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APPROXIMATIONS FOR MARKOV CHAINS WITH UPPER HESSENBERG TRANSITION MATRICES

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Abstract We present an approximation for the stationary distribution $\boldsymbol{\pi}$ of a countably infinite-state Markov chain with transition probability matrix $P = (p_{ij})$ of upper Hessenberg form. Our approximation makes use of an associated upper Hessenberg matrix which is spatially homogeneous one $P^{(N)}$ except for a finite number of rows obtained by letting $p_{ij} = p_{j-i+1}$, $i \geq N+1$, for some distribution $\boldsymbol{p} = \{p_j\}$ with mean $\rho < 1$, where $p_{-j} = 0$ for $j \geq 1$. We prove that there exists an optimal ρ, say $\rho^*(N)$ with which our method provides exact probabilities up to the level N. However, in general to find this optimal $\rho^*(N)$ is practically impossible unless one has the exact distribution $\boldsymbol{\pi}$. Therefore, we propose a number of approximations to $\rho^*(N)$ and prove that a better approximation than that given by finite truncation methods can be obtained in the sense of smaller l_1-distance between exact distribution of its approximation. Numerical experiments are implemented for the $M/M/1$ retrial queue.

1. Introduction

This paper is concerned with the approximating the stationary distribution $\boldsymbol{\pi} = \{\pi_j\}$ of a countably infinite-state Markov chain with transition probability matrix P of upper Hessenberg form, i.e., $P = (p_{ij})$ with $p_{ij} = 0$ for $i > j+1$, which are frequently encountered in variety of application areas, especially in queueing models. When the structure of P is propitious, the stationary distribution can be determined analytically. However, as far as we know, there are no explicit solutions provided in the literature to the stationary distributions of the Markov chains with upper Hessenberg transition probability matrices. Instead of determining the stationary distribution from the infinite matrix, one usually reduces the dimensions of the matrix and make it finite. To do this, several authors have presented augmented truncation methods: one truncates the chain to the first N states, makes the resulting matrix stochastic and irreducible in some convenient way, and then solves the finite system (eg. see Gibson and Seneta[3], Wolf[9], Heyman[5], Tweedie[8] and the references therein). Zhao and Liu[11] showed that the censored Markov chain provides the best approximation among finite truncation methods in the sense of minimal l_1-sum of errors between the exact distribution and approximation. However, the truncation method uses a Markov chain with a finite state space. So, if the stationary distribution of the infinite–state Markov chain has a long tail, and averages and variances are heavily affected by truncation, the truncation level may have to be very large to get a good approximation.

In this paper we present an approximation method which utilizes an appropriate Markov chain with infinite state space and prove that our method provides the better results than those of censored chain. Our approximation makes use of an associated upper Hessenberg matrix which is spatially homogeneous one $P^{(N)}$ except for a finite number of rows obtained by letting $p_{ij} = p_{j-i+1}$, $i \geq N+1$, for some distribution $\boldsymbol{p} = \{p_j\}$ with mean $\rho < 1$, where

$p_{-j} = 0$ for $j \geq 1$. We prove that there exists an optimal ρ, say $\rho^*(N)$ with which our method provides exact probabilities up to the level N. However, in general to find this optimal $\rho^*(N)$ is practically impossible unless one has the exact distribution $\boldsymbol{\pi}$. Therefore, we propose a number of approximations to $\rho^*(N)$ and prove that a better approximation than that given by finite truncation methods can be obtained in the sense of smaller l_1-distance between exact distribution of its approximation.

The rest of this paper is organized as follows. In Section 2, we give our basic assumptions for ergodicity of P and review the basic results of the censored Markov chain. In Section 3, we present our approximation. In Section 4, we propose some approximations for the optimal $\rho^*(N)$. Numerical examples are presented in Section 5.

2. The Censored Markov Chain

Consider a discrete time Markov chain with state space $S = \{0, 1, 2, \cdots\}$ and transition probability matrix $P = (p_{ij})$ of upper Hessenberg form *i.e.*, $p_{ij} = 0$ whenever $i > j + 1$. Let

$$\rho_i = \begin{cases} \sum_{j=0}^{\infty} j p_{0j}, & \text{if } i = 0 \\ \sum_{j=0}^{\infty} j p_{i,j+i-1}, & \text{if } i \geq 1. \end{cases} \tag{2.1}$$

Since we are only interested in the stationary distribution of the Markov chain, we will not distinguish between the Markov chain itself and its transition probability matrix. Throughout this paper we assume that the matrix P is irreducible. It follows from Crabill[1] that a sufficient condition for the Markov chain P to be positive recurrent is

$$\limsup_{i \to \infty} \rho_i < 1 \tag{2.2}$$

and $\rho_i < \infty$, $i \geq 0$. We also assume the ergodicity condition described above and let $\pi = \{\pi_i\}_{i=0}^{\infty}$ be the stationary distribution of P.

The censored Markov chain of P with censoring set $E \subset S$ is defined as the stochastic process which records transitions of P during visits E. In other words, the sample paths of the censored chain are obtained from the sample paths of P by omitting all those portions which are in the complement of E. It is well-known that the stationary distribution of censored chain is proportional to that of the original chain and the proportional constant is the inverse of the truncated sum in the censoring set of the stationary distribution of the original Markov chain (eg. see Zhao and Liu[11]). The following lemma is a mathematical representation of this statement for a special censoring set.

Lemma 2.1 *The censored Markov chain obtained by censoring the Markov chain P on the states $\{j,\ 0 \leq j \leq N\}$ has the transition probability matrix*

$$Q^{(N)} = \begin{pmatrix} p_{0,0} & p_{0,1} & \cdots & p_{0,N-1} & \bar{p}_{0,N} \\ p_{1,0} & p_{1,1} & \cdots & p_{1,N-1} & \bar{p}_{1,N} \\ 0 & p_{2,1} & \cdots & p_{2,N-1} & \bar{p}_{2,N} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & p_{N,N-1} & \bar{p}_{N,N} \end{pmatrix},$$

where $\bar{p}_{ik} = \sum_{j=k}^{\infty} p_{ij}$. The stationary distribution $\boldsymbol{\nu}_N = \{\nu_j^{(N)}\}_{j=0}^{N}$ of $Q^{(N)}$ is given by

$$\nu_j^{(N)} = \frac{\pi_j}{\beta(N)},\ j = 0, 1, \cdots, N, \tag{2.3}$$

where $\beta(N) = \sum_{i=0}^{N} \pi_i$ is the truncated sum of $\boldsymbol{\pi}$ up to N.

For the l_1-sum of errors between the exact distribution and its approximation, we define the l_1-distance $||\boldsymbol{u}-\boldsymbol{v}||_1$ and the truncated l_1-distance $||\boldsymbol{u}-\boldsymbol{v}||_1^{(N)}$ between two infinite vectors $\boldsymbol{u} = (u_0, u_1, \cdots)$ and $\boldsymbol{v} = (v_0, v_1, \cdots)$ by

$$||\boldsymbol{u}-\boldsymbol{v}||_1 = \sum_{i=0}^{\infty} |u_i - v_i|, \quad ||\boldsymbol{u}-\boldsymbol{v}||_1^{(N)} = \sum_{i=0}^{N} |u_i - v_i|.$$

The next lemma due to Zhao and Liu[11] shows that the censoring method provides the minimal l_1-sum of errors among the augmented truncation methods for the Markov chain with upper Hessenberg transition matrix.

Lemma 2.2 *Let $\{\tilde{\nu}_j^{(N)}, j = 0, 1, \cdots, N\}$ be a stationary distribution of the Markov chain obtained by the augmented truncation method from P with truncation level N. Then the following holds:*

$$||\boldsymbol{\pi} - \boldsymbol{\nu}^{(N)}||_1 = 2(1 - \beta(N)) \leq ||\boldsymbol{\pi} - \tilde{\boldsymbol{\nu}}^{(N)}||_1, \tag{2.4}$$

where $\boldsymbol{\nu}^{(N)} = (\nu_0^{(N)}, \nu_1^{(N)}, \cdots, \nu_N^{(N)}, 0, 0, \cdots)$, and similarly for $\tilde{\boldsymbol{\nu}}^{(N)}$.

3. Approximations

The first step of our method is to modify the spatially inhomogeneous matrix P to be homogeneous one $P^{(N)} = (p_{ij}^{(N)})$ except for the first $N+1$ rows with

$$p_{ij}^{(N)} = \begin{cases} p_{ij}, & \text{if } 0 \leq i \leq N \text{ or } i > j+1 \\ p_{j-i+1}, & \text{if } j \geq i-1 \text{ and } i \geq N+1 \end{cases} \tag{3.1}$$

for some a probability distribution $\boldsymbol{p} = \{p_j\}_{j=0}^{\infty}$ satisfying $p_0 > 0$ and $\rho = \sum_{k=0}^{\infty} k p_k < 1$. Since P is ergodic, $P^{(N)}$ is ergodic. Let $\boldsymbol{\pi}^{(N)} = \{\pi_i^{(N)}\}$ be the stationary distribution of $P^{(N)}$.

In the following, we represent $\boldsymbol{\pi}^{(N)}$ in terms of $\boldsymbol{\pi}$, and discuss the l_1-distance between $\boldsymbol{\pi}^{(N)}$ and $\boldsymbol{\pi}$. Let $p(z)$ and $\Pi^{(N)}(z)$ be the generating functions of $\boldsymbol{p}$ and $\boldsymbol{\pi}^{(N)}$, that is,

$$p(z) = \sum_{j=0}^{\infty} p_j z^j, \quad \Pi^{(N)}(z) = \sum_{j=0}^{\infty} \pi_j^{(N)} z^j$$

and define

$$p_i(z) = \begin{cases} \sum_{j=0}^{\infty} p_{0j} z^j, & \text{for } i = 0 \\ \sum_{j=0}^{\infty} p_{i,j+i-1} z^j, & \text{for } i \geq 1. \end{cases}$$

Then it is easily seen that $p_i'(1) = \rho_i$, $i \geq 0$ and $p'(1) = \rho$. We have from the definition of $P^{(N)}$ and the equation $\boldsymbol{\pi}^{(N)} P^{(N)} = \boldsymbol{\pi}^{(N)}$ that the following relation

$$\pi_j^{(N)} = \begin{cases} \sum_{i=0}^{j+1} \pi_i^{(N)} p_{ij}, & \text{if } 0 \leq j \leq N-1 \\ \sum_{i=0}^{N} \pi_i^{(N)} p_{ij} + \sum_{i=N+1}^{j+1} \pi_i^{(N)} p_{j+1-i}, & \text{if } j \geq N. \end{cases} \tag{3.2}$$

Routine calculation yields the generating function

$$\Pi^{(N)}(z) = \frac{1}{z - p(z)} \left(\pi_0^{(N)} (z-1) p_0(z) + \sum_{k=0}^{N} \pi_k^{(N)} z^k (p_k(z) - p(z)) \right). \tag{3.3}$$

Proposition 3.1 *The stationary distribution $\boldsymbol{\pi}^{(N)}$ of $P^{(N)}$ is given by*

$$\pi_j^{(N)} = C^{(N)}\nu_j^{(N)} = \frac{C^{(N)}}{\beta(N)}\pi_j,\ 0 \le j \le N, \tag{3.4}$$

where

$$C^{(N)} = \frac{1-\rho}{\kappa(N)-\rho}, \quad \kappa(N) = \frac{\delta(N)}{\beta(N)} = \nu_0^{(N)} + \sum_{j=0}^{N} \nu_j^{(N)}\rho_j$$

and

$$\delta(N) = \pi_0 + \sum_{j=0}^{N} \pi_j\rho_j.$$

Proof. Since $p_{ij}^{(N)} = Q_{ij}^{(N)}$, $0 \le i \le N$, $0 \le j \le N-1$, where $Q_{ij}^{(N)}$ is the (i,j)-entry of the matrix $Q^{(N)}$, the vector $(\pi_0^{(N)}, \cdots, \pi_N^{(N)})$ is a left invariant vector of the augmented matrix $Q^{(N)}$ and hence we have

$$\pi_j^{(N)} = C^{(N)}\nu_j^{(N)},\ 0 \le i \le N, \tag{3.5}$$

where $C^{(N)}$ is a constant. The constant $C^{(N)}$ is obtained by normalizing condition

$$1 = \Pi(1-) = C^{(N)}\frac{1-\rho}{\kappa(N)-\rho},$$

where we used (3.5) and (2.3). □

In order to stress the dependence on N and ρ, we write $\pi_j^{(N)}(\rho)$ and $C^{(N)}(\rho)$ instead of $\pi_j^{(N)}$ and $C^{(N)}$, respectively.

Proposition 3.2 *For each $0 \le \rho < 1$, $\lim_{N\to\infty} C^{(N)}(\rho) = 1$ and hence*

$$\lim_{N\to\infty} \pi_j^{(N)} = \pi_j,\ j = 0, 1, 2, \cdots.$$

Proof. It is clear from the definition of $\beta(N)$ that $\lim_{N\to\infty}\beta(N) = 1$. Thus it suffices to show that $\lim_{N\to\infty}\delta(N) = 1$. It is easily seen that the stationary distribution $\boldsymbol{\pi}$ of P satisfies

$$\pi_j = \frac{1}{p_{j,j-1}}\sum_{i=0}^{j-1}\pi_i\bar{p}_{ij},\ j = 1, 2, \cdots. \tag{3.6}$$

Multiplying both sides of (3.6) by $p_{j,j-1}$ and summing over j yield

$$\sum_{j=1}^{\infty}\pi_j p_{j,j-1} = \pi_0\sum_{j=1}^{\infty}\bar{p}_{0j} + \sum_{j=1}^{\infty}\pi_j\sum_{k=j}^{\infty}\bar{p}_{jk} - \sum_{j=1}^{\infty}\pi_j\bar{p}_{jj}. \tag{3.7}$$

By noting from the definition of ρ_i in (2.1) that

$$\rho_i = \begin{cases} \sum_{j=1}^{\infty}\bar{p}_{0j}, & \text{if } i = 0 \\ \sum_{j=0}^{\infty}\bar{p}_{i,j+i}, & \text{if } i \ge 1 \end{cases}$$

and $p_{j,j-1} + \bar{p}_{jj} = 1$, (3.7) becomes

$$\pi_0 + \sum_{i=0}^{\infty} \pi_i \rho_i = 1$$

which shows that $\lim_{N\to\infty} \delta(N) = 1$. ■

Proposition 3.2 also shows that the convergence of $\boldsymbol{\pi}^{(N)}$ to $\boldsymbol{\pi}$ does not depend on the choice of the probability distribution $\boldsymbol{p}$.

In the next proposition, we show that there exists an optimal ρ in the sense that the truncated l_1-distance between $\boldsymbol{\pi}$ and $\boldsymbol{\pi}^{(N)}$ is 0.

Proposition 3.3 *There is a $\rho^*(N)$ satisfying $\pi_j^{(N)}(\rho^*(N)) = \pi_j$, $j = 0, 1, 2, \cdots, N$, and it is given by*

$$\rho^*(N) = \frac{1-\delta(N)}{1-\beta(N)} = \frac{\sum_{i=N+1}^{\infty} \pi_i \rho_i}{\sum_{i=N+1}^{\infty} \pi_i}. \tag{3.8}$$

Proof. Since $C^{(N)}(\rho) = \sum_{j=0}^{N} \pi_j^{(N)}(\rho) < 1$, we have $\kappa(N) > 1$. By differentiating $C^{(N)}(\rho)$ with respect to ρ, it is easily seen that $C^{(N)}(\rho)$ is a strictly monotone decreasing function of ρ, $0 \le \rho \le 1$, for each fixed N. Since

$$\pi_j^{(N)}(1) = 0 < \pi_j < \frac{1}{\delta(N)}\pi_j = \pi_j^{(N)}(0),\ 0 \le j \le N,$$

there exists a unique $\rho^*(N)$ satisfying $\pi_j^{(N)}(\rho^*(N)) = \pi_j$, $0 \le j \le N$. The formula (3.8) is obtained by solving $C^{(N)}(\rho) = \beta(N)$ for ρ. □

Corollary 3.4 *If ρ satisfies*

$$\rho^*(N) < \rho < \frac{1}{2}(\kappa(N) + \rho^*(N)), \tag{3.9}$$

then

$$||\boldsymbol{\pi} - \boldsymbol{\pi}^{(N)}(\rho)||_1^{(N)} < ||\boldsymbol{\pi} - \boldsymbol{\nu}^{(N)}||_1^{(N)}. \tag{3.10}$$

Proof. Since $C^{(N)}(\rho)$ is a decreasing function of ρ, $C^{(N)}(\rho) < \beta(N)$ and hence we have $||\boldsymbol{\pi} - \boldsymbol{\pi}^{(N)}(\rho)||_1^{(N)} = \beta(N) - C^{(N)}(\rho)$. We note from $||\boldsymbol{\pi} - \boldsymbol{\nu}^{(N)}||_1^{(N)} = 1 - \beta(N)$ that (3.9) is equivalent to $2\beta(N) < 1 + C^{(N)}(\rho)$. Thus the corollary is proved by the fact that

$$1 + C^{(N)}\left(\frac{\kappa(N) + \rho^*(N)}{2}\right) = 2C^{(N)}(\rho^*(N)) = 2\beta(N)$$

and $C^{(N)}(\rho)$ is a decreasing function of ρ. □

Proposition 3.5 *We have, for each $0 \le \rho < 1$, that*

$$||\boldsymbol{\pi} - \boldsymbol{\pi}^{(N)}(\rho)||_1 = 2\left(1 - \sum_{i=0}^{\infty} \min(\pi_i^{(N)}(\rho), \pi_i)\right). \tag{3.11}$$

Proof.

$$\begin{aligned}
\|\boldsymbol{\pi} - \boldsymbol{\pi}^{(N)}(\rho)\|_1 &= \sum_{\substack{i=0 \\ \pi_i^{(N)}(\rho) > \pi_i}}^{\infty} (\pi_i^{(N)}(\rho) - \pi_i) + \sum_{\substack{i=0 \\ \pi_i^{(N)}(\rho) < \pi_i}}^{\infty} (\pi_i - \pi_i^{(N)}(\rho)) \\
&= 2\left(1 - \sum_{\substack{i=0 \\ \pi_i^{(N)}(\rho) > \pi_i}}^{\infty} \pi_i - \sum_{\substack{i=0 \\ \pi_i^{(N)}(\rho) < \pi_i}}^{\infty} \pi_i^{(N)}(\rho) - \sum_{\substack{i=0 \\ \pi_i^{(N)}(\rho) = \pi_i}}^{\infty} \pi_i\right) \\
&= 2\left(1 - \sum_{i=0}^{\infty} \min(\pi_i^{(N)}(\rho), \pi_i)\right).
\end{aligned}$$

□

Corollary 3.6 *(i) If $C^{(N)}(\rho) \geq \beta(N)$, that is, $\rho < \rho^*(N)$, then the following hold:*

$$\|\boldsymbol{\pi} - \boldsymbol{\pi}^{(N)}(\rho)\|_1 < \|\boldsymbol{\pi} - \boldsymbol{\nu}^{(N)}\|_1, \quad \|\boldsymbol{\pi} - \boldsymbol{\pi}^{(N)}(\rho)\|_1^{(N)} < \|\boldsymbol{\pi} - \boldsymbol{\nu}^{(N)}\|_1^{(N)}.$$

(ii) When $C^{(N)}(\rho) < \beta(N)$, that is, $\rho > \rho^*(N)$, a necessary and sufficient condition for

$$\|\boldsymbol{\pi} - \boldsymbol{\pi}^{(N)}(\rho)\|_1 < \|\boldsymbol{\pi} - \boldsymbol{\nu}^{(N)}\|_1$$

is that ρ satisfies

$$C^{(N)}(\rho) + \sum_{i=N+1}^{\infty} \min(\pi_i^{(N)}(\rho), \pi_i) > \beta(N).$$

Before closing this section, we note some useful results relating to the moments and the recursive formula for the probabilities $\pi_j^{(N)}$, $j \geq N+1$.

Once the probability distribution $\boldsymbol{p}$ is given, one can easily calculate the approximation of moments by differentiating the generating function $\Pi^{(N)}(z)$ at $z = 1$. We provide formulas for mean $L^{(N)}$ and variance $V^{(N)}$ of $\boldsymbol{\pi}^{(N)}$ as follows:

$$L^{(N)} = \frac{1}{2(1-\rho)}\left(2\pi_0^{(N)}\rho_0 + \sum_{k=0}^{N} \pi_k^{(N)}\left[2k(\rho_k - \rho) + p_k''(1) - p''(1)\right] + p''(1)\right),$$

$$V^{(N)} = L_2^{(N)} - [L^{(N)}]^2,$$

where

$$L_2^{(N)} = \frac{1}{3(1-\rho)}(3L^{(N)}p''(1) + p^{(3)}(1) + \mathbf{I}) + L^{(N)},$$

$$\begin{aligned}
\mathbf{I} = \sum_{k=0}^{N} \pi_k^{(N)}&\left[3k(k-1)(\rho_k - \rho) + 3k(p_k''(1) - p''(1)) + (p_k^{(3)}(1) - p^{(3)}(1))\right] \\
&+3\pi_0^{(N)}p_0''(1).
\end{aligned}$$

For the calculation of $\boldsymbol{\pi}^{(N)}$, one has to solve the linear equation $\boldsymbol{\nu}_N Q^{(N)} = \boldsymbol{\nu}_N$ with $\boldsymbol{\nu}_N \boldsymbol{e} = 1$. There are several ways of doing this, e.g. see Grasmann[4], Latouche et al.[6] and Zhao and Li[10]. Once $\boldsymbol{\nu}_N$ is obtained, one can calculate $\pi_j^{(N)}$, $0 \leq i \leq N$ using (3.4) in Proposition 3.1. The remaining probabilities $\pi_j^{(N)}$, $j \geq N+1$ can be calculated by the recursive formula (e.g. see Ramaswami[7])

$$\pi_j^{(N)} = \frac{1}{p_0}\left(\sum_{i=0}^{N} \pi_i^{(N)}\bar{p}_{ij} + \sum_{i=N+1}^{j-1} \pi_i^{(N)}\bar{p}_{j+1-i}\right).$$

4. Approximations for $\rho^*(N)$

The optimal $\rho^*(N)$ given in (3.8) is determined by $\{\pi_j, 0 \le j \le N\}$, which are unknown in general. We therefore suggest approximations $\hat{\rho}(N)$ of $\rho^*(N)$ with which $\boldsymbol{\pi}^{(N)}(\hat{\rho}(N))$ provides the smaller l_1-distance or truncated l_1-distance from the exact distribution than that given by the censored chain.

We introduce three approximations for $\rho^*(N)$ as follows:

$$\rho_1^*(N) = \sup_{j \ge N+1} \rho_j, \quad \rho_2^*(N) = \inf_{j \ge N+1} \rho_j, \quad \rho_3^*(N) = \frac{\rho_1^*(N) + \rho_2^*(N)}{2}.$$

Remark. If $\{\rho_n\}$ is monotone increasing (decreasing, respectively), except possibly for finitely many terms, $\rho_1^*(N) = \lim_{n\to\infty} \rho_n$ ($\rho_2^*(N) = \lim_{n\to\infty} \rho_n$, respectively).

Now we investigate the conditions under which each of $\rho_i^*(N)$ provides better approximations than those of censored chain. It is clear from the definition of $\rho^*(N)$ in (3.8) that

$$\rho_2^*(N) \le \rho^*(N) \le \rho_1^*(N). \tag{4.1}$$

Thus we see from (i) of Corollary 3.6 that

$$||\boldsymbol{\pi} - \boldsymbol{\pi}^{(N)}(\rho_2^*(N))||_1 < ||\boldsymbol{\pi} - \boldsymbol{\nu}^{(N)}||_1$$

and

$$||\boldsymbol{\pi} - \boldsymbol{\pi}^{(N)}(\rho_2^*(N))||_1^{(N)} < ||\boldsymbol{\pi} - \boldsymbol{\nu}^{(N)}||_1^{(N)}.$$

Let

$$\rho^+ = \limsup_{n\to\infty} \rho_n = \lim_{n\to\infty} \rho_1^*(n), \quad \rho^- = \liminf_{n\to\infty} \rho_n = \lim_{n\to\infty} \rho_2^*(n).$$

If $2\rho^+ \le (1 + \rho^-)$, then $\rho_1^*(N)$ satisfies the condition (3.9) for large N and hence we have from Corollary 3.4 that

$$||\boldsymbol{\pi} - \boldsymbol{\pi}^{(N)}(\rho_2^*(N))||_1^{(N)} < ||\boldsymbol{\pi} - \boldsymbol{\nu}^{(N)}||_1^{(N)}.$$

It is immediate from (4.1) that either $\rho_3^*(N) < \rho^*(N)$ or $\rho_3^*(N)$ satisfies the condition (3.9) for N with $\rho_1^*(N) < 1$.

Now we consider the way of choosing a probability distribution $\boldsymbol{p}_i^*(N)$ with mean $\rho_i^*(N)$, $i = 1, 2, 3$. If one can easily find $N_1 \ge N$ such that $\rho_1^*(N) = \rho_{N_1}$, then one takes $\boldsymbol{p}_1^*(N) = (p_{N_1,N_1-1}, p_{N_1,N_1}, p_{N_1,N_1+1}, \cdots)$, the nonzero part of the N_1th row of P. If it is difficult to find such N_1, then one can take $(\boldsymbol{p}_M, \rho_M)$ as $(\boldsymbol{p}_1^*(N), \rho_1^*(N))$, where $M = M(\epsilon)$ is an integer satisfying $|\rho_M - \rho_1^*(N)| < \epsilon$ with $M \ge N$ for a given $\epsilon > 0$. We can choose $\boldsymbol{p}_i^*(N)$, $i = 2, 3$ similarly.

Remark. Based on some numerical experiments, we suggest the following criteria for $(\boldsymbol{p}(N), \rho(N))$: If $\{\rho_n\}$ is monotone, use $(\boldsymbol{p}_{N+1}, \rho_{N+1})$. If $\{\rho_n\}$ is not monotone, use $(\boldsymbol{p}_3^*(N), \rho_3^*(N))$.

5. Numerical Example

Here we consider the $M/M/1$ retrial queue whose behaviors are as follows. Customers arrive according to a Poisson process with rate λ and service times which are exponentially distributed with mean $\frac{1}{\mu}$. Each arriving customer checks the state of the server. If the server is idle, the customer seizes the channel and his service begins. If the server is busy, the customer joins the retrial group and starts generating requests for service according to a Poisson process of rate θ until he finds a free server. For comprehensive surveys of retrial queues, see Falin and Templeton[2]. Let X_n be the number of customers in the system at the instant of nth service completion, $n = 1, 2, \cdots$. Then the Markov chain $\{X_n\}$ is ergodic if $\rho = \dfrac{\lambda}{\mu} < 1$, and the transition probabilities $p_{ij} = P(X_{n+1} = j | X_n = i)$ of $\{X_n\}$ are given by

$$p_{ij} = \begin{cases} 0, & i > j+1 \\ pq^j, & i = 0,\ j \geq 0 \\ \frac{i\theta}{\lambda + i\theta} p, & i \geq 1,\ j = i-1 \\ \frac{\lambda}{\lambda+i\theta} pq^{j-i} + \frac{i\theta}{\lambda+i\theta} pq^{j-i+1}, & i \geq 1,\ j \geq i, \end{cases}$$

where $p = \dfrac{1}{1+\rho}$, $q = 1-p$. The ρ_i, $i \geq 0$ are given by

$$\rho_i = \begin{cases} \rho, & i = 0 \\ \rho + \frac{\lambda}{\lambda+i\theta}, & i \geq 1. \end{cases}$$

The generating function $\Pi(z)$ of the stationary distribution $\boldsymbol{\pi}$ of P is given by

$$\Pi(z) = \left(\frac{1-\rho}{1-\rho z} \right)^{c+1},$$

where $c = \frac{\lambda}{\theta}$. By taking the Taylor series expansion of $\Pi(z)$, we have that

$$\pi_k = (1-\rho)^{c+1} \binom{-c-1}{k} (-1)^k \rho^k,\ k \geq 0.$$

Hence we get the recursive formula

$$\pi_{j+1} = \rho \left(1 + \frac{c}{j+1} \right) \pi_j,\ j \geq 0$$

with $\pi_0 = (1-\rho)^{c+1}$.

In Table 1, we list the l_1-distances $\mathrm{Er}(\boldsymbol{\nu}^{(n)}) = ||\boldsymbol{\pi} - \boldsymbol{\nu}^{(n)}||_1$ and $\mathrm{Er}(\boldsymbol{\pi}^{(n)}) = ||\boldsymbol{\pi} - \boldsymbol{\pi}^{(n)}||_1$ for various truncation levels n and system parameters ρ and θ with $\lambda = 1.0$. Since $\{\rho_n, n \geq 1\}$ is monotone decreasing, we use $\boldsymbol{\pi}^{(n)}(\rho_1^*(n))$ with $\boldsymbol{p} = (p_{n+1,n}, p_{n+1,n+1}, p_{n+1,n+2}, \cdots)$ to approximate $\boldsymbol{\pi}$ for each level n. From Table 1 we see that as ρ increases and θ decreases, that is, the tail of $\boldsymbol{\pi}$ becomes heavier, our approximation becomes more effective than the truncation method. Table 1 also shows that $\boldsymbol{\pi}^{(n)}$ converges more rapidly to $\boldsymbol{\pi}$ than $\boldsymbol{\nu}^{(n)}$ when θ increases. A reason why this happens is the differences $|p_{ij} - p_{n+1,j}|$, $i \geq n+1$, becomes smaller as θ increases.

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Table 1. l_1-distances $\mathrm{Er}(\boldsymbol{\nu}^{(n)}) = ||\boldsymbol{\pi} - \boldsymbol{\nu}^{(n)}||_1$ and $\mathrm{Er}(\boldsymbol{\pi}^{(n)}) = ||\boldsymbol{\pi} - \boldsymbol{\pi}^{(n)}||_1$ in $M/M/1$ retrial queue with $\lambda = 1.0$

	$\theta = 0.1$			$\theta = 1.0$			$\theta = 10.0$		
ρ	n	$\mathrm{Er}(\boldsymbol{\nu}^{(n)})$	$\mathrm{Er}(\boldsymbol{\pi}^{(n)})$	n	$\mathrm{Er}(\boldsymbol{\nu}^{(n)})$	$\mathrm{Er}(\boldsymbol{\pi}^{(n)})$	n	$\mathrm{Er}(\boldsymbol{\nu}^{(n)})$	$\mathrm{Er}(\boldsymbol{\pi}^{(n)})$
	20	$7.1 \cdot 10^{-2}$	$7.9 \cdot 10^{-3}$	5	$1.3 \cdot 10^{-1}$	$6.0 \cdot 10^{-3}$	5	$3.8 \cdot 10^{-2}$	$1.5 \cdot 10^{-4}$
0.5	30	$1.5 \cdot 10^{-3}$	$5.5 \cdot 10^{-5}$	10	$6.4 \cdot 10^{-3}$	$9.8 \cdot 10^{-5}$	10	$1.2 \cdot 10^{-3}$	$1.7 \cdot 10^{-6}$
	40	$1.5 \cdot 10^{-5}$	$2.7 \cdot 10^{-7}$	20	$1.1 \cdot 10^{-5}$	$4.7 \cdot 10^{-8}$	15	$4.0 \cdot 10^{-5}$	$2.7 \cdot 10^{-8}$
	30	$5.5 \cdot 10^{-1}$	$2.2 \cdot 10^{-1}$	20	$8.2 \cdot 10^{-3}$	$1.4 \cdot 10^{-4}$	10	$4.8 \cdot 10^{-2}$	$2.2 \cdot 10^{-4}$
0.7	50	$2.3 \cdot 10^{-2}$	$1.6 \cdot 10^{-3}$	30	$3.3 \cdot 10^{-4}$	$2.6 \cdot 10^{-6}$	20	$1.4 \cdot 10^{-3}$	$2.1 \cdot 10^{-6}$
	80	$3.0 \cdot 10^{-5}$	$6.1 \cdot 10^{-7}$	40	$1.2 \cdot 10^{-5}$	$5.3 \cdot 10^{-8}$	30	$4.2 \cdot 10^{-5}$	$2.9 \cdot 10^{-8}$
	150	$1.3 \cdot 10^{-1}$	$1.9 \cdot 10^{-2}$	30	$3.1 \cdot 10^{-1}$	$2.7 \cdot 10^{-2}$	10	$7.0 \cdot 10^{-1}$	$1.9 \cdot 10^{-2}$
0.9	200	$8.4 \cdot 10^{-3}$	$4.9 \cdot 10^{-4}$	50	$5.7 \cdot 10^{-2}$	$1.9 \cdot 10^{-3}$	40	$3.3 \cdot 10^{-2}$	$1.3 \cdot 10^{-4}$
	250	$3.4 \cdot 10^{-4}$	$1.0 \cdot 10^{-5}$	80	$3.6 \cdot 10^{-3}$	$4.9 \cdot 10^{-5}$	70	$1.5 \cdot 10^{-3}$	$2.1 \cdot 10^{-6}$
	300	$9.5 \cdot 10^{-6}$	$1.7 \cdot 10^{-7}$	120	$7.6 \cdot 10^{-5}$	$4.7 \cdot 10^{-7}$	100	$6.4 \cdot 10^{-5}$	$4.9 \cdot 10^{-8}$

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