Lemma([8] pp.653), we have that for every fixed $z_{1}$ with $\left|z_{1}\right| \leq 1$, the equation

$$
\begin{equation*}
z_{2}-\hat{B}_{2}\left(\lambda_{1}\left(1-z_{1}\right)+\lambda_{2}\left(1-z_{2}\right)\right)=0 \tag{3.1}
\end{equation*}
$$

has exactly one root in the region $\left|z_{2}\right| \leq 1$. Actually, the root satisfies

$$
\begin{equation*}
z_{2}=\hat{V}\left(\lambda_{1}\left(1-z_{1}\right)\right) \tag{3.2}
\end{equation*}
$$

where $\hat{V}(s)$ is the $L S T$ of the busy period distribution of an M/G/1 queue with arrival rate $\lambda_{2}$ and service time distribution $B_{2}(\cdot)$. Denoting this root by $z_{2}=\eta\left(z_{1}\right)$, we have $\eta\left(z_{1}\right)=\hat{V}\left(\lambda_{1}\left(1-z_{1}\right)\right)$. Furthermore, $\eta(1)=1$, and

$$
\begin{equation*}
\left.\frac{d}{d z_{1}} \eta\left(z_{1}\right)\right|_{z_{1}=1}=\frac{\lambda_{1} b_{2}}{1-\rho_{2}},\left.\quad \frac{d^{2}}{d z_{1}^{2}} \eta\left(z_{1}\right)\right|_{z_{1}=1}=\frac{\lambda_{1}^{2} b_{2}^{(2)}}{\left(1-\rho_{2}\right)^{3}} . \tag{3.3}
\end{equation*}
$$

Since $\Phi^{(2)}\left(z_{1}, z_{2}\right)$ should be regular for $\left|z_{2}\right|<1$, and continuous for $\left|z_{2}\right| \leq 1$, for every fixed $z_{1}$ with $\left|z_{1}\right| \leq 1$, the numerator of (2.15) must vanish at $z_{2}=\eta\left(z_{1}\right)$. As the equations
discussed hereafter are mainly those about the argument $z_{1}$, we write $z$ instead of $z_{1}$ for simplicity in this section. Substituting the root $z_{2}=\eta(z)$ into (2.15), we have

$$
\begin{align*}
& \left(B_{1}^{*}(z, \eta(z))-z\right) S_{1}^{*}(z, \eta(z)) \sum_{m=0}^{N}\left(\varphi_{m}(z)-\varphi_{m}(0)\right) \eta^{m}(z)+\left(B_{1}^{*}(z, \eta(z)) S_{1}^{*}(z, \eta(z))-z\right) \\
& \times \sum_{m=0}^{M}\left(\psi_{m}(z)-\psi_{m}(0)\right) \eta^{m}(z)+z\left(r_{2} \eta(z)+r_{1} B_{1}^{*}(z, \eta(z))-1\right) S_{1}^{*}(z, \eta(z)) \varphi_{0}(0) \\
& +z\left(r_{2} \eta(z)+r_{1} B_{1}^{*}(z, \eta(z)) S_{1}^{*}(z, \eta(z)) S_{2}^{*}(z, \eta(z))-1\right) \psi_{0}(0)=0 \tag{3.4}
\end{align*}
$$

Next we rewrite (2.17) as follows

$$
\begin{gather*}
-\sum_{j=0}^{m-1} G_{1, m-j}(z) \varphi_{j}(z)+\left(z-G_{1,0}(z)\right) \varphi_{m}(z)=\sum_{j=0}^{\min \{m, M\}} H_{1, m-j}(z)\left(\psi_{j}(z)-\psi_{j}(0)\right)-\sum_{j=1}^{m} G_{1, m-j}(z) \\
\times \varphi_{j}(0)+\left(r_{1} z-1\right) G_{1, m}(z) \varphi_{0}(0)+r_{1} z H_{1, m}(z) \psi_{0}(0), \quad 0 \leq m \leq N \tag{3.5}
\end{gather*}
$$

Define the vectors

$$
\begin{equation*}
\boldsymbol{\varphi}(z)=\left(\varphi_{0}(z), \varphi_{1}(z), \cdots, \varphi_{N}(z)\right)^{\tau}, \quad \boldsymbol{\psi}(z)=\left(\psi_{0}(z), \psi_{1}(z), \cdots, \psi_{M}(z)\right)^{\tau} \tag{3.6}
\end{equation*}
$$

Then the equations (3.5) can be represented by matrix form

$$
\begin{equation*}
\boldsymbol{\Lambda}_{1}(z) \boldsymbol{\varphi}(z)=\mathbf{H}_{1}(z)(\boldsymbol{\psi}(z)-\boldsymbol{\psi}(0))+\mathbf{G}_{1}(z) \boldsymbol{\varphi}(0)+\psi_{0}(0) \mathbf{v}_{1}(z) \tag{3.7}
\end{equation*}
$$

where $\boldsymbol{\Lambda}_{1}(z)$ is the $(N+1) \times(N+1)$ matrix

$$
\boldsymbol{\Lambda}_{1}(z)=\left[\begin{array}{cccc}
z-G_{1,0}(z) & & & \\
-G_{1,1}(z) & z-G_{1,0}(z) & & \\
\vdots & \vdots & \ddots & \\
-G_{1, N}(z) & -G_{1, N-1}(z) & \cdots & z-G_{1,0}(z)
\end{array}\right]
$$

$\mathbf{H}_{1}(z)$ is the $(N+1) \times(M+1)$ matrix

$$
\mathbf{H}_{1}(z)=\left[\begin{array}{cccc}
H_{1,0}(z) & & & \\
H_{1,1}(z) & H_{1,0}(z) & & \\
\vdots & \vdots & \ddots & \\
H_{1, M}(z) & H_{1, M-1}(z) & \cdots & H_{1,0}(z) \\
\vdots & \vdots & \vdots & \vdots \\
H_{1, M}(z) & H_{1, M-1}(z) & \cdots & H_{1,0}(z)
\end{array}\right]
$$

$\mathrm{G}_{1}(z)$ is the $(N+1) \times(N+1)$ matrix

$$
\mathbf{G}_{1}(z)=\left[\begin{array}{cccc}
\left(r_{1} z-1\right) G_{1,0}(z) & & & \\
\left(r_{1} z-1\right) G_{1,1}(z) & -G_{1,0}(z) & & \vdots \\
\vdots & \vdots & \ddots & \vdots \\
\left(r_{1} z-1\right) G_{1, N}(z) & -G_{1, N-1}(z) & \cdots & -G_{1,0}(z)
\end{array}\right]
$$

and $\mathbf{v}_{1}(z)$ is the $(N+1) \times 1$ vector

$$
\mathbf{v}_{1}(z)=\left(r_{1} z H_{1,0}(z), r_{1} z H_{1,1}(z), \cdots, r_{1} z H_{1, N}(z)\right)^{\tau}
$$

Moreover, note that (2.22) holds for all $z$ with $|z| \leq 1$. We get $\psi_{m}(0)$ by substituting $z=0$ into (2.22),

$$
\begin{gather*}
\psi_{m}(0)=\sum_{j=1}^{m+1} H_{2, m-j+1}(0) \varphi_{j}(0)+\sum_{j=1}^{m+1} G_{2, m-j+1}(0) \psi_{j}(0)+r_{2} H_{2, m}(0) \varphi_{0}(0)+r_{2} G_{2, m}(0) \psi_{0}(0) \\
0 \leq m \leq M-1 \tag{3.8}
\end{gather*}
$$

Hence, we have

$$
\begin{gather*}
\psi_{m}(z)-\psi_{m}(0)=\sum_{j=1}^{m+1}\left(H_{2, m-j+1}(z)-H_{2, m-j+1}(0)\right) \varphi_{j}(0)+\sum_{j=1}^{m+1}\left(G_{2, m-j+1}(z)-G_{2, m-j+1}(0)\right) \psi_{j}(0) \\
+r_{2}\left(H_{2, m}(z)-H_{2, m}(0)\right) \varphi_{0}(0)+r_{2}\left(G_{2, m}(z)-G_{2, m}(0)\right) \psi_{0}(0), \quad 0 \leq m \leq M-1 \tag{3.9}
\end{gather*}
$$

In order to derive a complete expression of the vector $\boldsymbol{\psi}(z)-\boldsymbol{\psi}(0)$ by the vectors $\boldsymbol{\varphi}(z)$, $\varphi(0)$ and $\boldsymbol{\psi}(0)$, we also need an equation for the function $\psi_{M}(z)-\psi_{M}(0)$. This can be obtained from (3.4) and (3.9). Since $z=1$ is the unique, simple root of the equation $z-B_{1}^{*}(z, \eta(z)) S_{1}^{*}(z, \eta(z))=0$ and $\eta(z) \neq 0$ for all $z$ with $|z| \leq 1$, substituting $\psi_{m}(z)-$ $\psi_{m}(0), 0 \leq m \leq M-1$ in (3.9) into (3.4), we get after a long calculation that for $z$ with $|z| \leq 1$,

$$
\begin{align*}
\psi_{M}(z)-\psi_{M}(0)=\alpha(z) \sum_{j=0}^{N} \varphi_{j}(z) \eta^{j-M}(z) & -\sum_{j=1}^{N} \varphi_{j}(0) k_{j}(z)-\sum_{j=1}^{M} \psi_{j}(0) l_{j}(z)+\beta_{1}(z) \varphi_{0}(0) \\
+ & \beta_{2}(z) \psi_{0}(0) \tag{3.10}
\end{align*}
$$

where for $z \neq 1$,

$$
\begin{align*}
& \alpha(z) \equiv \frac{\left(B_{1}^{*}(z, \eta(z))-z\right) S_{1}^{*}(z, \eta(z))}{z-B_{1}^{*}(z, \eta(z)) S_{1}^{*}(z, \eta(z))},  \tag{3.11}\\
& k_{j}(z) \equiv \begin{cases}\alpha(z) \eta^{j-M}(z)+\sum_{m=j-1}^{M-1}\left(H_{2, m-j+1}(z)-H_{2, m-j+1}(0)\right) \eta^{m-M}(z) & \text { if } 1 \leq j \leq M \\
\alpha(z) \eta^{j-M}(z) & \text { if } M<j \leq N,\end{cases} \\
& l_{j}(z) \equiv \sum_{m=j-1}^{M-1}\left(G_{2, m-j+1}(z)-G_{2, m-j+1}(0)\right) \eta^{m-M}(z), \quad 1 \leq j \leq M,  \tag{3.12}\\
& \beta_{1}(z) \equiv \frac{\left(r_{2} z \eta(z)+\left(r_{1} z-1\right) B_{1}^{*}(z, \eta(z))\right) S_{1}^{*}(z, \eta(z))}{\left(z-B_{1}^{*}(z, \eta(z)) S_{1}^{*}(z, \eta(z))\right) \eta^{M}(z)}-r_{2} \sum_{m=0}^{M-1}\left(H_{2, m}(z)-H_{2, m}(0)\right) \eta^{m-M}(z),  \tag{3.13}\\
& \beta_{2}(z) \equiv \frac{\left(r_{2} \eta(z)+r_{1} B_{1}^{*}(z, \eta(z)) S_{1}^{*}(z, \eta(z)) S_{2}^{*}(z, \eta(z))-1\right) z}{\left(z-B_{1}^{*}(z, \eta(z)) S_{1}^{*}(z, \eta(z))\right) \eta^{M}(z)}-r_{2} \sum_{m=0}^{M-1}\left(G_{2, m}(z)-G_{2, m}(0)\right)  \tag{3.14}\\
& \times \eta^{m-M}(z), \tag{3.15}
\end{align*}
$$

and for $z=1$,

$$
\begin{equation*}
\alpha(1)=\lim _{z \rightarrow 1} \alpha(z)=-\frac{1-\rho_{1}-\rho_{2}}{1-\rho_{1}-\rho_{2}-\lambda_{1} s_{1}} \tag{3.16}
\end{equation*}
$$

$$
\begin{aligned}
& k_{j}(1)=\lim _{z \rightarrow 1} k_{j}(z)= \begin{cases}\alpha(1)+\sum_{m=j-1}^{M-1}\left(H_{2, m-j+1}(1)-H_{2, m-j+1}(0)\right) & \text { if } 1 \leq j \leq M \\
\alpha(1) & \text { if } M<j \leq N,\end{cases} \\
& l_{j}(1)=\lim _{z \rightarrow 1} l_{j}(z)=\sum_{m=j-1}^{M-1}\left(G_{2, m-j+1}(1)-G_{2, m-j+1}(0)\right), \quad 1 \leq j \leq M,
\end{aligned}, \begin{aligned}
& \beta_{1}(1)=\lim _{z \rightarrow 1} \beta_{1}(z)=\frac{1-r_{2}\left(\rho_{1}+\rho_{2}\right)}{1-\rho_{1}-\rho_{2}-\lambda_{1} s_{1}}-r_{2} \sum_{m=0}^{M-1}\left(H_{2, m-j+1}(1)-H_{2, m-j+1}(0)\right) \\
& \beta_{2}(1)=\lim _{z \rightarrow 1} \beta_{2}(z)=\frac{r_{1}\left(\rho_{1}+\rho_{2}+\lambda_{1}\left(s_{1}+s_{2}\right)\right)}{1-\rho_{1}-\rho_{2}-\lambda_{1} s_{1}}-r_{2} \sum_{m=0}^{M-1}\left(G_{2, m-j+1}(1)-G_{2, m-j+1}(0)\right) .
\end{aligned}
$$

Writing (3.9) and (3.10) in matrix form, we get an expression of $\boldsymbol{\psi}(z)-\boldsymbol{\psi}(0)$ as

$$
\begin{equation*}
\boldsymbol{\psi}(z)-\boldsymbol{\psi}(0)=\boldsymbol{\Lambda}_{2}(z) \boldsymbol{\varphi}(z)+\mathbf{H}_{2}(z) \boldsymbol{\varphi}(0)+\mathbf{G}_{2}(z) \boldsymbol{\psi}(0) \tag{3.17}
\end{equation*}
$$

where $\boldsymbol{\Lambda}_{2}(z)$ is the $(M+1) \times(N+1)$ matrix

$$
\boldsymbol{\Lambda}_{2}(z)=\left[\begin{array}{cccc}
\mathbf{0} \\
\alpha(z) \eta^{-M}(z) & \alpha(z) \eta^{1-M}(z) & \cdots & \alpha(z) \eta^{N-M}(z)
\end{array}\right]
$$

where $\mathbf{0}$ denotes a zero matrix. $\mathbf{H}_{2}(z)$ is the $(M+1) \times(N+1)$ matrix

$$
\mathbf{H}_{2}(z)=\left[\begin{array}{ccc}
r_{2}\left(H_{2,0}(z)-H_{2,0}(0)\right) & H_{2,0}(z)-H_{2,0}(0) \\
r_{2}\left(H_{2,1}(z)-H_{2,1}(0)\right) & H_{2,1}(z)-H_{2,1}(0) \\
\vdots & \vdots \\
r_{2}\left(H_{2, M-1}(z)-H_{2, M-1}(0)\right) & H_{2, M-1}(z)-H_{2, M-1}(0) \\
\beta_{1}(z) & -k_{1}(z) \\
0 & & \\
H_{2,0}(z)-H_{2,0}(0) & \ddots & \\
\vdots & \cdots & H_{2,0}(z)-H_{2,0}(0) \\
& \left.\begin{array}{cccc} 
\\
H_{2, M-2}(z)-H_{2, M-2}(0) & \cdots & & \\
-k_{2}(z) & \cdots & -k_{M}(z) & -k_{M+1}(z) \\
\cdots & -k_{N}(z)
\end{array}\right],
\end{array}\right.
$$

and $\mathbf{G}_{2}(z)$ is the $(M+1) \times(M+1)$ matrix
$\mathbf{G}_{2}(z)=\left[\begin{array}{cclc}r_{2}\left(G_{2,0}(z)-G_{2,0}(0)\right) & G_{2,0}(z)-G_{2,0}(0) & & \\ \vdots & \vdots & \ddots & \\ r_{2}\left(G_{2, M-1}(z)-G_{2, M-1}(0)\right) & G_{2, M-1}(z)-G_{2, M-1}(0) & \cdots & G_{2,0}(z)-G_{2,0}(0) \\ \beta_{2}(z) & -l_{1}(z) & \cdots & -l_{M}(z)\end{array}\right]$
Furthermore, we rewrite (3.8) as follows:

$$
\begin{aligned}
& \left(1-r_{2} G_{2,0}(0)\right) \psi_{0}(z)-G_{2,0}(0) \psi_{1}(0)=r_{2} H_{2,0}(0) \varphi_{0}(0)+H_{2,0}(0) \varphi_{1}(0) \\
& -r_{2} G_{2, m}(0) \psi_{0}(0)-\sum_{j=1}^{m-1} G_{2, m-j+1}(0) \psi_{j}(0)+\left(1-G_{2,1}(0)\right) \psi_{m}(0)-G_{2,0}(0) \psi_{m+1}(0) \\
& =\sum_{j=1}^{m+1} H_{2, m-j+1}(0) \varphi_{j}(0)+r_{2} H_{2, m}(0) \varphi_{0}(0), \quad 1 \leq m \leq M-2
\end{aligned}
$$

$$
\begin{gather*}
-r_{2} G_{2, M}(0) \psi_{0}(0)-\sum_{j=1}^{M-2} G_{2, m-j+1}(0) \psi_{j}(0)+\left(1-G_{2,1}(0)\right) \psi_{M-1}(0)=\sum_{j=1}^{M} H_{2, m-j+1}(0) \varphi_{j}(0) \\
+r_{2} H_{2, M-1}(0) \varphi_{0}(0)+G_{2,0}(0) \psi_{M}(0) \tag{3.18}
\end{gather*}
$$

Define the vectors

$$
\begin{equation*}
\hat{\varphi}(0)=\left(\varphi_{0}(0), \varphi_{1}(0), \cdots, \varphi_{M}(0)\right)^{\tau}, \quad \hat{\psi}(0)=\left(\psi_{0}(0), \psi_{1}(0), \cdots, \psi_{M-1}(0)\right)^{\tau} \tag{3.19}
\end{equation*}
$$

It follows that the relation

$$
\begin{equation*}
\hat{\varphi}(0)=\mathbf{J} \varphi(0) \tag{3.20}
\end{equation*}
$$

holds, where $\mathbf{J}=\left(\mathbf{I}_{M+1}, \mathbf{0}\right)$ is the $(M+1) \times(N+1)$ matrix, and $\mathbf{I}_{M+1}$ is the $(M+1) \times(M+1)$ unit matrix. Then from (3.18) we get the matrix expression for $\hat{\psi}(0)$ as

$$
\begin{equation*}
\hat{\boldsymbol{\psi}}(0)=\mathbf{G}^{-1}(0)\left[\mathbf{H}(0) \hat{\boldsymbol{\varphi}}(0)+\psi_{M}(0) \mathbf{v}\right]=\mathbf{G}^{-1}(0)\left[\mathbf{H}(0) \mathbf{J} \boldsymbol{\varphi}(0)+\psi_{M}(0) \mathbf{v}\right], \tag{3.21}
\end{equation*}
$$

where $\mathbf{G}(0)$ is the $M \times M$ matrix

$$
\mathbf{G}(0)=\left[\begin{array}{ccccc}
1-r_{2} G_{2,0}(0) & -G_{2,0}(0) & & & \\
-r_{2} G_{2,1}(0) & 1-G_{2,1}(0) & -G_{2,0}(0) & & \\
\vdots & \vdots & \vdots & \ddots & \\
-r_{2} G_{2, M-2}(0) & -G_{2, M-2}(0) & -G_{2, M-3}(0) & \cdots & -G_{2,0}(0) \\
-r_{2} G_{2, M-1}(0) & -G_{2, M-1}(0) & -G_{2, M-2}(0) & \cdots & 1-G_{2,1}(0)
\end{array}\right],
$$

$\mathbf{H}(0)$ is the $M \times(M+1)$ matrix

$$
\mathbf{H}(0)=\left[\begin{array}{ccccc}
r_{2} H_{2,0}(0) & H_{2,0}(0) & & & \\
r_{2} H_{2,1}(0) & H_{2,1}(0) & H_{2,0}(0) & & \\
\vdots & \vdots & \vdots & \ddots & \\
r_{2} H_{2, M-1}(0) & H_{2, M-1}(0) & H_{2, M-2}(0) & \cdots & H_{2,0}(0)
\end{array}\right]
$$

and $\mathbf{v}$ is the $(M+1) \times 1$ vector

$$
\mathbf{v}=\left(0, \cdots, 0, G_{2,0}(0)\right)^{\tau}
$$

Remark 3.1. In (3.21) we have assumed the case where the matrix $\mathbf{G}(0)$ is invertible. Actually this assumption is not essential. Note that $\mathbf{G}(0)$ is a semi-down-triangular matrix, and $1-r_{2} G_{2,0}(0)>0,1-G_{2,1}(0)>0$, and $G_{2,0}(0)>0$, the rank of $\mathbf{G}(0)$ is $M-1$ at least. That is, the first $M-1$ rows of $\mathbf{G}(0)$ are linearly independent at least. Thus when $\mathbf{G}(0)$ is not invertible, $\psi_{M-1}(0)$ is a linear combination of $\psi_{0}(0), \psi_{1}(0), \cdots, \psi_{M-2}(0)$. From this linear combination and $(3.18), \psi_{M-1}(0)$ can still be represented by $\varphi(0)$ and $\psi_{M}(0)$. Again we have an expression for $\hat{\boldsymbol{\psi}}(0)$ similar to (3.21).

Especially from (3.21), there exist a row vector $\mathbf{u}=\left(u_{0}, u_{1}, \cdots, u_{N}\right)$ and a constant $a$ such that

$$
\begin{equation*}
\psi_{0}(0)=\mathbf{u} \varphi(0)+a \psi_{M}(0) \tag{3.22}
\end{equation*}
$$

Furthermore, using $\hat{\boldsymbol{\psi}}(0)$ and $\psi_{M}(0)$, we can represent the vector $\mathbf{G}_{2}(z) \boldsymbol{\psi}(0)$ as follows

$$
\begin{equation*}
\mathbf{G}_{2}(z) \boldsymbol{\psi}(0)=\hat{\mathbf{G}}_{2}(z) \hat{\boldsymbol{\psi}}(0)+\psi_{M}(0) \mathbf{v}_{2}(z) \tag{3.23}
\end{equation*}
$$

where $\hat{\mathbf{G}}_{2}(z)$ is the $(M+1) \times M$ matrix

$$
\hat{\mathbf{G}}_{2}(z)=\left[\begin{array}{cclc}
r_{2}\left(G_{2,0}(z)-G_{2,0}(0)\right) & 0 & & \\
r_{2}\left(G_{2,1}(z)-G_{2,1}(0)\right) & G_{2,1}(z)-G_{2,1}(0) & & \\
\vdots & \vdots & \ddots & \\
r_{2}\left(G_{2, M-1}(z)-G_{2, M-1}(0)\right) & G_{2, M-1}(z)-G_{2, M-1}(0) & \cdots & G_{2,1}(z)-G_{2,1}(0) \\
\beta_{2}(z) & -l_{1}(z) & \cdots & -l_{M-1}(z)
\end{array}\right]
$$

and $\mathbf{v}_{2}(z)$ is the $(M+1) \times 1$ vector

$$
\mathbf{v}_{2}(z)=\left(0, \cdots, 0, G_{2,0}(z)-G_{2,0}(0),-l_{M}(z)\right)^{\tau}
$$

Substituting (3.21) into (3.23), and then substituting (3.23) into (3.17), we have

$$
\begin{align*}
& \boldsymbol{\psi}(z)-\boldsymbol{\psi}(0)  \tag{3.24}\\
= & \boldsymbol{\Lambda}_{2}(z) \boldsymbol{\varphi}(z)+\mathbf{H}_{2}(z) \boldsymbol{\varphi}(0)+\hat{\mathbf{G}}_{2}(z)\left[\mathbf{G}^{-1}(0) \mathbf{H}(0) \hat{\boldsymbol{\varphi}}(0)+\psi_{M}(0) \mathbf{G}^{-1}(0) \mathbf{v}\right]+\psi_{M}(0) \mathbf{v}_{2}(z) \\
= & \boldsymbol{\Lambda}_{2}(z) \boldsymbol{\varphi}(z)+\left(\mathbf{H}_{2}(z)+\hat{\mathbf{G}}_{2}(z) \mathbf{G}^{-1}(0) \mathbf{H}(0) \mathbf{J}\right) \boldsymbol{\varphi}(0)+\psi_{M}(0)\left(\hat{\mathbf{G}}_{2}(z) \mathbf{G}^{-1}(0) \mathbf{v}+\mathbf{v}_{2}(z)\right) .
\end{align*}
$$

Moreover, substituting (3.24) and (3.22) into (3.7) yields

$$
\begin{align*}
\boldsymbol{\Lambda}_{\mathbf{1}}(z) \boldsymbol{\varphi}(z)= & \mathbf{H}_{1}(z)\left\{\boldsymbol{\Lambda}_{2}(z) \boldsymbol{\varphi}(z)+\left(\mathbf{H}_{2}(z)+\hat{\mathbf{G}}_{2}(z) \mathbf{G}^{-1}(0) \mathbf{H}(0) \mathbf{J}\right) \boldsymbol{\varphi}(0)+\psi_{M}(0)\left(\hat{\mathbf{G}}_{2}(z)\right.\right. \\
& \left.\left.\times \mathbf{G}^{-1}(0) \mathbf{v}+\mathbf{v}_{2}(z)\right)\right\}+\mathbf{G}_{1}(z) \boldsymbol{\varphi}(0)+\left(\mathbf{u} \boldsymbol{\varphi}(0)+a \psi_{M}(0)\right) \mathbf{v}_{\mathbf{1}}(z) \tag{3.25}
\end{align*}
$$

Since the vector $\mathbf{u} \varphi(0) \mathbf{v}_{\mathbf{1}}(z)$ can be represented as $\mathbf{u} \varphi(0) \mathbf{v}_{\mathbf{1}}(z)=\mathbf{L}(z) \boldsymbol{\varphi}(0)$, where $\mathbf{L}(z)$ is a $(N+1) \times(N+1)$ matrix

$$
\mathbf{L}(z)=\left[\begin{array}{ccc}
r_{1} z H_{1,0}(z) u_{0} & \cdots & r_{1} z H_{1,0}(z) u_{N} \\
\vdots & \ddots & \vdots \\
r_{1} z H_{1, N}(z) u_{0} & \cdots & r_{1} z H_{1, N}(z) u_{N}
\end{array}\right]
$$

we obtain the final matrix equation:

$$
\begin{equation*}
\mathcal{N}(z) \boldsymbol{\varphi}(z)=\mathcal{M}(z) \boldsymbol{\varphi}(0)+\psi_{M}(0) \mathbf{m}(z) \tag{3.26}
\end{equation*}
$$

where $\mathcal{M}(z)$ is $(N+1) \times(N+1)$ matrix, $\mathbf{m}(z)$ is $(N+1) \times 1$ vector: $\mathcal{M}(z) \equiv \mathbf{H}_{1}(z)\left(\mathbf{H}_{2}(z)+\right.$ $\left.\hat{\mathbf{G}}_{2}(z) \mathbf{G}^{-1}(0) \mathbf{H}(0) \mathbf{J}\right)+\mathbf{G}_{1}(z)+\mathbf{L}(z), \mathbf{m}(z) \equiv \mathbf{H}_{1}(z)\left(\hat{\mathbf{G}}_{2}(z) \mathbf{G}^{-1}(0) \mathbf{v}+\mathbf{v}_{2}(z)\right)+a \mathbf{v}_{1}(z)$, and $\mathcal{N}(z)$ is the $(N+1) \times(N+1)$ matrix

$$
\mathcal{N}(z) \equiv \Lambda_{1}(z)-\mathbf{H}_{1}(z) \Lambda_{2}(z)=\left[\begin{array}{cc}
\mathcal{N}_{1}(z) & 0 \\
\mathcal{N}_{2}(z) & \mathcal{N}_{3}(z)
\end{array}\right]
$$

and $\mathcal{N}_{i}(z), i=1,2,3$ are the $M \times M,(N-M+1) \times M,(N-M+1) \times(N-M+1)$ matrices, respectively,

$$
\begin{aligned}
& \mathcal{N}_{1}(z)=\left[\begin{array}{cccc}
z-G_{1,0}(z) & & & \\
-G_{1,1}(z) & z-G_{1,0}(z) & & \\
\vdots & \vdots & \ddots & \\
-G_{1, M-1}(z) & -G_{1, M-2}(z) & \cdots & z-G_{1,0}(z)
\end{array}\right], \\
& \mathcal{N}_{2}(z)=\left[\begin{array}{cccc}
-G_{1, M}(z)-\alpha(z) H_{1,0}(z) \eta^{-M}(z) & \cdots & -G_{1,1}(z)-\alpha(z) H_{1,0}(z) \eta^{-1}(z) \\
-G_{1, M+1}(z)-\alpha(z) H_{1,0}(z) \eta^{-M}(z) & \cdots & -G_{1,2}(z)-\alpha(z) H_{1,0}(z) \eta^{-1}(z) \\
\vdots & & \ddots & \vdots \\
-G_{1, N}(z)-\alpha(z) H_{1,0}(z) \eta^{-M}(z) & \cdots & -G_{1, N-M+1}(z)-\alpha(z) H_{1,0}(z) \eta^{-1}(z)
\end{array}\right],
\end{aligned}
$$

$$
\mathcal{N}_{3}(z)=\left[\begin{array}{ccc}
z-G_{1,0}(z)-\alpha(z) H_{1,0}(z) & \cdots & -\alpha(z) H_{1,0}(z) \eta^{N-M}(z) \\
-G_{1,1}(z)-\alpha(z) H_{1,0}(z) & \cdots & -\alpha(z) H_{1,0}(z) \eta^{N-M}(z) \\
\vdots & \ddots & \vdots \\
-G_{1, N-M}(z)-\alpha(z) H_{1,0}(z) & \cdots & z-G_{1,0}(z)-\alpha(z) H_{1,0}(z) \eta^{N-M}(z)
\end{array}\right]
$$

Whenever $\mathcal{N}(z)$ is non-singular, the solutions of (3.26) are given by
$\boldsymbol{\varphi}(z)=\mathcal{N}^{-1}(z)\left\{\mathcal{M}(z) \boldsymbol{\varphi}(0)+\psi_{M}(0) \mathbf{m}(z)\right\}=\frac{[\operatorname{adj} \mathcal{N}(z)]\left\{\mathcal{M}(z) \boldsymbol{\varphi}(0)+\psi_{M}(0) \mathbf{m}(z)\right\}}{\operatorname{det} \mathcal{N}(z)}$.
Since we seek $\varphi(z)$ which is analytic in $|z|<1$, and continuous in $|z| \leq 1$, the numerator of the right-hand side of (3.27) must vanish with sufficient order at the zeros of $\operatorname{det} \mathcal{N}(z)$ in the unit disk. This will give us a system of equations characterizing the constant vector $\varphi(0)$. Thus a first step in solving (3.26) is to identify the singularities of the matrix $\mathcal{N}(z)$, i.e., determine the number of zeros of $\operatorname{det} \mathcal{N}(z)$. To do this, note that $\mathcal{N}_{1}(z)$ is a triangular matrix, $\operatorname{det} \mathcal{N}_{1}(z)=\left(z-G_{1,0}(z)\right)^{M}$. We have that $\operatorname{det} \mathcal{N}(z)=\operatorname{det} \mathcal{N}_{1}(z) \operatorname{det} \mathcal{N}_{3}(z)=$ $\left(z-G_{1,0}(z)\right)^{M} \operatorname{det} \mathcal{N}_{3}(z)$. For $z$ with $|z|=1$,

$$
\left|G_{1,0}(z)\right|=\left|\int_{0}^{\infty} e^{-\left(\lambda_{2}+\lambda_{1}(1-z)\right) t} d B_{1}(t)\right|<\sum_{i=0}^{\infty}\left|\int_{0}^{\infty} \frac{\left(\lambda_{2} t\right)^{i} e^{-\lambda_{2} t}}{i!} e^{\left.-\lambda_{1}(1-z)\right) t} d B_{1}(t)\right| \leq 1=|z|
$$

An easy application of Rouché's theorem shows that the equation $z-G_{1,0}(z)=0$ has exactly one root, say $z_{0}$, in $|z| \leq 1$. Furthermore since $\left|G_{1,0}(z)\right|<1,\left|z_{0}\right|<1$. The following lemma holds.

Lemma 3.1. $z_{0}$ is a multiplicity $M$ zero of $\operatorname{det} \mathcal{N}(z)$ in the open unit disk $|z|<1$.
Next we consider the singularities of the matrix $\mathcal{N}_{3}(z)$. Note that $G_{1, j}(z), j=0, \cdots, N$, $H_{1,0}(z)=G_{1,0}(z) S_{2}^{*}(z, 0)$ and $\eta(z)$ get their maximum norm at $z=1$, and $\left|H_{1,0}(1)\right|<1$ $\sum_{j=0}^{N-M} G_{1, j}(1) \mid<\sum_{i=0}^{\infty} \int_{0}^{\infty}\left(\lambda_{2}\right.$
$t)^{i} / i!e^{-\lambda_{2} t} d B_{1}(t)=1$ and $\eta(1)=1$. Moreover, $\alpha(z)$ also gets its maximum norm at $z=1$, and $\alpha(-1)<1, \alpha(0)=1$ and $\alpha(1)=\left(1-\rho_{1}-\rho_{2}\right) /\left(1-\rho_{1}-\rho_{2}-\lambda_{1} s_{1}\right)>1$. Therefore, there is a possibility that $\sum_{j=0}^{N-M} G_{1, j}(1)+(N-M+1) \alpha(1) H_{1,0}(1) \leq 1$ from choices of the parameters of $B_{i}(\cdot), S_{i}(\cdot)$ and $\lambda_{i}$ for $i=1,2$. In this case, the number of zeros of $\operatorname{det} \mathcal{N}_{3}(z)$ in the unit disk is easy to be determined by the homotopy type of argument(see [16] for the details).
Theorem 3.2. If $\sum_{j=0}^{N-M} G_{1, j}(1)+(N-M+1) \alpha(1) H_{1,0}(1) \leq 1$, then $\operatorname{det} \mathcal{N}_{3}(z)$ has exactly $N-M+1$ zeros (counting multiplicities) in the open unit disk $|z|<1$ and no zeros on $|z|=1$.

Proof. Write $\operatorname{det} \mathcal{N}_{3}(z)=z \mathbf{I}_{N-M+1}-\mathcal{M}_{3}(z)$ and $\operatorname{define} \operatorname{det} \mathcal{N}_{3}^{t}(z)=z \mathbf{I}_{N-M+1}-t \mathcal{M}_{3}(z), 0 \leq$ $t \leq 1$, where $\mathbf{I}_{N-M+1}$ denotes the $(N-M+1) \times(N-M+1)$ unit matrix. Then $t \mathcal{M}_{3}(z)$ is analytic in $|z|<1$ and continuous in $|z| \leq 1$ according to the corresponding properties of $\mathcal{N}_{3}(z)$. The $k t h$ row sum of the entries of $t \mathcal{M}_{3}(z)$ is $t\left[\sum_{j=0}^{k} G_{1, j}(z)+\right.$ $\left.\alpha(z) H_{1,0}(z) \sum_{j=0}^{N-M} \eta^{j}(z)\right], k=0,1, \cdots, N-M$. For $0 \leq t<1$, these sums are strictly less than 1 absolutely value for all $|z| \leq 1$ because $G_{1, j}(1)>0, \alpha(1) H_{1,0}(1)>0$, and $t\left|\sum_{j=0}^{k} G_{1, j}(z)+\alpha(z) H_{1,0}(z) \sum_{j=0}^{N-M} \eta^{j}(z)\right| \leq \sum_{j=0}^{k} G_{1, j}(1)+(N-M+1) \alpha(1) H_{1,0}(1)<$ $\sum_{j=0}^{N-M} G_{1, j}(1)+(N-M+1) \alpha(1) H_{1,0}(1) \leq 1$. For $t=1$, the matrix $\mathcal{M}_{3}(z)$ has spectral radius less than 1 when $z$ with $|z|=1$, since the maximum row sum of the absolute value of the entries of $\mathcal{M}_{3}^{N-M+1}(z)$ does not exceed those of $\mathcal{M}_{3}^{N-M+1}(1)$, and these also
are strictly less than 1 because $\mathcal{M}_{3}(1)$ is irreducible substochastic under the condition $\sum_{j=0}^{N-M} G_{1, j}(1)+(N-M+1) \alpha(1) H_{1,0}(1) \leq 1$. Thus $n(t)$, the number of zeros of det $\mathcal{N}_{3}^{t}(z)$ in $|z| \leq 1$ counting multiplicities, is a continuous integer-valued function of $t$ for $0 \leq t \leq 1$. Therefore, $n(1)=n(t)=n(0)=N-M+1$.

In the general case, it is hard to determine the number of zeros of $\operatorname{det} \mathcal{N}_{3}(z)$ in $|z| \leq 1$ by a direct method because the verification of the conditions needed to apply Rouchés theorem is quite difficult. This can be seen from the following result of the function $\operatorname{det} \mathcal{N}_{3}(z)$ obtained by a direct calculation. We present it here because the resulting formula is very useful in understanding the construction of $\operatorname{det} \mathcal{N}_{3}(z)$ and determining its zeros numerically. Let

$$
\begin{aligned}
\mathcal{A}^{(l)}(z) & =\left[\begin{array}{ccc}
z-G_{1,0}(z)-\alpha(z) H_{1,0}(z) & \cdots & -\alpha(z) H_{1,0}(z) \eta^{l}(z) \\
-G_{1,1}(z)-\alpha(z) H_{1,0}(z) & \cdots & -\alpha(z) H_{1,0}(z) \eta^{l}(z) \\
\vdots & \ddots & \vdots \\
-G_{1, l}(z)-\alpha(z) H_{1,0}(z) & \cdots & z-G_{1,0}(z)-\alpha(z) H_{1,0}(z) \eta^{l}(z)
\end{array}\right], \\
\mathcal{B}^{(l)}(z) & =\left[\begin{array}{cccc}
z-G_{1,0}(z) & & & 1 \\
-G_{1,1}(z) & z-G_{1,0}(z) & & 1 \\
\vdots & \vdots & \ddots & \vdots \\
-G_{1, l-1}(z) & -G_{1, l-2}(z) & \cdots & z-G_{1,0}(z) \\
-G_{1, l}(z) & -G_{1, l-1}(z) & \cdots & -G_{1,0}(z) \\
1
\end{array}\right], \quad l=0,1, \cdots, N-M .
\end{aligned}
$$

Lemma 3.3. (i) $\operatorname{det} \mathcal{A}^{(l)}(z)=\left(z-G_{1,0}(z)\right) \operatorname{det} \mathcal{A}^{(l-1)}(z)-\alpha(z) H_{1,0}(z) \eta^{l}(z) \operatorname{det} \mathcal{B}^{(l)}(z)$,
(ii) $\operatorname{det} \mathcal{B}^{(l)}(z)=\mathbf{x}_{l l}^{(l-1)}(z)$, where $\mathbf{x}_{l i}^{(-1)}(z)=1, i=0,1, \cdots, l$, and for $k=0,1, \cdots, l-1$

$$
\mathbf{x}_{l i}^{(k)}(z)=\left\{\begin{array}{cl}
0 & \text { if } i=0,1, \cdots, k  \tag{3.28}\\
G_{1, i-k}(z) \mathbf{x}_{l k}^{(k-1)}(z)+\left(z-G_{1,0}(z)\right) \mathbf{x}_{l i}^{(k-1)}(z) & \text { if } i=k+1, \cdots, l
\end{array}\right.
$$

(iii) $\operatorname{det} \mathcal{A}^{(l)}(z)=\left(z-G_{1,0}(z)\right)^{l+1}-\alpha(z) H_{1,0}(z)\left\{\left(z-G_{1,0}(z)\right)^{l}-\eta(z)\left(z-G_{1,0}(z)\right)^{l-1} \mathbf{x}_{11}^{(0)}(z)\right.$

$$
\left.-\cdots-\eta^{l-1}(z)\left(z-G_{1,0}(z)\right) \mathbf{x}_{(l-1)(l-1)}^{(l-2)}(z)-\eta^{l}(z) \mathbf{x}_{l l}^{(l-1)}(z)\right\}
$$

Proof. (i) Using the linear decomposition property of determinant, we can write $\operatorname{det} \mathcal{A}^{(l)}(z)$ as follows:

$$
\begin{array}{r}
\operatorname{det} \mathcal{A}^{(l)}(z)=\left|\begin{array}{cccc}
z-G_{1,0}(z)-\alpha(z) H_{1,0}(z) & -\alpha(z) H_{1,0}(z) \eta(z) & \cdots & 0 \\
-G_{1,1}(z)-\alpha(z) H_{1,0}(z) & z-G_{1,0}(z)-\alpha(z) H_{1,0}(z) \eta(z) & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
-G_{1, l}(z)-\alpha(z) H_{1,0}(z) & -G_{1, l-1}(z)-\alpha(z) H_{1,0}(z) \eta(z) & \cdots & z-G_{1,0}(z)
\end{array}\right| \\
-\alpha(z) H_{1,0}(z) \eta^{l}(z)\left|\begin{array}{cccc}
z-G_{1,0}(z)-\alpha(z) H_{1,0}(z) & -\alpha(z) H_{1,0}(z) \eta(z) & \cdots & 1 \\
-G_{1,1}(z)-\alpha(z) H_{1,0}(z) & z-G_{1,0}(z)-\alpha(z) H_{1,0}(z) \eta(z) & \cdots & 1 \\
\vdots & \vdots & \ddots & \vdots \\
-G_{1, l}(z)-\alpha(z) H_{1,0}(z) & -G_{1, l-1}(z)-\alpha(z) H_{1,0}(z) \eta(z) & \cdots & 1
\end{array}\right|,
\end{array}
$$

For the second term, multiplying the lth column by $\alpha(z) H_{1,0}(z) \eta^{i}(z)$, and then adding it to the $i$ th column for $i=0,1, \cdots, l-1$ yields $\alpha(z) H_{1,0}(z) \eta^{l}(z) \operatorname{det} \mathcal{B}^{(l)}(z)$. Then, taking the Laplace expansion along column $l$ in the first term, we get (i). (ii) can be derived by repeating the following procedure: (1) multiplying the first column by $-\left(z-G_{1,0}(z)\right)^{-1}$, and adding it to the $l t h$ column; then taking the factor $\left(z-G_{1,0}(z)\right)^{-1}$ out from the $l t h$ column,
(2) multiplying the second column by $\left(G_{1,1}(z)+\left(z-G_{1,0}(z)\right)\right)\left(z-G_{1,0}(z)\right)^{-1}$, and adding it to the $l$ th column; then taking the factor $\left(z-G_{1,0}(z)\right)^{-1}$ out from the $l$ th column, etc. Finally, note that

$$
\begin{aligned}
& \operatorname{det} \mathcal{A}^{(1)}(z)=\left|\begin{array}{cc}
z-G_{1,0}(z)-\alpha(z) H_{1,0}(z) & -\alpha(z) H_{1,0}(z) \eta(z) \\
-G_{1,1}(z)-\alpha(z) H_{1,0}(z) & z-G_{1,0}(z)-\alpha(z) H_{1,0}(z) \eta(z)
\end{array}\right| \\
&=\left(z-G_{1,0}(z)\right)^{2}-\alpha(z) H_{1,0}(z)\left(z-G_{1,0}(z)\right)-\alpha(z) H_{1,0}(z) \eta(z)\left(G_{1,1}(z)+\left(z-G_{1,0}(z)\right)\right) .
\end{aligned}
$$

Substituting (ii) into (i) and repeating it, we get (iii).
Since for any $k<l$, the submatrix consisting of the first $k-1$ row and the first $k-1$ column of $B^{(k)}(z)$ is the same as that consisting of the first $k-1$ row and the first $k-1$ column of $B^{(l)}(z)$, and the calculation procedure is same, it holds that $\mathbf{x}_{k k}^{(k-1)}(z)=\mathbf{x}_{l k}^{(k-1)}(z)$. Furthermore, repeating the result (3.28) and rearranging the terms, $\mathbf{x}_{l k}^{(k-1)}(z)$ can be determined directly by the following Lemma 3.4.

Lemma 3.4. For $l=1, \cdots, N-M$,
(i) $\mathbf{x}_{k k}^{(k-1)}(z)=\mathbf{x}_{l k}^{(k-1)}(z) ; k=0,1, \cdots, l-1$,
(ii) $\mathbf{x}_{l 0}^{(-1)}(z)=1, \quad \mathbf{x}_{l 1}^{(0)}(z)=G_{1,1}(z)+\left(z-G_{1,0}(z)\right) \quad$ and for $k=2, \cdots, l$,

$$
\begin{aligned}
\mathbf{x}_{l k}^{(k-1)}(z)= & \left(z-G_{1,0}(z)\right)^{k}+\left((k-1) G_{1,1}(z)+G_{1, k}(z)\right)\left(z-G_{1,0}(z)\right)^{k-1} \\
& +G_{1,1}(z)\left(a_{11}^{(k)} G_{1,1}(z)+a_{12}^{(k)} G_{1,2}(z)+\cdots+a_{1(k-1)}^{(k)} G_{1, k-1}(z)\right)\left(z-G_{1,0}(z)\right)^{k-2} \\
& +\cdots \cdots \\
& +G_{1,1}^{k-3}(z)\left(a_{(k-2) 1}^{(k)} G_{1,1}(z)+a_{(k-2) 2}^{(k)} G_{1,2}(z)+a_{(k-2) 3}^{(k)} G_{1,3}(z)\right)\left(z-G_{1,0}(z)\right)^{2} \\
& +G_{1,1}^{k-2}(z)\left((k-1) G_{1,1}(z)+G_{1,2}(z)\right)\left(z-G_{1,0}(z)\right)+G_{1,1}^{k}(z)
\end{aligned}
$$

where $a_{m i}^{(k)}, m=1, \cdots, k-2 ; i=1, \cdots, k-m-1$ all are positive integers and defined as follows:

$$
\begin{aligned}
& a_{11}^{(1)}=a_{12}^{(1)} 1, \quad a_{0 i}^{(2)}=0, \text { for } i=1,2,3 \\
& a_{11}^{(k)}=(k-2)+a_{11}^{(k-1)}, a_{1 i}^{(k)}=1, \text { for } i=2, \cdots, k-1, \\
& a_{21}^{(k)}=a_{11}^{(k-1)}+a_{12}^{(k-1)}, a_{2 i}^{(k)}=k-i-1, \text { for } i=2, \cdots, k-2 \\
& a_{m i}^{(k)}=a_{(m-1) i}^{(k-1)}+a_{m i}^{(k-1)}, \quad a_{m(k-m)}^{(k)}=1, \text { for } m=3, \cdots, k-3, \quad i=1, \cdots, k-m-1, \\
& a_{(k-2) 1}^{(k)}=(k-2)+a_{(k-3) 1}^{(k-1)}, a_{(k-2) 2}^{(k)}=1+a_{(k-3) 2}^{(k-1)}, a_{(k-2) 3}^{(k)}=1
\end{aligned}
$$

Note that $\mathcal{N}_{3}(z)=\mathcal{A}^{(N-M)}(z)$. Substituting $\mathbf{x}_{(N-M) k}^{(k-1)}(z), k=0,1, \cdots, N-M$ into $\operatorname{det} \mathcal{A}^{(N-M)}(z)$ in Lemma 3.3 (iii), we obtain $\operatorname{det} \mathcal{N}_{3}(z)$.

## Theorem 3.5.

$$
\begin{aligned}
\operatorname{det} \mathcal{N}_{3}(z) & =\left(z-G_{1,0}(z)\right)^{N-M+1}-\alpha(z) H_{1,0}(z) \sum_{m=0}^{N-M} \eta^{m}(z)\left(z-G_{1,0}(z)\right)^{N-M}-\alpha(z) H_{1,0}(z) \eta(z) \\
& \times\left\{\left(1+\sum_{m=1}^{N-M-1} m \eta^{m}(z)\right) G_{1,1}(z)+\sum_{m=2}^{N-M} \eta^{m-1}(z) G_{1, m}(z)\right\}\left(z-G_{1,0}(z)\right)^{N-M-1} \\
& -\alpha(z) H_{1,0}(z) \eta^{2}(z) G_{1,1}(z)\left\{\left(1+\sum_{m=3}^{N-M} a_{11}^{(m)} \eta^{m-2}(z)\right) G_{1,1}(z)+\sum_{m=1}^{N-M-2} \eta^{m}(z) G_{1,2}(z)\right. \\
& \left.+\sum_{m=2}^{N-M-2} \eta^{m}(z) G_{1,3}(z)+\cdots+\eta^{N-M-2}(z) G_{1, N-M-1}(z)\right\}\left(z-G_{1,0}(z)\right)^{N-M-2}
\end{aligned}
$$

$$
\begin{align*}
& -\alpha(z) H_{1,0}(z) \eta^{N-M-2}(z) G_{1,1}^{N-M-3}(z)\left\{\left(1+(N-M-2) \eta(z)+a_{(N-M-2) 1}^{(N-M)} \eta^{2}(z)\right) \times\right. \\
& \left.G_{1,1}(z)+\left(1+a_{(N-M-2) 2}^{(N-M)} \eta(z)\right) \eta(z) G_{1,2}(z)+a_{(N-M-2) 3}^{(N-M)} \eta^{2}(z) G_{1,3}(z)\right\}\left(z-G_{1,0}(z)\right)^{2} \\
& -\alpha(z) H_{1,0}(z) \eta^{N-M-1}(z) G_{1,1}^{N-M-2}(z)\left\{\left(1+\left(N-M-\delta_{\{N-M \geq 2\}}\right) \eta(z)\right) G_{1,1}(z)\right. \\
& \left.+\eta(z) G_{1,2}(z) \delta_{\{N-M \geq 2\}}\right\}\left(z-G_{1,0}(z)\right)-\alpha(z) H_{1,0}(z) \eta^{N-M}(z) G_{1,1}^{N-M}(z) \tag{3.29}
\end{align*}
$$

where $\delta_{A}$ denotes the indicator function of the set $A$.
Corollary 3.6. $z=z_{0}$ is not a zero of $\operatorname{det} \mathcal{N}_{3}(z)$, and $z=1$ is not always a zero of $\operatorname{det} \mathcal{N}_{3}(z)$. Proof. Recall that $z=z_{0}$ is the zero of $z-G_{1,0}(z)$. Substituting it into (3.29) yields that

$$
\operatorname{det} \mathcal{N}_{3}\left(z_{0}\right)=-\alpha\left(z_{0}\right) H_{1,0}\left(z_{0}\right) \eta^{N-M}\left(z_{0}\right) G_{1,1}^{N-M}\left(z_{0}\right)
$$

which, obviously, is not equal to zero. For $z=1$, we claim that depending on the choice of parameters, it might not be a zero of $\operatorname{det} \mathcal{N}_{3}(z)$. First from Theorem 3.2, we have seen that in the case $\sum_{j=0}^{N-M} G_{1, j}(1)+(N-M+1) \alpha(1) H_{1,0}(1) \leq 1$, $\operatorname{det} \mathcal{N}_{3}(z)$ has no zeros on $|z|=1$. Hence, $z=1$ is not a zero of $\operatorname{det} \mathcal{N}_{3}(z)$. In the case $\sum_{j=0}^{N-M} G_{1, j}(1)+(N-M+1) \alpha(1) H_{1,0}(1)>$ 1 , if $G_{1,0}(1)+(N-M+1) \alpha(1) H_{1,0}(1)>1$ also holds, substituting $z=1$ into (3.29), we have

$$
\operatorname{det} \mathcal{N}_{3}(1)<\left(1-G_{1,0}(1)-(N-M+1) \alpha(1) H_{1,0}(1)\right)\left(1-G_{1,0}(1)\right)^{N-M}<0
$$

Therefore, only when $0 \leq 1-G_{1,0}(1)-(N-M+1) \alpha(1) H_{1,0}(1)<\sum_{j=1}^{N-M} G_{1, j}(1), z=1$ might becomes a candidate zero of $\operatorname{det} \mathcal{N}_{3}(z)$.

Since $\operatorname{det} \mathcal{N}_{3}(z)$ is analytic in $|z|<1$ and continuous in $|z| \leq 1$, and $\left|\operatorname{det} \mathcal{N}_{3}\left(z_{0}\right)\right|>0$, we conclude that $\operatorname{det} \mathcal{N}_{3}(z)$ has finitely many zeros in the unit disk(otherwise the analytic function $\operatorname{det} \mathcal{N}_{3}(z)$ must be vanished in the unit disk). Let $z_{1}, z_{2}, \cdots, z_{\kappa}$ be the distinct zeros of $\operatorname{det} \mathcal{N}_{3}(z)$ and $d_{i}$ the multiplicity of $z_{i}$. Then $\operatorname{det} \mathcal{N}(z)$ has zeros $z_{0}, z_{1}, \cdots, z_{\kappa}$ with the multiplicities $d_{0}, d_{1}, \cdots, d_{\kappa}$, where $d_{0}=M$. From the analyticity of $\varphi(z)$, the numerator of the right-hand side of (3.28) must become zero with sufficient order at these zeros $z_{0}, z_{1}, \cdots, z_{\kappa}$. Therefore, for $i=0,1, \cdots, \kappa$ we have

$$
\begin{equation*}
\frac{d^{j}}{d z^{j}}\left\{[\operatorname{adj} \mathcal{N}(z)]\left[\mathcal{M}(z) \varphi(0)+\psi_{M}(0) \mathbf{m}(z)\right]\right\}_{z=z_{i}}=0, \quad 0 \leq j<d_{i} . \tag{3.30}
\end{equation*}
$$

Define $(N+1) d_{j} \times(N+2)$ matrix $\mathcal{C}\left(z_{i}\right), i=0,1, \cdots, \kappa$ as

$$
\begin{array}{r}
\mathcal{C}\left(z_{i}\right)=\left[[\operatorname{adj} \mathcal{N}(z) \mathcal{M}(z), \operatorname{adj} \mathcal{N}(z) \mathbf{m}(z)]_{z=z_{i}}, \frac{d}{d z}[\operatorname{adj} \mathcal{N}(z) \mathcal{M}(z), \operatorname{adj} \mathcal{N}(z) \mathbf{m}(z)]_{z=z_{i}}, \cdots\right. \\
\left.\cdots, \frac{d^{d_{i}-1}}{d z^{d_{i}-1}}[\operatorname{adj} \mathcal{N}(z) \mathcal{M}(z), \operatorname{adj} \mathcal{N}(z) \mathbf{m}(z)]_{z=z_{i}}\right]^{\tau} \tag{3.31}
\end{array}
$$

and $\mathcal{C}=\left[\mathcal{C}\left(z_{0}\right), \mathcal{C}\left(z_{1}\right), \cdots, \mathcal{C}\left(z_{\kappa}\right)\right]^{\tau}$. Then, we can write the systems (3.30) as

$$
\begin{equation*}
\mathcal{C}\left[\varphi(0), \psi_{M}(0)\right]^{\tau}=0 \tag{3.32}
\end{equation*}
$$

Theorem 3.7. (i) For each $i=0,1, \cdots, \kappa$, the rank of $\mathcal{C}\left(z_{i}\right)$ is exactly $d_{i}$.
(ii) The rank of $\mathcal{C}$ is exactly $\sum_{i=0}^{\kappa} d_{i}=M+\sum_{i=1}^{\kappa} d_{i}$.

Proof is omitted here because of the space limitation. A similarly detailed proof can be found in [16], Secs. 5 and 6, where the relation of the spectral basis and singularities of $\mathcal{N}(z)$ is used to determine the ranks of $\mathcal{C}\left(z_{i}\right)$ and $\mathcal{C}$ (note that $\mathcal{N}(z), \mathcal{C}\left(z_{i}\right)$ and $\mathcal{C}$ respectively correspond to $A(z), C\left(\xi_{i}\right)$ and $\hat{B}$ there). From (3.32), hence, we can only obtain $M+\sum_{i=1}^{\kappa} d_{i}$ linearly independent equations with respect to $N+2$ boundary probabilities $\varphi_{i}(0), i=0,1, \cdots, N$ and $\psi_{M}(0)$. Utilizing the analyticity of the function $\varphi(z)$ and the fact that $z_{i}$ is a zero of $\operatorname{det} \mathcal{N}(z)$ with multiplicity $d_{i}$, we have obtained the conclusion that $z_{i}$ must be a zero of the numerator of the right-hand side of (3.28) with the same order $d_{i}$. Again from these facts, we have the following corollary.

Corollary 3.8. For $j=0,1, \cdots, d_{i}-1$, the matrix equation $d^{j} / d z^{j}[\operatorname{adj} \mathcal{N}(z) \mathcal{M}(z), \operatorname{adj} \mathcal{N}(z)$ $\mathbf{m}(z)]_{z=z_{i}}\left[\boldsymbol{\varphi}(0), \psi_{M}(0)\right]^{\top}=0$ only gives one linearly independent equation.

Proof. Write $\mathcal{K}(z) \equiv[\operatorname{adj} \mathcal{N}(z) \mathcal{M}(z), \operatorname{adj} \mathcal{N}(z) \mathbf{m}(z)]$ and $\mathcal{K}^{(j)}(z) \equiv d^{j} / d z^{j}[\operatorname{adj} \mathcal{N}(z) \mathcal{M}(z), \operatorname{adj}$ $\mathcal{N}(z) \mathbf{m}(z)$ ]. By the definition of $\mathcal{N}(z), \mathcal{M}(z)$ and $\mathbf{m}(z)$, the every entry of $\mathcal{K}(z)$ is analytic in $|z|<1$ and continuous in $|z| \leq 1$. We first consider the case $\left|z_{i}\right|<1$. Suppose that for some a number $j<d_{i}$, the rank of the matrix $\mathcal{K}^{(j)}\left(z_{i}\right)$ is larger than 1 . Since $\mathcal{C}_{i}$ has exactly the rank $d_{i}$, there must exists a number $l<d_{i}$ such that $\mathcal{K}^{(l)}\left(z_{i}\right)=\sum_{k=1}^{l} \mathbf{C}_{k} \mathcal{K}^{\left(j_{k}\right)}\left(z_{i}\right)$, where $\mathbf{C}_{k}$ is a constant matrix and $j_{k}<l$ for $k=1,2, \cdots, l$. By the analyticity of the entries of $\mathcal{K}(z)$, we have that there exists a neibourhood $U\left(z_{i}\right)$ of $z_{i}$ such that for all $z \in U\left(z_{i}\right), \mathcal{K}^{(l)}(z)=\sum_{k=1}^{l} \mathbf{C}_{k} \mathcal{K}^{\left(j_{k}\right)}(z)$. Then for $z \in U\left(z_{i}\right), \mathcal{K}^{(2 l)}(z)=d^{l} / d z^{l}\left(\mathcal{K}^{(l)}(z)\right)=$ $d^{l} / d z^{l}\left(\sum_{k_{1}=1}^{l} \mathbf{C}_{k_{1}} \mathcal{K}^{\left(j_{k_{1}}\right)}(z)\right)=\sum_{k_{1}=1}^{l} \mathbf{C}_{k_{1}} d^{j_{k_{1}}} / d z^{j_{k_{1}}}\left(\mathcal{K}^{(l)}(z)\right)=\sum_{k_{1}=1}^{l} \sum_{k_{2}=1}^{l} \mathbf{C}_{k_{1}} \mathbf{C}_{k_{2}} \mathcal{K}^{\left(j_{k_{1}}+j_{k_{2}}\right)}(z)$. If $j_{k_{1}}+j_{k_{2}} \geq l$, then $\mathcal{K}^{\left(j_{k_{1}}+j_{k_{2}}\right)}(z)=\sum_{k_{3}=1}^{l} \mathrm{C}_{k_{3}} \mathcal{K}^{\left(j_{k_{1}}+j_{k_{1}}+j_{k_{3}}-l\right)}(z)$. Hence, repeating this procedure, we obtain $\mathcal{K}^{(2 l)}(z)=\sum_{k_{1}=1}^{l} \sum_{k_{2}=1}^{l} \cdots \sum_{k_{m}=1}^{l} \mathbf{C}_{k_{1}} \mathbf{C}_{k_{2}} \cdots \mathbf{C}_{k_{m}} \mathcal{K}^{\left(j_{k_{1}}+j_{k_{2}}+\cdots+j_{k_{m}-(m-2) l}\right)}(z)$ for $z \in U\left(z_{i}\right)$, where $0 \leq j_{k_{1}}+j_{k_{2}}+\cdots+j_{k_{m}}-(m-2) l<l$ for all $k_{1}, k_{2}, \cdots, k_{m}$. Substituting $z=z_{i}$ and using the fact that $\mathcal{K}^{(j)}\left(z_{i}\right)\left[\boldsymbol{\varphi}(0), \psi_{M}(0)\right]^{\tau}=0$ for $j<d_{i}$ yield that $\mathcal{K}^{(2 l)}\left(z_{i}\right)\left[\boldsymbol{\varphi}(0), \psi_{M}(0)\right]^{\tau}=0$. By inductive reasoning, we can obatin $\mathcal{K}^{(n l)}\left(z_{i}\right)\left[\varphi(0), \psi_{M}(0)\right]^{\tau}=$ 0 for any $n \geq 0$. This is contradictory to the fact that $z_{i}$ is a zero of the numerator of the right-hand side of (3.28) with order $d_{i}$. The conclusion in the case $\left|z_{i}\right|=1$ can be proved similarly.

In general, which row should be chosen in determining $M+\sum_{i=1}^{\kappa} d_{i}$ linearly independent equations, depends on the concrete formation of the matrix $d^{j} / d z^{j}[\operatorname{adj} \mathcal{N}(z) \mathcal{M}(z), \operatorname{adj} \mathcal{N}(z)$ $\mathbf{m}(z)]_{z=z_{i}}$. Here, without loss of generality, suppose that for $z_{i}$, the $i$ th row of $d^{j} / d z^{j}[\operatorname{adj} \mathcal{N}(z)$ $\left.\mathcal{M}(z), \operatorname{adj} \mathcal{N}(z) \mathbf{m}\left(z_{i}\right)\right]_{z=z_{i}}$ is what we need. Let $\mathbf{e}_{i}=(0, \cdots, 1, \cdots, 0)$, i.e., the entry in $i$ th position is 1 , others are zero. Define $\left(M+\sum_{i=1}^{\kappa} d_{i}\right) \times(N+1)$ matrix $\mathcal{F}$ and $\left(M+\sum_{i=1}^{\kappa} d_{i}\right) \times 1$ vector $\mathcal{E}$ by taking the $i$ row of the matrix $d^{j} / d z^{j}[\operatorname{adj} \mathcal{N}(z) \mathcal{M}(z)]_{z=z_{i}}$ and the $i$ th entry of the vector $d^{j} / d z^{j}[\operatorname{adj} \mathcal{N}(z) \mathbf{m}(z)]_{z=z_{i}}$ for $j=0,1, \cdots, d_{i}-1 ; i=0,1, \cdots, \kappa$, namely,

$$
\begin{gathered}
\mathcal{F}=\left[\mathbf{e}_{0}[\operatorname{adj} \mathcal{N}(z) \mathcal{M}(z)]_{z=z_{0}}, \cdots, \mathbf{e}_{0} \frac{d^{d_{0}-1}}{d z^{d_{0}-1}}[\operatorname{adj} \mathcal{N}(z) \mathcal{M}(z)]_{z=z_{0}}, \cdots \cdots,\right. \\
\left.\mathbf{e}_{\kappa}[\operatorname{adj} \mathcal{N}(z) \mathcal{M}(z)]_{z=z_{\kappa}}, \cdots, \mathbf{e}_{\kappa} \frac{d^{d_{\kappa}-1}}{d z^{d_{\kappa}-1}}[\operatorname{adj} \mathcal{N}(z) \mathcal{M}(z)]_{z=z_{\kappa}}\right]^{\tau}, \\
\mathcal{E}=-\left[\mathbf{e}_{0}[\operatorname{adj} \mathcal{N}(z) \mathbf{m}(z)]_{z=z_{0}}, \cdots, \mathbf{e}_{0} \frac{d^{d_{0}-1}}{d z^{d_{0}-1}}[\operatorname{adj} \mathcal{N}(z) \mathbf{m}(z)]_{z=z_{0}}, \cdots \cdots,\right. \\
\left.\mathbf{e}_{\kappa}[\operatorname{adj} \mathcal{N}(z) \mathbf{m}(z)]_{z=z_{\kappa}}, \cdots, \mathbf{e}_{\kappa} \frac{d^{d_{\kappa}-1}}{d z^{d_{\kappa}-1}}[\operatorname{adj} \mathcal{N}(z) \mathbf{m}(z)]_{z=z_{\kappa}}\right]^{\tau},
\end{gathered}
$$

Then, $M+\sum_{i=1}^{\kappa} d_{i}$ linearly independent equations from (3.32) with respect to $\varphi_{i}(0), i=$ $0,1, \cdots \cdots, N$ and $\psi_{M}(0)$ can be expressed as

$$
\begin{equation*}
\mathcal{F} \varphi(0)-\psi_{M}(0) \mathcal{E}=0 \tag{3.33}
\end{equation*}
$$

In the case $\sum_{j=0}^{N-M} G_{1, j}(1)+(N-M+1) \alpha(1) H_{1,0}(1) \leq 1$, we have proved that $\operatorname{det} \mathcal{N}(z)$ has exactly $N+1$ zeros(counting multiplicity) in the open disk, namely, $M+\sum_{j=1}^{\kappa} d_{j}=N+1$. Also note that $z=1$ is not a zero of $\operatorname{det} \mathcal{N}(z)$ in this case. For every $\psi_{M}(0)$ fixed, therefore, (3.33) gives exactly $N+1$ linearly independent equations to determine $\varphi_{i}(0), i=0,1, \cdots \cdots, N$.

Remark 3.2. In the case $\sum_{j=0}^{N-M} G_{1, j}(1)+(N-M+1) \alpha(1) H_{1,0}(1)>1$, determining the value of $\sum_{i=1}^{\kappa} d_{i}$ by a direct method still is an opening problem. Here, basing on our numerical calculation results and the fact that under the ergodicity condition, the inherent Kolmogorov equations for the stationary state probabilities have an unique, absolutely convergent solution, we conjecture that the analytic function $\operatorname{det} \mathcal{N}_{3}(z)$ has the specified number of zeros in the required domain (see [4,9] for the detailed arguments). That is, when $z=1$ is not a zero of $\operatorname{det} \mathcal{N}_{3}(z), \sum_{i=1}^{\kappa} d_{i}=N-M+1$, and when $z=1$ is a zero of $\operatorname{det} \mathcal{N}_{3}(z)$, $\sum_{i=1}^{\kappa} d_{i}=N-M+2$ (in this case, there exists a equation equivalent to the normalizing condition $\left.\Phi^{(1)}(1,1)+\Phi^{(2)}(1,1)=1\right)$. Indeed, the Kolmogorov equations for the equilibrium distribution of the Markov chain $\left\{\left(X_{k}^{(1)}, X_{k}^{(2)}, J_{k}\right)\right\}_{k \geq 1}$, along with the normalizing condition $\Phi^{(1)}(1,1)+\Phi^{(2)}(1,1)=1$, have a unique absolutely convergent solution, and using generating functions, we have transformed those Kolmogorov equation into the ( $N+1$ )-dimensional matrix equation (3.26), plus the normalizing condition. Hence, if $M+\sum_{i=1}^{\kappa} d_{i}+1=N+2$, i.e., $\sum_{i=1}^{\kappa} d_{i}=N-M+1$, then as there exists an unique solution, the equations (3.33), plus the normalizing condition must be independent. Now suppose that $\sum_{i=1}^{\kappa} d_{i}<N-M+1$. Then we would obtain too few equations to determine all $N+2$ unknown constants uniquely, and we would find multiple solutions for them-which seems to be impossible. Finally, if $\sum_{i=1}^{\kappa} d_{i}>N-M+1$, then we would find too many equations for the $N+2$ unknown constants. Again from the fact that there is a unique solution, there must be exactly $N+2$ independent equations amongst those (3.33), plus the normalizing condition.

According to the above arguments, in practice, the matrix $\mathcal{F}$ is made up of $N+1$ linearly independent rows. The remaining work is to determine the unknown constant $\psi_{M}(0)$. First we write (3.33) as

$$
\begin{equation*}
\boldsymbol{\varphi}(0)=\psi_{M}(0) \boldsymbol{\xi} \tag{3.34}
\end{equation*}
$$

where $\boldsymbol{\xi}=\left(\xi_{0}, \cdots, \xi_{N}\right)^{\tau} \equiv \mathcal{F}^{-1} \mathcal{E}$. For $|z| \leq 1$, define $\boldsymbol{\varphi}(z)$ by substituting (3.34) into (3.27) as

$$
\begin{equation*}
\boldsymbol{\varphi}(z)=\psi_{M}(0) \mathcal{N}^{-1}(z)\{\mathcal{M}(z) \boldsymbol{\xi}+\mathbf{m}(z)\}=\psi_{M}(0) \frac{[\operatorname{adj} \mathcal{N}(z)]\{\mathcal{M}(z) \boldsymbol{\xi}+\mathbf{m}(z)\}}{\operatorname{det} \mathcal{N}(z)} \tag{3.35}
\end{equation*}
$$

Then $\varphi(z)$ is clearly analytic in $|z|<1$ except on the zero set of $\operatorname{det} \mathcal{N}(z)$. Since the equation (3.34) holds, the numerator of (3.35) vanishes on the zero set of $\operatorname{det} \mathcal{N}(z)$. Thus $\varphi(z)$ may be extended to a function which is analytic in $|z|<1$ and continuous in $|z| \leq 1$ by Riemann removable singularity theorem. For simplicity, we still use the notation $\varphi(z)$ to denote the extended function. When $z=1$, in particular, we have,

$$
\begin{equation*}
\boldsymbol{\varphi}(1)=\psi_{M}(0) \mathcal{N}^{-1}(1)\{\mathcal{M}(1) \boldsymbol{\xi}+\mathbf{m}(1)\} . \tag{3.36}
\end{equation*}
$$

Hence

$$
\begin{equation*}
\boldsymbol{\varphi}(1)-\boldsymbol{\varphi}(0)=\psi_{M}(0) \mathcal{N}^{-1}(1)\{[\mathcal{M}(1)+\mathcal{N}(1)] \boldsymbol{\xi}+\mathbf{m}(1)\} \equiv \psi_{M}(0) \mathbf{w}_{\varphi} \tag{3.37}
\end{equation*}
$$

Substituting $z=1$ into (3.24), and then substituting (3.34) and (3.36) into the resulting formula, we have

$$
\begin{align*}
\boldsymbol{\psi}(1)-\boldsymbol{\psi}(0)= & \psi_{M}(0)\left\{\left[\boldsymbol{\Lambda}_{2}(1) \mathcal{N}^{-1}(1) \mathcal{M}(1)+\mathbf{H}_{2}(1)+\hat{\mathbf{G}}_{2}(1) \mathbf{G}^{-1}(0) \mathbf{H}(0) \mathbf{J}\right] \boldsymbol{\xi}\right. \\
& \left.+\boldsymbol{\Lambda}_{2}(1) \mathcal{N}^{-1}(1) \mathbf{m}(1)+\hat{G}_{2}(1) \mathbf{G}^{-1}(0) \mathbf{v}+\mathbf{v}_{2}(1)\right\} \equiv \psi_{M}(0) \mathbf{w}_{\psi} \tag{3.38}
\end{align*}
$$

Substituting $z_{1}=1$ and $z_{2}=1$ into (2.14), we have

$$
\begin{equation*}
\Phi^{(1)}(1,1)=r_{1}\left(\varphi_{0}(0)+\psi_{0}(0)\right)+\sum_{m=0}^{N}\left(\varphi_{m}(1)-\varphi_{m}(0)\right)+\sum_{m=0}^{M}\left(\psi_{m}(1)-\psi_{m}(0)\right) \tag{3.39}
\end{equation*}
$$

Substituting $z_{1}=1$ into (2.15) and subsequently letting $z_{2} \rightarrow 1$, we have

$$
\begin{gather*}
\Phi^{(2)}(1,1)=\frac{r_{2}+r_{1} \lambda_{2} b_{1}}{1-\rho_{2}} \varphi_{0}(0)+\frac{r_{2}+r_{1} \lambda_{2}\left(b_{1}+s_{1}+s_{2}\right)}{1-\rho_{2}} \psi_{0}(0)+\frac{\lambda_{2} b_{1}}{1-\rho_{2}} \sum_{m=0}^{N}\left(\varphi_{m}(1)\right. \\
\left.-\varphi_{m}(0)\right)+\frac{\lambda_{2}\left(b_{1}+s_{1}\right)}{1-\rho_{2}} \sum_{m=0}^{M}\left(\psi_{m}(1)-\psi_{m}(0)\right) \tag{3.40}
\end{gather*}
$$

Therefore, using the normalization condition:

$$
\begin{equation*}
\Phi^{(1)}(1,1)+\Phi^{(2)}(1,1)=1, \tag{3.41}
\end{equation*}
$$

the relations (3.37), (3.38) and $\psi_{0}(0)=\mathbf{u} \varphi(0)+a \psi_{M}(0)=\psi_{M}(0)(\mathbf{u} \boldsymbol{\xi}+a), \varphi_{0}(0)=\psi_{M}(0) \xi_{0}$, the unknown constant $\psi_{M}(0)$ may be determined by

$$
\begin{equation*}
\psi_{M}(0)=\left(1-\rho_{2}\right) K^{-1} \tag{3.42}
\end{equation*}
$$

here, $K \equiv\left[r_{1}\left(1-\rho_{2}+\lambda_{2} b_{1}\right)+r_{2}\right] \xi_{0}+\left[r_{1}\left(1-\rho_{2}+\lambda_{2}\left(b_{1}+s_{1}+s_{2}\right)\right)+r_{2}\right](\mathbf{u} \boldsymbol{\xi}+a)+\left(1-\rho_{2}+\lambda_{2} b_{1}\right)<$ $\left.\left.\mathbf{w}_{\varphi}, 1\right\rangle+\left(1-\rho_{2}+\lambda_{2}\left(b_{1}+s_{1}\right)\right)<\mathbf{w}_{\psi}, \mathbf{1}\right\rangle, \quad$ where $<\cdot, \cdot>$ represents the internal product of vectors, and 1 the unit vector.

## Algorithm

1. Determine all zeros of $\operatorname{det} \mathcal{N}(z)$ :
(i) Note that the unique zero $z_{0}$ of the function $f(z)=z-G_{1,0}(z)$ is real, and $f(-1)<$ $0, f(1)>0$. Calculate the value of $z_{0}$ by using bisection method in the interval $(-1,1)$.
(ii) Choose lattice points on the unit disk by a suitable interval $\epsilon$. And solve the equation $\operatorname{det} \mathcal{N}_{3}(z)=0(|z| \leq 1)$ by using these lattice points as the initial values of Newton's method. Namely, setting $y_{0}$ as one of these lattice points, we get the sequence $\left\{y_{k}\right\}$ such that

$$
y_{k+1}=y_{k}-\frac{\operatorname{det} \mathcal{N}_{3}\left(y_{k}\right)}{\left.\frac{d}{d z} \operatorname{det} \mathcal{N}_{3}(z)\right|_{z=y_{k}}} .
$$

Here, calculate the values of $\operatorname{det} \mathcal{N}_{3}\left(y_{k}\right)$ and $\left.\frac{d}{d z} \operatorname{det} \mathcal{N}_{3}(z)\right|_{z=y_{k}}$ by using (3.29).
(iii) If the number of zeros of $\operatorname{det} \mathcal{N}_{3}(z)$ in the unit disk obtained at step (ii) is less than $N-M+1$, narrow intervals of lattice points(e.g. put $\epsilon \rightarrow \epsilon / 2$ ) and return to step (ii).
2. Calculate the boundary probabilities $\boldsymbol{\varphi}(0), \psi_{M}(0)$ :
(i) For $i=0,1, \cdots, \kappa$, calculate

$$
\begin{aligned}
& \mathbf{e}_{i}[\operatorname{adj} \mathcal{N}(z) \mathcal{M}(z)]_{z=z_{i}}, \mathbf{e}_{i} \frac{d}{d z}[\operatorname{adj} \mathcal{N}(z) \mathcal{M}(z)]_{z=z_{i}}, \cdots, \mathbf{e}_{i} \frac{d^{d_{i}-1}}{d z^{d_{i}-1}}[\operatorname{adj} \mathcal{N}(z) \mathcal{M}(z)]_{z=z_{i}}, \\
& \mathbf{e}_{i}[\operatorname{adj} \mathcal{N}(z) \mathbf{m}(z)]_{z=z_{i}}, \mathbf{e}_{i} \frac{d}{d z}[\operatorname{adj} \mathcal{N}(z) \mathbf{m}(z)]_{z=z_{i}}, \cdots, \mathbf{e}_{i} \frac{d^{d_{i}-1}}{d z^{d_{i}-1}}[\operatorname{adj} \mathcal{N}(z) \mathbf{m}(z)]_{z=z_{i}}
\end{aligned}
$$

where $\mathbf{e}_{i}=(0, \cdots, 1, \cdots, 0)$.
(ii) Calculate $\psi_{M}(0)=\left(1-\rho_{2}\right) K^{-1}$, and solve the system of $N+1$ linearly independent equations:

$$
K \mathcal{F} \varphi(0)=(1-\rho) \mathcal{E}
$$

## 4. Waiting Times

In this section we consider the $L S T$ of the waiting time distributions and the mean waiting times at $Q_{i}, i=1,2$. Let $W_{i}$ represent the waiting time at $Q_{i}$, and $\hat{W}_{i}(s)$ its $L S T$ for $i=1,2$. Since the customers present in $Q_{i}$ just after the instant of service completion of a type $i$ customer are just the customers who had arrived during the waiting time and service time of that customer, we have the following relations:

$$
\begin{array}{ll}
r_{1} \hat{W}_{1}\left(\lambda_{1}\left(1-z_{1}\right)\right) \hat{B}_{1}\left(\lambda_{1}\left(1-z_{1}\right)\right)=\Phi^{(1)}\left(z_{1}, 1\right), & \left|z_{1}\right| \leq 1, \\
r_{2} \hat{W}_{2}\left(\lambda_{2}\left(1-z_{2}\right)\right) \hat{B}_{2}\left(\lambda_{2}\left(1-z_{2}\right)\right)=\Phi^{(2)}\left(1, z_{2}\right), & \left|z_{2}\right| \leq 1 . \tag{4.2}
\end{array}
$$

Therefore

$$
\begin{equation*}
E\left[W_{1}\right]=\left.\frac{1}{r_{1} \lambda_{1}} \frac{d}{d z_{1}} \Phi^{(1)}\left(z_{1}, 1\right)\right|_{z_{1}=1}-b_{1}, \quad E\left[W_{2}\right]=\left.\frac{1}{r_{2} \lambda_{2}} \frac{d}{d z_{2}} \Phi^{(2)}\left(1, z_{2}\right)\right|_{z_{2}=1}-b_{2} . \tag{4.3}
\end{equation*}
$$

Substituting $z_{2}=1$ into (2.14), and then differentiating in $z_{1}$, we get

$$
\begin{equation*}
\left.\frac{d}{d z_{1}} \Phi^{(1)}\left(z_{1}, 1\right)\right|_{z_{1}=1}=r_{1}\left(\varphi_{0}(0)+\left(1+\lambda_{2} s_{2}\right) \psi_{0}(0)\right)-\left(1-\rho_{1}\right) \Phi^{(1)}(1,1)+\sum_{m=0}^{N} \varphi_{m}^{\prime}(1)+\sum_{m=0}^{N} \psi_{m}^{\prime}(1) \tag{4.4}
\end{equation*}
$$

Next, substituting $z_{1}=1$ into (2.15), we get $\Phi^{(2)}\left(1, z_{2}\right)=\zeta\left(z_{2}\right) / \nu\left(z_{2}\right)$ where

$$
\begin{aligned}
& \nu\left(z_{2}\right) \equiv z_{2}-\hat{B}_{2}\left(\lambda_{2}\left(1-z_{2}\right)\right) \\
& \zeta\left(z_{2}\right) \equiv \hat{B}_{2}\left(\lambda_{2}\left(1-z_{2}\right)\right)\left\{\left(r_{2} z_{2}+r_{1} \hat{B}_{1}\left(\lambda_{2}\left(1-z_{2}\right)\right)-1\right) \hat{S}_{1}\left(\lambda_{2}\left(1-z_{2}\right)\right) \varphi_{0}(0)+\left(r_{2} z_{2}+r_{1} \hat{B}_{1}\left(\lambda_{2}\left(1-z_{2}\right)\right)\right.\right. \\
& \left.\quad \times \hat{S}_{1}\left(\lambda_{2}\left(1-z_{2}\right)\right) \hat{S}_{2}\left(\lambda_{2}\left(1-z_{2}\right)\right)-1\right) \psi_{0}(0)+\left(\hat{B}_{1}\left(\lambda_{2}\left(1-z_{2}\right)\right)-1\right) \hat{S}_{1}\left(\lambda_{2}\left(1-z_{2}\right)\right) \sum_{m=0}^{N}\left(\varphi_{m}(1)\right. \\
& \left.\left.\quad-\varphi_{m}(0)\right) z_{2}^{m}+\left(\hat{B}_{1}\left(\lambda_{2}\left(1-z_{2}\right)\right) \hat{S}_{1}\left(\lambda_{2}\left(1-z_{2}\right)\right)-1\right) \sum_{m=0}^{M}\left(\psi_{m}(1)-\psi_{m}(0)\right) z_{2}^{m}\right\} .
\end{aligned}
$$

Since $\nu(1)=\zeta(1)=0$, using L'Hospital's rule, we get

$$
\begin{equation*}
\left.\frac{d}{d z_{2}} \Phi^{(2)}\left(1, z_{2}\right)\right|_{z_{2}=1}=\frac{\zeta^{\prime \prime}(1) \nu^{\prime}(1)-\zeta^{\prime}(1) \nu^{\prime \prime}(1)}{2\left(\nu^{\prime}(1)\right)^{2}} \tag{4.5}
\end{equation*}
$$

where $\nu^{\prime}(1)=1-\rho_{2}, \nu^{\prime \prime}(1)=-\lambda_{2} b_{2}^{(2)}$ and

$$
\zeta^{\prime}(1)=\left(r_{2}+r_{1} \lambda_{2} b_{1}\right) \varphi_{0}(0)+\left(r_{2}+r_{1} \lambda_{2}\left(b_{1}+s_{1}+s_{2}\right) \phi_{0}(0)+\lambda_{2} b_{1} \sum_{m=0}^{N}\left(\varphi_{m}(1)-\varphi_{m}(0)\right)\right.
$$

$$
\begin{aligned}
& +\lambda_{2}\left(b_{1}+s_{1}\right) \sum_{m=0}^{M}\left(\psi_{m}(1)-\psi_{m}(0)\right), \\
\zeta^{\prime \prime}(1)= & {\left[r_{1} \lambda_{2}^{2} b_{1}^{(2)}+2\left(r_{2}+r_{1} \lambda_{2} b_{1}\right) \lambda_{2} s_{1}+2 \rho_{2}\left(r_{2}+r_{1} \lambda_{2} b_{1}\right)\right] \varphi_{0}(0)+\left[r_{1} \lambda_{2}^{2}\left(b_{1}+s_{1}+s_{2}\right)^{(2)}\right.} \\
& +2 \rho_{2}\left(r_{2}+r_{1} \lambda_{2}\left(b_{1}+s_{1}+s_{2}\right)\right] \psi_{0}(0)+\left[\lambda_{2}^{2}\left(b_{1}^{(2)}+2 b_{1} s_{1}\right)+2 \rho_{2} \lambda_{2} b_{1}\right] \sum_{m=0}^{N}\left(\varphi_{m}(1)\right. \\
& \left.-\varphi_{m}(0)\right)+\left[\lambda_{2}^{2}\left(b_{1}+s_{1}\right)^{(2)}+2 \rho_{2} \lambda_{2}\left(b_{1}+s_{1}\right)\right] \sum_{m=0}^{N}\left(\psi_{m}(1)-\psi_{m}(0)\right) \\
& +2 \lambda_{2} b_{1} \sum_{m=1}^{N} m\left(\varphi_{m}(1)-\varphi_{m}(0)\right)+2 \lambda_{2}\left(b_{1}+s_{1}\right) \sum_{m=1}^{N} m\left(\psi_{m}(1)-\psi_{m}(0)\right) .
\end{aligned}
$$

As shown in (4.4), the differential values $\varphi_{m}^{\prime}(1), 0 \leq m \leq N ; \psi_{m}^{\prime}(1), 0 \leq m \leq M$ are necessary to obtain $d \Phi^{(1)}\left(z_{1}, 1\right) /\left.d z_{1}\right|_{z_{1}=1}$. From (2.22), one can easily calculate $\psi_{m}^{\prime}(1)$ as

$$
\begin{gather*}
\psi_{m}^{\prime}(1)=\sum_{j=1}^{m+1} H_{2, m-j+1}^{\prime}(1) \varphi_{j}(0)+\sum_{j=1}^{m+1} G_{2, m-j+1}^{\prime}(1) \psi_{j}(0)+r_{2} H_{2, m}^{\prime}(1) \varphi_{0}(0)+r_{2} G_{2, m}^{\prime}(1) \psi_{0}(0) \\
1 \leq m \leq M-1 \tag{4.6}
\end{gather*}
$$

Furthermore, differentiating (3.11) in $z$ and then letting $z \rightarrow 1$, one can get $\psi_{M}^{\prime}(1)$

$$
\begin{align*}
\psi_{M}^{\prime}(1)=\alpha^{\prime}(1) \sum_{j=0}^{N} \varphi_{j}(1) & +\alpha(1) \sum_{j=0}^{N}\left[\varphi_{j}^{\prime}(1)+\varphi_{j}(1)(j-M) \eta^{\prime}(1)\right]-\sum_{j=1}^{N} \varphi_{j}(0) k_{j}^{\prime}(1)-\sum_{j=1}^{M} \psi_{j}(0) l_{j}^{\prime}(1) \\
+ & \beta_{1}^{\prime}(1) \varphi_{0}(0)+\beta_{2}^{\prime}(1) \psi_{0}(0) \tag{4.7}
\end{align*}
$$

where

$$
\begin{gathered}
\alpha^{\prime}(1)=\frac{\left[\lambda_{2} s_{1} \lambda_{1}^{2} b_{2}^{(2)}-\left(2 \lambda_{1} s_{1}\left(1-\rho_{2}\right)+\lambda_{1}^{2} s_{1}^{(2)}\right)\left(1-\rho_{2}\right)\right]\left(1-\rho_{1}-\rho_{2}\right)}{2\left(1-\rho_{1}-\rho_{2}-\lambda_{1} s_{1}\right)^{2}\left(1-\rho_{2}\right)^{2}} \\
+\frac{\left[\left(\lambda_{1}^{2} b_{2}^{(2)}-2 \lambda_{1} s_{1}\left(1-\rho_{1}-\rho_{2}\right)\left(1-\rho_{2}\right)-\lambda_{2} b_{1} \lambda_{1}^{2} b_{2}^{(2)}\right] \lambda_{1} s_{1}\right.}{2\left(1-\rho_{1}-\rho_{2}-\lambda_{1} s_{1}\right)^{2}\left(1-\rho_{2}\right)^{2}} \\
k_{j}^{\prime}(1)= \begin{cases}\alpha^{\prime}(1)+\alpha(1)(j-M) \eta^{\prime}(1)+\sum_{m=j-1}^{M-1}\left[H_{2, m-j+1}^{\prime}(1)\right. \\
\left.+\left(H_{2, m-j+1}(1)-H_{2, m-j+1}(0)\right)(m-M) \eta^{\prime}(1)\right] & \text { if } 1 \leq j \leq M \\
\alpha^{\prime}(1)+\alpha(1)(j-M) \eta^{\prime}(1) & \text { if } M<j \leq N,\end{cases} \\
l_{j}^{\prime}(1)=\sum_{m=j-1}^{M-1}\left[G_{2, m-j+1}^{\prime}(1)+\left(G_{2, m-j+1}(1)-G_{2, m-j+1}(0)\right)(m-M) \eta^{\prime}(1)\right], \quad 1 \leq j \leq M
\end{gathered},
$$

and

$$
\begin{aligned}
\beta_{1}^{\prime}(1)= & \frac{2\left(r_{1} \rho_{1}+r_{2} \lambda_{1} b_{2}\right)\left(1-\rho_{2}\right)+2 \lambda_{1} s_{1}\left(1-\rho_{1}-\rho_{2}+r_{2} \lambda_{1} b_{2}\right)-r_{2} \lambda_{1}^{(2)} b_{1}^{(2)}}{2\left(1-\rho_{1}-\rho_{2}-\lambda_{1} s_{1}\right)\left(1-\rho_{2}\right)} \\
& +\frac{r_{2} \lambda_{1}^{2} b_{2}^{(2)}\left(1+\lambda_{2} b_{1}\right)}{2\left(1-\rho_{1}-\rho_{2}-\lambda_{1} s_{1}\right)\left(1-\rho_{2}\right)^{2}}-\left\{\frac{2 M \lambda_{1} b_{2}\left(1-\rho_{1}-\rho_{2}-\lambda_{1} s_{1}\right)-\lambda_{1}^{2}\left(b_{1}+s_{1}\right)^{(2)}}{2\left(1-\rho_{1}-\rho_{2}-\lambda_{1} s_{1}\right)^{2}\left(1-\rho_{2}\right)}\right. \\
& \left.+\frac{\lambda_{2}\left(b_{1}+s_{1}\right) \lambda_{1}^{2} b_{2}^{(2)}}{2\left(1-\rho_{1}-\rho_{2}-\lambda_{1} s_{1}\right)^{2}\left(1-\rho_{2}\right)^{2}}\right\}\left(1-\rho_{1}-\rho_{2}+r_{2} \lambda_{1} b_{2}\right)-r_{2} \sum_{m=0}^{M-1}\left[H_{2, m}^{\prime}(1)+\right. \\
& \left.\left(H_{2, m}(1)-H_{2, m}(0)\right)(m-M) \frac{\lambda_{1} b_{1}}{1-\rho_{2}}\right],
\end{aligned}
$$

$$
\begin{aligned}
\beta_{2}^{\prime}(1)= & \frac{2\left(r_{1} \lambda_{1}\left(b_{1}+s_{1}+s_{2}\right)+r_{2} \lambda_{1} b_{1}\right)\left(1-\rho_{2}\right)+r_{1} \lambda_{1}^{2}\left(b_{1}+s\right)^{(2)}}{2\left(1-\rho_{1}-\rho_{2}-\lambda_{1} s_{1}\right)\left(1-\rho_{2}\right)} \\
& +\frac{\lambda_{1}^{2} b_{2}^{(2)}\left(r_{2}+r_{1} \lambda_{2}\left(b_{1}+s_{1}+s_{2}\right)\right.}{2\left(1-\rho_{1}-\rho_{2}-\lambda_{1} s_{1}\right)\left(1-\rho_{2}\right)^{2}}-\left\{\frac{2 M \lambda_{1} b_{2}\left(1-\rho_{1}-\rho_{2}-\lambda_{1} s_{1}\right)-\lambda_{1}^{2}\left(b_{1}+s_{1}\right)^{(2)}}{2\left(1-\rho_{1}-\rho_{2}-\lambda_{1} s_{1}\right)^{2}\left(1-\rho_{2}\right)}\right. \\
& \left.+\frac{\lambda_{2}\left(b_{1}+s_{1}\right) \lambda_{1}^{2} b_{2}^{(2)}}{2\left(1-\rho_{1}-\rho_{2}-\lambda_{1} s_{1}\right)^{2}\left(1-\rho_{2}\right)^{2}}\right\}\left(r_{1} \lambda_{1}\left(b_{1}+s_{1}+s_{2}\right)+r_{2} \lambda_{1} b_{2}\right)-r_{2} \sum_{m=0}^{M-1}\left[G_{2, m}^{\prime}(1)+\right. \\
& \left.\left(G_{2, m}(1)-G_{2, m}(0)\right)(m-M) \frac{\lambda_{1} b_{1}}{1-\rho_{2}}\right] .
\end{aligned}
$$

Finally, differentiating (3.33) in $z$ and then letting $z \rightarrow 1$, we can get $d /\left.d z \varphi(z)\right|_{z=1}$ as follows

$$
\begin{aligned}
& \left.\frac{d}{d z} \boldsymbol{\varphi}(z)\right|_{z=1}=\mathcal{N}^{-1}(z)\left\{\psi _ { M } ( 0 ) \left[\frac{d}{d z} \mathbf{H}_{1}(z)\left(\mathbf{H}_{2}(z)+\hat{\mathbf{G}}_{2}(z) \mathbf{G}^{-1}(0) \mathbf{H}(0) \mathbf{J}\right)+\mathbf{H}_{\mathbf{1}}(z)\left(\frac{d}{d z} \mathbf{H}_{2}(z)\right.\right.\right. \\
& \left.\left.+\frac{d}{d z} \hat{\mathbf{G}}_{2}(z) \mathbf{G}^{-1}(0) \mathbf{H}(0) \mathbf{J}\right)+\frac{d}{d z} \mathbf{G}_{\mathbf{1}}(z)+\frac{d}{d z} \mathbf{L}_{1}(z)\right] \boldsymbol{\xi}+\frac{d}{d z} \mathbf{H}_{1}(z)\left(\hat{G}_{2}(z) \mathbf{G}^{-1}(0) \mathbf{v}+\right. \\
& \left.\left.\mathbf{v}_{2}(z)\right)+\mathbf{H}_{1}(z)\left(\frac{d}{d z} \hat{G}_{2}(z) \mathbf{G}^{-1}(0) \mathbf{v}+\frac{d}{d z} \mathbf{v}_{2}(z)\right)+a \frac{d}{d z} \mathbf{v}_{\mathbf{1}}(z)-\left(\frac{d}{d z} \mathcal{N}(z)\right) \boldsymbol{\varphi}(z)\right\}\left.\right|_{z=1} .
\end{aligned}
$$

## 5. Numerical Results

In this section we present some numerical examples for varying values of the threshold $M$ and $N$ to illustrate their effects on the mean waiting times and the tail distributions of the queue lengths. In these numerical examples we assume that service and switching times of both queues are exponentially distributed. First we consider a symmetry case. The parameter values are $\lambda_{1}=\lambda_{2}=1, b_{1}=b_{2}=1 / 3$ and $s_{1}=s_{2}=0$ (the case that switching times in both queues are zero). The mean waiting times $E\left[W_{1}\right]$ and $E\left[W_{2}\right]$ are given in Figures 1 (a) and $1(\mathrm{~b})$, respectively. The results show that $E\left[W_{1}\right]$ is monotonically decreasing, and $E\left[W_{2}\right]$ is monotonically increasing in $N$ and $M$. But for sufficient large values of $N$ and $M$, the variation of these values becomes small because the utilizations $\rho_{1}=\rho_{2}=1 / 3$ in this case are relatively small, and the queue $Q_{1}$ is essentially behaving as one served with exhaustive service schedule for sufficient large values of $N$ and $M$. Therefore, small values of the threshold $N$ and $M$ should be considered when the utilizations are relatively small.


Figure 1(a)


Figure 1(b)

In the second case the parameter values are $\lambda_{1}=\lambda_{2}=1, b_{1}=1 / 4, b_{2}=1 / 2$ and $s_{1}=s_{2}=1 / 10$. The mean waiting times $E\left[W_{1}\right]$ and $E\left[W_{2}\right]$ are given in Figure 2. There are a relatively small utilization $\rho_{1}=1 / 4$ in $Q_{1}$ and a relatively large utilization $\rho_{2}=1 / 2$ in $Q_{2}$.

The result indicates that the effect of $N$ and $M$ on $E\left[W_{2}\right]$ is greater than that on $E\left[W_{1}\right]$. In this case, therefore, the small values of $N$ and $M$ should be considered. The results of case 3 are shown in Figure 3. The parameter values are $\lambda_{1}=\lambda_{2}=1, b_{1}=1 / 2, b_{2}=1 / 4$ and $s_{1}=s_{2}=1 / 10$. Contrary to the case 2 , adjusting the values of $N$ and $M$ gives a greater effect on $E\left[W_{1}\right]$, but not on $E\left[W_{2}\right]$. So the large values of values of $N$ and $M$ should be considered in this case.


Figure 2


Figure 3

In Figure 4, we show the tail distributions of the queue-lengths of both queues. The parameter values are set to be the same as those in the case 3 . Here $N=6, M=0$ and $M=3$ respectively. When the value of $M$ becomes larger, the tail distribution of the queuelength in $Q_{1}$ decreases, and the tail distribution of the queue-length in $Q_{2}$ increases. As one can expect, the above results show that the mean waiting time and the tail distribution of the queue-length in $Q_{1}$ are decreasing, and those in $Q_{2}$ are increasing in the values of $N$ and $M$. This is because that for the large value of $N$ and $M$, the queue $Q_{1}$ gets a higher priority over the queue $Q_{2}$.


Figure 4

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