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A UNIFIED MODEL AND ANALYSIS FOR AHP AND ANP

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Abstract Saaty proposes the analyzing methods for AHP using the principal eigenvector of the comparison matrix, and for ANP (analytic network process) using the limiting process method of the powers of the supermatrix. These methods are based on the irreducibility of the evaluation matrices. We develop the unified method solving both AHP and ANP based on Frobenius min-max theorem. Further this method is extended to the solving method of the general (not necessarily irreducible) evaluation matrix, by the graphic decomposition process.

1. Introduction

The eigenvector method in AHP (Analytic Hierarchy Process) is to evaluate the weights of objects (criteria and alternatives) by a comparison matrix C, and ANP(Analytic Network Process) analysis is to evaluate the weights of objects by the so called supermatrix S.

Each element of C is obtained by paired comparison of objects and Saaty [3] proposed the estimation method of weights of objects by the elements of the principal eigenvector of C. Sekitani and Yamaki [5] gave the mathematical foundation of Saaty's eigenvector method through Frobenius min-max theorem and developed the several mathematical programming models whose solutions are equivalent to the principal eigenvector of C.

In order to calculate the weights of objects in ANP, Saaty [4] uses the limiting process $\lim_{\nu\to\infty} S^{\nu}$, but it is clear that the weight vector u of objects in ANP is the solution of the equation $Su = u$ if S is a stochastic matrix (see for details in section 3.1), which is always required in ANP. So the principal eigenvalue of S is unity, and we can regard the solution u of $Su = u$ as the principal eigenvector of S.

Thus in AHP and ANP we estimate weights of objects by the principal eigenvector of C and S, respectively. In our general evaluation model we extend C and S to the total evaluation matrix A, whose principal eigenvector is also a basic tool in our analysis. In ordinary sense the off-diagonal entries of C are obtained by the direct pairwise comparisons but those of S are mainly obtained through several steps of calculation. Hence S is considered to be a higher level matrix than C (The name "supermatrix" itself reveals the fact). But we do develop the the same unifying method which is applicable to both AHP and ANP.

In general the comparison matrix C in AHP is required to be a reciprocal matrix and the supermatrix S in ANP must be a stochastic matrix. Furthermore in both cases C and S must be irreducible. Considering the arc (i,j) if and only if (i,j) element of a matrix is nonzero, we have a directed graph corresponding to the matrix. Then irreducibility of a matrix is equivalent to strong connectivity of its graph. (Saaty [4] treats the case of the general (not necessarily irreducible) S, but the results are not perfect and not acceptable).

Here we propose a new model of the general evaluation including AHP and ANP. It is free from the above mentioned various restrictions imposed on the matrices C and S. The only restriction of our total evaluation matrix A is nonnegativity, that is, all entries a_{ij} of $A = [a_{ij}]$ are nonnegative; $a_{ij} \geq 0$.

Our main analyzing method is to decompose the graph corresponding to A into some strongly connected components, (each of those is called a cluster of objects), and then to apply the principal eigenvector method to each cluster.

To unify AHP, ANP and their variations, we reconsider both the standardization of their evaluation matrices and their structures of evaluation in section 2. In section 3 the analysis of the case that structure of evaluation is strongly connected, that is, the evaluation matrix is irreducible, are stated. In section 4 the general method is stated, which is the main purpose of this research. To illustrate the effect of the general method, we show two examples in section 5.

2. Evaluation Matrices of AHP, ANP and Their Variations

We place this section for preliminary discussions for the following sections. In section 2.1 we explain the concept of the standardizing problem which plays important role in the general solving method of ANP stated in section 4. In section 2.2 a graphics representation of evaluation structures is stated. If the graph of an evaluation structure is strongly connected, then the solving method is very simple (see the details in section 3), but there are many practical problems without the strongly connected structure. These examples are shown in section 2.2.

2.1. Standardizing problem

The analysis of AHP is based on the comparison matrix $C = [c_{ij}]$, whose (i, j) element c_{ij} is considered to be the ratio evaluation of the j^{th} object to the i^{th} object. Furthermore, it can be said that the j^{th} object is evaluated by the i^{th} object based on a criterion and its result value is c_{ij}.

In AHP the reciprocal condition

$$c_{ij} = \frac{1}{c_{ji}} \tag{2.1}$$

is commonly assumed, but here we do not assume this condition. Because generally the evaluation value of the i^{th} object by j does not necessarily coincide with that of j by i. Further in conventional AHP we have $c_{ii} = 1$, but here we can assume that $c_{ii} = 0$. When all diagonal elements of the matrix C are 1, the matrix C has the same principal eigenvector as the matrix C whose diagonal elements c_{ii} are replaced with 0. As for ANP, there are many types of structures of evaluation. So here we select the simplest one like the following example in [4], and discuss this as a typical type of ANP.

Example 1 *In USA there are three big fast-food companies, McDonald's(M), Burger King(B) and Wendy's(W). Assume that they are evaluated by two criteria advertisement(A) and service(S) like Table 1, and each of M,B and W has its management policy with weights for A and S like Table 1. In a word (A,S) evaluate (M,B,W) and at the same time (M,B,W) evaluate (A,S). The evaluation matrix of (M,B,W) by (A,S) is*

$$U = \begin{bmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \\ u_{31} & u_{32} \end{bmatrix} \tag{2.2}$$

Table 1: Evaluation values of McDonald's, Burger King and Wendy's for two criteria

	A	S
M	u_{11}	u_{12}
B	u_{21}	u_{22}
W	u_{31}	u_{32}

Table 2: Management Policy for advertisement and service

	M	B	W
A	w_{11}	w_{12}	w_{13}
S	w_{21}	w_{22}	w_{23}

and that of (A,S) by (M,B,W) is

$$W = \begin{bmatrix} w_{11} & w_{12} & w_{13} \\ w_{21} & w_{22} & w_{23} \end{bmatrix}, \tag{2.3}$$

where standardizing conditions

$$\sum_{i=1}^{3} u_{ij} = 1 \quad j = 1,2, \qquad \sum_{i=1}^{2} w_{ij} = 1 \quad i = 1,2,3 \tag{2.4}$$

are assumed.

Saaty's supermatrix for this ANP is

$$S = \begin{bmatrix} 0 & W \\ U & 0 \end{bmatrix}. \tag{2.5}$$

Because of (2.4) S is a stochastic matrix. The analysis of ANP are always based on stochasticness of supermatrix. Therefore, the principal eigenvalue of S is always 1. ■

Generally the principal eigenvalue $\lambda_{\max}$ of a comparison matrix C in AHP is not 1, and let

$$\bar{C} = \frac{1}{\lambda_{\max}} C \tag{2.6}$$

be called the standardized comparison matrix. Of course $\lambda_{\max}$ is real and positive because of the irreducibility of C(see Theorem 1 of section 3 or [8] for details). From (2.6), C and $\bar{C}$ have the same principal eigenvector. Using $\bar{C}$ instead of C, we can discuss various problems in different fields by the unified measurement.

Kinoshita and Nakanishi [2] discuss the problem with the same structure as Example 1, but they select the specific alternative/criterion as dominant one and evaluate other alternatives/criteria based on the dominant alternative/criterion whose evaluation value is always 1. Thus we have the evaluation matrix like that of Table 3 or Table 4, where alternative M (criterion S) is selected as the dominant one.

For this problem

$$S = \begin{bmatrix} 0 & 0 & 1/4 & 3 & 2 \\ 0 & 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 & 0 \\ 2 & 1/2 & 0 & 0 & 0 \\ 3 & 1/6 & 0 & 0 & 0 \end{bmatrix} \tag{2.7}$$

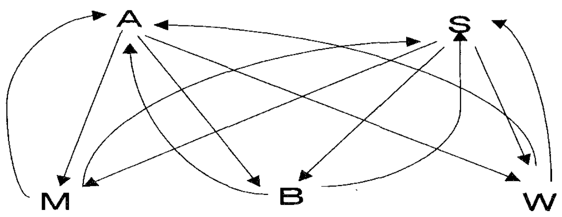

Figure 2.1: Graphic representation of supermatrix in Example 1

is no longer a stochastic matrix, and S has the principal eigenvalue $\lambda_{\max} = 3.65$ and $\bar{S} = S/3.65$ is the standardized supermatrix which is not stochastic but has the principal eigenvalue of 1.

Table 3: Evaluation values of dominant alternative

	A	S
M	1	1
B	2	1/2
W	3	1/6

Table 4: Evaluation values of dominant criteria

	M	B	W
A	1/4	3	2
S	1	1	1

2.2. The graphic structure of the evaluation problem

Consider Example 1 of ANP in section 2.1. The first column of the matrix in Table 1 can be considered to be the vector whose elements are the values of alternatives M, B and W evaluated by criterion A, and the second column values of the alternatives evaluated by criterion S. We represent this fact by the arcs from A and S to M, B and W respectively shown in Figure 2.1. As the same way, we represent Table 2 by the arc from M, B and W to A and S representing shown in Figure 2.1.

We can consider the graphic representation of the comparison matrix C of AHP, where the i^{th} column is the evaluation vector of other objects by the i^{th} object. In general comparison matrix $C = [c_{ij}]$, for any i and j $(i \neq j)$, c_{ij} and c_{ji} have positive values, so we always have arcs (i, j) and (j, i) in the graph corresponding to C, but we have no loop (i, i) because of the agreement mentioned in section 2.1.

We often encounter the case of incomplete information in AHP, where some elements of C are missing. If the (i, j) element of C is missing then we have no arc (i, j) in the corresponding graph. As the incomplete information AHP whose corresponding graph is disconnected is meaningless, we have only to consider the problem with the connected graph.

In AHP, even if it is the incomplete case, if c_{ij} is positive, then $c_{ji} = 1/c_{ij}$ is also positive, so if its graph is connected, then it is automatically strongly connected. The directed graph, where for any i and j $(i \neq j)$ there is a directed path from the i^{th} node to the j^{th} node, is called strongly connected.

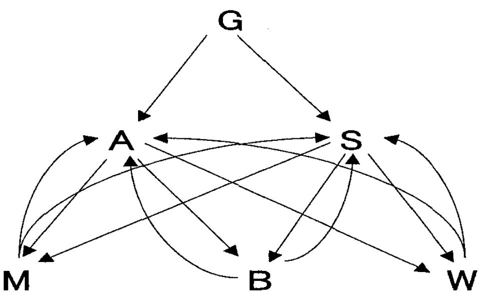

Figure 2.2: Graphic representation of Example 2 with general evaluations for criteria

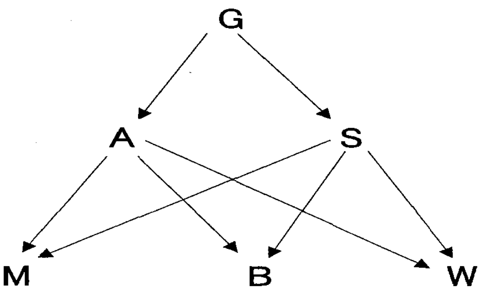

Figure 2.3: Graphic representation of AHP

But in ANP we often encounter the supermatrix whose graph is connected but not strongly connected like that of the following example.

Example 2 *Consider again Example 1 (in section 2.1). Assume that general evaluations of A and S in fast food business world are v_1 and v_2, respectively and let $v = [v_1, v_2]^T$, then the supermatrix is*

$$S = \begin{bmatrix} 0 & 0 & 0 \\ v & 0 & W \\ 0 & U & 0 \end{bmatrix} \tag{2.8}$$

and the corresponding graph is that in Figure 2.2. This is clearly connected but not strongly connected. ■

In the hierarchy structure of AHP we unify multi-evaluation of alternatives by the several criteria. Let W_i be the evaluation vector of alternatives by the i^{th} criterion. Then $W = [W_1, \ldots, W_n]$ is the evaluation matrix of alternatives by the criteria. Let v be the weight vector of the criteria, then the total evaluation matrix is

$$S = \begin{bmatrix} 0 & 0 & 0 \\ v & 0 & 0 \\ 0 & W & 0 \end{bmatrix}. \tag{2.9}$$

And its graph is like that of Figure 2.3 where the number of criteria is 2 and the number of alternatives is 3. This graph is also connected but not strongly connected.

We introduce a general evaluation problem including each of ANP and AHP as a special case. Let its evaluation matrix be

$$A = [a_{ij}] \tag{2.10}$$

of order n, where a_{ij} is a ratio evaluation of the j^{th} object by the i^{th} object for $i, j = 1, \ldots, n$. And we assume that evaluation values are always nonnegative and self-evaluations are not carried out, that is

$$a_{ij} \geq 0 \qquad \text{for all } i, j = 1, \ldots, n, \qquad a_{ii} = 0 \quad \text{for all } i = 1, \ldots, n. \tag{2.11}$$

We have the supermatrix S in ANP or the comparison matrix C in AHP as a special case of general evaluation matrix A. But we do not impose any other restrictions except (2.11) on A, such as the stochasticness or the reciprocity.

Considering an arc (i, j) if and only if $a_{ij} > 0$, we have the graph corresponding to A. The solving method of general evaluation problems varies depending on the structure of its graph.

If the graph is strongly connected, then its solving method is very simple and it is almost equivalent to the classical or conventional ANP or AHP method. However, if the graph is not strongly connected, then there has been no satisfactory solving method. The main purpose of this paper is to propose the method to solve the general problems with not necessarily strongly connected graphs.

3. Analysis for an Irreducible Evaluation Matrix

In this section we discuss the solving method with the supermatrix S or more general evaluation problem with the matrix A of (2.11), when S or A is irreducible(that is the evaluation structure is strongly connected). It is simply to find the principal eigenvector of S or A. The main purpose of section 3.1 is to show that our solution coincides with that of Saaty's limiting process. And in section 3.2 we show that the mathematical foundation of our method is based on the averaging principle of Sekitani and Yamaki [5]. This part is essentially restatement of [5] for our context.

3.1. Mathematical foundation of Saaty's ANP for the irreducible supermatrix

We illustrate that Perron-Frobenius' Theorem is mathematical foundation of Saaty's ANP for the irreducible supermatrix as well as AHP.

Theorem 1 (Perron-Frobenius' Theorem (See [8])) *Suppose that D is an irreducible nonnegative matrix. Then there are an eigenvalue λ and the corresponding eigenvector w satisfying the following two conditions:*

(1) $\lambda > 0$, $w > 0$ *and* $\lambda \geq |\alpha|$ *for every eigenvalue* α *of the matrix* D.

(2) λ *is a single root of the characteristic equation of* D.

The simple and largest eigenvalue is referred to as the principal eigenvalue. From (2) of Theorem 1, it follows directly that a principal eigenvector of the irreducible nonnegative matrix is essentially (except for a scalar multiple) unique.

Saaty [4] proposes the following solving method ANP; If the supermatrix S is (stochastic) irreducible and primitive, then $\lim_{\nu \to \infty} S^\nu$ converges to a S^∞, each column of which is the same vector w, that is ,

$$\lim_{\nu \to \infty} S^\nu = S^\infty = [w, \ldots, w] \tag{3.1}$$

and the elements of w are the desired evaluation weights of objects. For the imprimitive case, he introduces the index c of imprimitivity (or cycle index) which is defined by the greatest common divisor of lengths of all cycles of the corresponding graph for S. And he shows that if S is irreducible and it has the cycle index c, then S^c (with appropriately rearranged rows and columns) is a block diagonal matrix, each diagonal component of which is irreducible, and $\lim_{\nu \to \infty} (S^c)^\nu$ converges to a block diagonal matrix S^∞ such that each diagonal block consists of the same column vectors. Therefore, for the irreducible supermatrix S with the cycle index c,

$$S^c = \begin{bmatrix} S(1) & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & S(c) \end{bmatrix}, \tag{3.2}$$

$$\lim_{\nu\to\infty} (S^c)^\nu = \begin{bmatrix} S(1)^\infty & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & S(c)^\infty \end{bmatrix}, \tag{3.3}$$

and each column in the i^{th} diagonal block of S^∞ is the same vector $w(i)$, that is

$$S(i)^\infty = [w(i), \ldots, w(i)] \quad \text{for all } i = 1, \ldots, c. \tag{3.4}$$

The elements of $w(i)$ are desired evaluation weights of objects in the i^{th} block.

Here firstly we show that the principal eigenvector of S coincides with w in (3.1) for the case of primitivity.

Theorem 2 *Let S be a stochastic irreducible matrix. If S is primitive, then its principal eigenvector u, that is the solution u of*

$$Su = u, \tag{3.5}$$

coincides (except for a scalar multiple) with w in (3.1).

Proof: Since S is stochastic, its principal eigenvalue is 1, and the solution of (3.5) is the principal eigenvector of S. Note that S is nonnegative and irreducible, so the solution of (3.5) is unique (except for a scalar multiple) by Theorem 4. Let

$$S^\nu = [w, \ldots, w] + E_\nu \tag{3.6}$$

$$S^{\nu+1} = [w, \ldots, w] + E_{\nu+1}. \tag{3.7}$$

It follows from (3.1) that

$$\lim_{\nu\to\infty} E_\nu = \lim_{\nu\to\infty} E_{\nu+1} = 0. \tag{3.8}$$

Multiplying (3.6) by S, we have

$$S([w, \ldots, w] + E_\nu) = S^{\nu+1} = [w, \ldots, w] + E_{\nu+1},$$

so we have

$$S[w, \ldots, w] = [w, \ldots, w] \tag{3.9}$$

in the limiting case of $\nu \to \infty$. The left hand side of (3.9) can be written as $[Sw, \cdots, Sw]$, so we have

$$Sw = w. \tag{3.10}$$

But the solution of (3.5) is unique (except for a scalar multiple) so $w = u$. □

Theorem 3 *Let S and c be a stochastic irreducible matrix and its cycle index, respectively. Let u be a positive principal eigenvector of S (that is the solution of (3.5)). If all components of u are rearranged by the same way as $[w(1)^T, \ldots, w(c)^T]^T$ in (3.4) and we take an appropriate scalar multiple of each $w(i)$ for $i = 1, \ldots, c$, then, u coincides with $[w(1)^T, \ldots, w(c)^T]^T$. That is*

$$u = \begin{bmatrix} w(1) \\ \vdots \\ w(c) \end{bmatrix}. \tag{3.11}$$

Proof: Let

$$
\begin{aligned}
(S^c)^\nu &= \begin{bmatrix} S(1)^\infty & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & S(c)^\infty \end{bmatrix} + E_\nu, \\
(S^c)^{\nu+1} &= \begin{bmatrix} S(1)^\infty & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & S(c)^\infty \end{bmatrix} + E_{\nu+1}.
\end{aligned}
$$

It follows from (3.3) that

$$\lim_{\nu\to\infty} E_\nu = \lim_{\nu\to\infty} E_{\nu+1} = 0$$

and

$$S^c\left(\begin{bmatrix} S(1)^\infty & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & S(c)^\infty \end{bmatrix} + E_\nu\right) = \begin{bmatrix} S(1)^\infty & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & S(c)^\infty \end{bmatrix} + E_{\nu+1}.$$

So we have

$$\begin{bmatrix} S(1)^c & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & S(c)^c \end{bmatrix}\begin{bmatrix} S(1)^\infty & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & S(c)^\infty \end{bmatrix} = \begin{bmatrix} S(1)^\infty & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & S(c)^\infty \end{bmatrix},$$

which shows

$$
\begin{aligned}
S(i)^c S(i)^\infty &= S(i)^\infty && \text{for all } i = 1, \ldots, c, \\
S(i)^c[w(i), \ldots, w(i)] &= [w(i), \ldots, w(i)] && \text{for all } i = 1, \ldots, c, \\
S(i)^c w(i) &= w(i) && \text{for all } i = 1, \ldots, c.
\end{aligned} \tag{3.12}
$$

Since $S(i)^c$ is stochastic and irreducible, it follows from (3.12) that $w(i)$ is a positive principal eigenvector of $S(i)^c$.

On the other hand, a principal eigenvector u of S satisfies the equation $S^c u = u$. In fact, it follows from the stochastic matrix S that $S^2 u = Su = u$. Suppose that $S^k u = u$ for some $k \geq 2$, then $S^{k+1}u = S^k S u = S^k u = u$. Since all components of u are rearranged by the same way as $[w(1)^T, \ldots, w(c)^T]^T$, we have $u = [u(1)^T, \ldots, u(c)^T]^T$. Therefore,

$$S(i)^c u(i) = u(i) \quad \text{for all } i = 1, \ldots, c.$$

This implies from the stochastic irreducible matrix $S(i)^c$ that $u(i)$ is also a positive principal eigenvector of $S(i)^c$.

Since both of $u(i)$ and $w(i)$ are a positive principal eigenvector of $S(i)^c$, it follows from Theorem 1 that $u(i)$ coincides with $w(i)$(except for a scalar multiple). □

Theorem 3 shows that in order to solve ANP with an irreducible supermatrix (strongly connected structure) we have only to find the solution of (3.5). The supermatrix S is a stochastic matrix whose principal eigenvalue is 1, so the solution of (3.5) is also the principal eigenvector of S. So we can state that the solution of ANP with the strongly connected structure is the principal eigenvector of its supermatrix.

3.2. Averaging principle analysis for the irreducible evaluation matrix

As shown in section 3.1, the solving method of ANP with the irreducible evaluation matrix is to find the principal eigenvector of the supermatrix S. Hence solving methods of both AHP and ANP are commonly based on the eigenvector method. The eigenvetor method could be the key of the unified approach of AHP, ANP and their variations.

Sekitani and Yamaki [5] focus on the irreducibility and nonnegativity of the evaluation matrix in AHP and develop a principle for eigenvector method of AHP. The principle not only gives us the foundation for eigenvector method of AHP but also makes us free from the conventional restrictions such as reciprocity or stochasticness of the evaluation matrix.

Firstly we extend the principle for the general evaluation problem with the following irreducible evaluation matrix $A = [a_{ij}]$ of order n. As the same assumption as [5] we suppose that every object evaluates itself and gives itself the evaluation value. Then w_i is the self-evaluation value of the i^{th} object and $a_{ij}w_j$ is the evaluation value of the i^{th} object from viewpoint of the j^{th} object. We denote the i^{th} row vector of A by a_i and the number of positive entry in a_i by M_i for $i = 1, \ldots, n$. So the average of the external evaluation of the i^{th} object is $a_i w/M_i$. If all a_{ij}'s are evaluated or calculated under the considerable and consistent judgment of the decision maker, then there exists a self-evaluation vector $\bar{w}$ such that $a_{ij}\bar{w}_j = \bar{w}_i$ for all $i, j = 1, \ldots, n$ and hence, $\bar{w}_i = a_i\bar{w}/M_i$ for all $i = 1, \ldots, n$. However, almost all of A in practice are inconsistent and we have some gap between w_i and $a_i w/M_i$. Therefore all w_i's minimizing overall discrepancies between w_i and $a_i w/M_i$ for each $i = 1, \ldots, n$ must be desirable estimation of weights of object.

This idea is represented by the following mathematical language:

$$\min_{w>0} \max \left\{ \left| \frac{a_i w}{M_i w_i} - 1 \right| \,\middle|\, i = 1, \ldots, n \right\}. \tag{3.13}$$

The following Frobenius' theorem and corollary just meet to get a solution of Problem (3.13).

Theorem 4 (Frobenius' Theorem (See [1])) *Suppose that D is a nonnegative matrix of order n and that $\lambda_{\max}$ is the principal eigenvalue of D. Let d_i be the i^{th} row vector of D for $i = 1, \ldots, n$. Then for every positive vector u,*

$$\min \left\{ \frac{d_i u}{u_i} \,\middle|\, i = 1, \ldots, n \right\} \leq \lambda_{\max} \leq \max \left\{ \frac{d_i u}{u_i} \,\middle|\, i = 1, \ldots, n \right\}. \tag{3.14}$$

Furthermore, if the matrix D is irreducible,

$$\max_{u>0} \min \left\{ \frac{d_i u}{u_i} \,\middle|\, i = 1, \ldots, n \right\} = \lambda_{\max} = \min_{u>0} \max \left\{ \frac{d_i u}{u_i} \,\middle|\, i = 1, \ldots, n \right\}. \tag{3.15}$$

A nonzero vector u is a principal eigenvector of A if and only if

$$\lambda_{\max} = \frac{d_1 u}{u_1} = \cdots = \frac{d_n u}{u_n}. \tag{3.16}$$

Furthermore, for every positive vector u except a principal eigenvector of D

$$\min \left\{ \frac{d_i u}{u_i} \,\middle|\, i = 1, \ldots, n \right\} < \lambda_{\max} < \max \left\{ \frac{d_i u}{u_i} \,\middle|\, i = 1, \ldots, n \right\}. \tag{3.17}$$

Proof: See [6] for the proof. □

From the above theorem we have the following corollary:

Corollary 5 *Suppose that D is a nonnegative irreducible matrix of order n, and that $\lambda_{\max}$ is the principal eigenvalue of D. Let d_i be the i^{th} row vector of D for every $i = 1, \ldots, n$. Then*

$$\min_{u>0} \max \left\{ \left| \frac{d_i u}{u_i} - 1 \right| \middle| \, i = 1, \ldots, n \right\} = |\lambda_{\max} - 1|, \tag{3.18}$$

where the equality in (3.18) holds for every positive eigenvector u of D corresponding to $\lambda_{\max}$. For every positive vector u except for an eigenvector of D corresponding to $\lambda_{\max}$

$$\max \left\{ \left| \frac{d_i u}{u_i} - 1 \right| \middle| \, i = 1, \ldots, n \right\} > |\lambda_{\max} - 1|. \tag{3.19}$$

Proof: See [5] for details. □

Let

$$\hat{A} = \begin{bmatrix} a_1/M_1 \\ \vdots \\ a_n/M_n \end{bmatrix} \tag{3.20}$$

for the irreducible evaluation matrix A, then $\hat{A}$ is also irreducible and it follows from Corollary 5 that a positive principal eigenvector of $\hat{A}$ is a solution of (3.13).

In this study we propose the averaging principle for the irreducible evaluation matrix A as follows:

Step 1: Generate the matrix $\hat{A}$ defined by (3.20) for the evaluation matrix A.

Step 2: Find a positive principal eigenvector $\hat{w}$ of $\hat{A}$ and define $\hat{w}$ as a weight vector of the evaluation matrix A.

From the irreducibility of the evaluation matrix A we see that M_i is a natural number for every $i = 1, \ldots, n$ and that $\hat{A}$ is well defined in Step 1. Theorem 1 guarantees the existence and the uniqueness of $\hat{w}$ of Step 2. Taking $D = \hat{A}$ in Corollary 5, we see that an optimal solution of Problem (3.13) is identical to a positive principal eigenvector of $\hat{A}$.

Example 3 *Consider Example 1 and the corresponding evaluation matrix (2.7). We apply the averaging principle to the evaluation matrix (2.7). In step 1 we have $M_1 = M_2 = 3$ and $M_3 = M_4 = M_5 = 2$ and divide the i^{th} row vector of (2.7) by M_i for every $i = 1, \ldots, 6$. Then we get*

$$\hat{A} = \begin{bmatrix} 0 & 0 & 1/12 & 1 & 2/3 \\ 0 & 0 & 1/3 & 1/3 & 1/3 \\ 1/2 & 1/2 & 0 & 0 & 0 \\ 1 & 1/4 & 0 & 0 & 0 \\ 3/2 & 1/12 & 0 & 0 & 0 \end{bmatrix}$$

and find a positive principal eigenvector of $\hat{A}$ in Step 2. A principal eigenvector of $\hat{A}$ is $\bar{w} = [0.262,\ 0.135,\ 0.133,\ 0.199,\ 0.271]^T$. The weights of A and S are 0.660 *and* 0.340*, respectively and the weights of M, B, W are* 0.221, 0.330 *and* 0.449*, respectively.* □

Here, we compare the averaging principle with ANP for the Saaty's supermatrix (2.5).

Theorem 6 *Consider the stochastic evaluation matrix A with an $n_1 \times n_2$ submatrix W and an $n_2 \times n_1$ submatrix U i.e.,*

$$A = \begin{bmatrix} 0 & W \\ U & 0 \end{bmatrix}.$$

If U and W are positive, then the weight vector for the evaluation matrix A by ANP is identical to that by the averaging principle.

Proof: Let U be the evaluation matrix of n_2 alternatives by n_1 criteria and let W be the evaluation matrix of n_1 criteria by n_2 alternatives. Since both U and W are positive matrices, A is irreducible. Let x and y be the n_1 weights of criteria and the n_2 weights of alternatives, respectively. Then from Theorem 3 we have

$$WUy = y \text{ and } UWx = x$$

and hence, x is a positive principal eigenvector of UW and y is a positive principal eigenvector of WU.

On the other hand, in step 1 of the averaging principle we get

$$\hat{A} = \begin{bmatrix} 0 & 1/n_2 W \\ 1/n_1 U & 0 \end{bmatrix},$$

since U and W are positive.

In step 2 we find a positive principal vector $[v^T, z^T]^T$ and the principal eigenvalue $\lambda_{\max}$ of $\hat{A}$, that is, we solve

$$\begin{bmatrix} 0 & 1/n_2 W \\ 1/n_1 U & 0 \end{bmatrix} \begin{bmatrix} v \\ z \end{bmatrix} = \lambda_{\max} \begin{bmatrix} v \\ z \end{bmatrix}.$$

This implies that

$$WUz = (n_1 n_2 \lambda_{\max}^2) z \text{ and } UWv = (n_1 n_2 \lambda_{\max}^2) v.$$

We see from Theorem 1 that z is a principal eigenvector of WU and that v is a principal eigenvector of UW.

Therefore, we have completed the proof. □

Since each column of U and W is often calculated by AHP, both U and W are positive matrices and the weights for the Saaty's supermatrix (2.5) by ANP are often identical to those by the averaging principle.

4. Averaging Principle Analysis for a Reducible Evaluation Matrix

In section 3 we stated the solving method of the evaluation problem with the strongly connected structure, that is, the evaluation matrix A of (2.11) is irreducible. Here we state the general solving method of the evaluation problem with the reducible evaluation matrix A. This is the main purpose of our paper.

4.1. Cluster decomposition and standardization

We consider a reducible evaluation matrix whose graph is not strongly connected but connected. If the corresponding graph is not connected, the reducible evaluation matrix is decomposed into some irreducible evaluation submatrices and each of all submatrices should be evaluated individually and independently. Here, we assume that the reducible evaluation matrix corresponds the connected graph but not strongly connected one.

In graph theory it is well known that a connected graph can be decomposed into strongly connected components which are topologically ordered. According to the topological order we can put a linear order for all strongly connected components. Let L be the number of the strongly connected components in the graph corresponding to the reducible evaluation

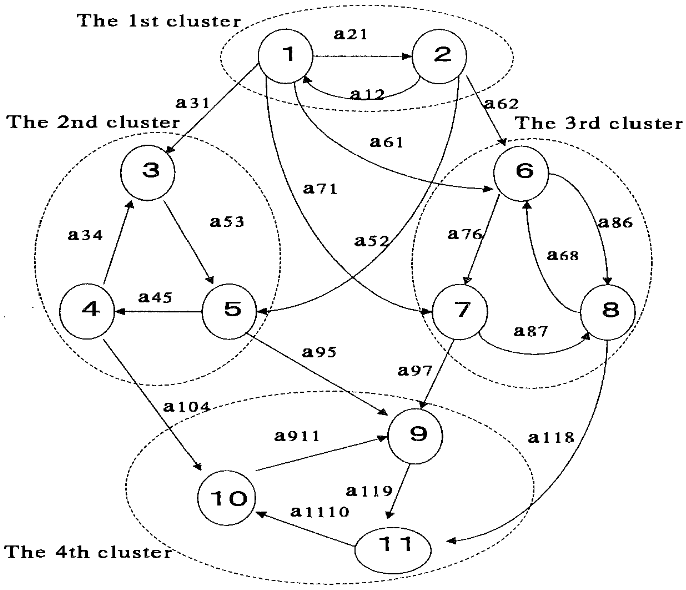

Figure 4.4: Decomposition of the overall evaluation structure into 4 clusters

matrix and let all strongly connected components be ordered linearly. Then we can represent the reducible evaluation matrix A as follows:

$$A = \begin{bmatrix} A^1 & 0 & \cdots & 0 \\ B^{21} & A^2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ B^{L1} & \cdots & B^{LL-1} & A^L \end{bmatrix}. \tag{4.1}$$

Let $n(l)$ be the number of nodes in the l^{th} strongly connected component for $l = 1, \dots, L$. Then the order of the submatrix A^l of (4.1) is $n(l)$ and the submatrix B^{lk} of (4.1) is an $n(l) \times n(k)$ matrix for every $(l, k) \in \{\, (l, k) \mid 1 \le l < k \le L\}$. The l^{th} cluster is called the set of an object corresponding to a node in the l^{th} strongly connected component. The submatrix A^l of (4.1) is the evaluation matrix of the objects in the l^{th} strongly connected component by themselves and it is called the internal evaluation matrix of the l^{th} cluster. The submatrix B^{lk} of (4.1) is the evaluation matrix of the objects in the l^{th} cluster by those in the k^{th} cluster.

Example 4 *We consider an evaluation system with 11 objects and the following reducible evaluation matrix (4.1) :*

$$A = \left[\begin{array}{cc|ccc|ccc|ccc} & a_{12} & & & & & & & & & \\ a_{21} & & & & & & & & & & \\ \hline a_{31} & & & a_{34} & & & & & & & \\ & & & & a_{45} & & & & & & \\ & a_{52} & a_{53} & & & & & & & & \\ \hline a_{61} & a_{62} & & & & & & a_{68} & & & \\ a_{71} & & & & & a_{76} & & & & & \\ & & & & & a_{86} & a_{87} & & & & \\ \hline & & & & a_{95} & & a_{97} & & & a_{910} & \\ & & & a_{104} & & & & & & & a_{1011} \\ & & & & & & & a_{118} & a_{119} & & \end{array}\right].$$

The evaluation structure consists of 4 clusters, the first cluster $\{1,2\}$*, the second cluster* $\{3,4,5\}$*, the third cluster* $\{6,7,8\}$ *and the forth cluster* $\{9,10,11\}$*. Figure 4.4 shows connectivity of these 4 closters. We have* $n(1)=2$*,* $n(2)=n(3)=3$ *and* $n(4)=3$*, and the following 10 submatrices:*

$$A^1=\begin{bmatrix}0 & a_{12}\\ a_{21} & 0\end{bmatrix},$$
$$A^2=\begin{bmatrix}0 & a_{34} & 0\\ 0 & 0 & a_{45}\\ a_{53} & 0 & 0\end{bmatrix}, B^{21}=\begin{bmatrix}a_{31} & 0\\ 0 & 0\\ 0 & a_{52}\end{bmatrix},$$
$$A^3=\begin{bmatrix}0 & 0 & a_{68}\\ a_{76} & 0 & 0\\ a_{86} & a_{87} & 0\end{bmatrix}, B^{31}=\begin{bmatrix}a_{61} & a_{62}\\ a_{71} & 0\\ 0 & 0\end{bmatrix}, \quad B^{32}=\begin{bmatrix}0&0&0\\0&0&0\\0&0&0\end{bmatrix}$$
$$A^4=\begin{bmatrix}0 & a_{910} & 0\\ 0 & 0 & a_{1011}\\ a_{119} & 0 & 0\end{bmatrix}, B^{41}=\begin{bmatrix}0&0\\0&0\\0&0\end{bmatrix}, B^{42}=\begin{bmatrix}0 & 0 & a_{95}\\ 0 & a_{104} & 0\\ 0 & 0 & 0\end{bmatrix} \text{ and } B^{43}=\begin{bmatrix}0 & a_{97} & 0\\ 0 & 0 & 0\\ 0 & 0 & a_{108}\end{bmatrix}.$$

■

For every internal evaluation matrix A^l of (4.1) we state some properties as follows:

Lemma 7 *Let* A *be the reducible evaluation matrix (4.1). If* $n(l)\geq 2$*, the internal evaluation matrix* A^l *of the* l^{th} *cluster is irreducible.*

Lemma 8 *Let* A *be the reducible evaluation matrix (4.1). If* $n(l)=1$*, the internal evaluation matrix* A^l *of the* l^{th} *cluster is a scaler* 0.

If there exists an arc from the k^{th} cluster to the l^{th} cluster, we say that the k^{th} cluster precedes the l^{th} cluster, and that the k^{th} cluster is a predecessor of the l^{th} cluster.

Lemma 9 *Suppose that* A *is the reducible evaluation matrix (4.1). The submatrix* $\left[B^{l1},\ldots,B^{ll-1}\right]$ *of* A *is a nonnegative and nonzero matrix if and only if there exists a predecessor of the* l^{th} *cluster.*

If a cluster does not have a predecessor, it is called a source cluster. Notice that the first cluster is a source cluster and that the last cluster has a predecessor in the case of $L\geq 2$.

For the evaluation matrix A defined by (4.1), we propose the cluster-wise standardization other than Saaty's column-wise standardization such that each column-sum of A is 1. The submatrix

$$\begin{bmatrix}0\\ A^l\\ B^{l+1l}\\ \vdots\\ B^{Ll}\end{bmatrix} \tag{4.2}$$

of the evaluation matrix A is a set of evaluation data by the l cluster. For all $l=1,\ldots,L$, we standardize each submatrix (4.2) of A as follows:

Let λ^l be the principal eigenvalue of A^l with $n(l)\geq 2$ and let $\lambda^l=1$ in the case of $n(l)=1$. For all $l=1,\ldots,L$, we divide the internal evaluation matrix A^l of the l^{th} cluster and $B^{l+1l},\ldots,B^{Ll}$ by λ^l. Let $\bar{A}^l=A^l/\lambda^l$ and $\bar{B}^{kl}=B^{kl}/\lambda^l$ for $k=l+1,\ldots,L$ and

$l = 1, \ldots, L$, then a standardized evaluation matrix $\bar{A}$ of A is defined as

$$\bar{A} = \begin{bmatrix} \bar{A}^1 & 0 & \cdots & 0 \\ \bar{B}^{21} & \bar{A}^2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ \bar{B}^{L1} & \cdots & \bar{B}^{LL-1} & \bar{A}^L \end{bmatrix}. \tag{4.3}$$

Because it follows from Theorem 1 that $\lambda^k > 0$ for $k = 1, \ldots, L$, $\bar{A}$ is also a nonnegative matrix. The topology of the graph corresponding to the evaluation matrix A are identical to that to $\bar{A}$. Therefore, we have the same properties for the standardized evaluation matrix $\bar{A}$ as Lemmas 7, 8 and 9.

Lemma 10 *Let $\bar{A}$ be the standardized reducible evaluation matrix (4.3). If $n(l) \geq 2$, the standardized internal evaluation matrix $\bar{A}^l$ of the l^{th} cluster is irreducible and its principal eigenvalue is* 1.

Lemma 11 *Let $\bar{A}$ be the standardized reducible evaluation matrix (4.3). If $n(l) = 1$, the standardized internal evaluation matrix $\bar{A}^l$ of the l^{th} cluster is a scaler* 0.

Lemma 12 *Let $\bar{A}$ be the standardized reducible evaluation matrix (4.3). The submatrix $\left[\bar{B}^{l1}, \ldots, \bar{B}^{ll-1}\right]$ of $\bar{A}$ is a nonnegative and nonzero matrix if and only if there exists a predecessor of the l^{th} cluster.*

Let M_i^l be the number of positive elements in the i^{th} row vector of the submatrix $\left[\bar{B}^{l1}, \ldots, \bar{B}^{ll-1}, \bar{A}^l\right]$ of the standardized reducible evaluation matrix (4.3), then we have the following lemma:

Lemma 13 *Let $\bar{A}$ be the standardized reducible evaluation matrix (4.3). Suppose that there exists a predecessor of the l^{th} cluster or that $n(l) \geq 2$, then $M_i^l \geq 1$ for every $i = 1, \ldots, n(l)$.*

Proof: When $n(l) = 1$, the l^{th} cluster has at least one predecessor and hence, $M_1^l = M_{n(l)}^l \geq 1$. When $n(l) \geq 2$, each node of the l^{th} cluster has at least one arc from the other nodes of the l^{th} cluster since the l^{th} cluster is strongly connected. Therefore, $M_i^l \geq 1$ for every $i = 1, \ldots, n(l)$. □

4.2. Cluster-wise averaging principle for the reducible evaluation matrix

We develop the cluster-wise averaging principle for the standardized reducible evaluation matrix (4.3) that is sequentially applied to weighting all objects of the cluster according the topological order of clusters.

Let $\bar{A}$ be the standardized reducible evaluation matrix (4.3). For every source cluster with a single object, for example the first cluster $\{G\}$ of Example 2, the cluster-wise averaging principle determines the weight of the single object as 1. Let $\hat{w}^l$ be the weight vector of the l^{th} cluster, then the l^{th} cluster with $n(l) = 1$ has $\hat{w}^l = \hat{w}_1^l = 1$.

For all clusters except source clusters with a single object, the cluster-wise averaging principle provides the weight vector of each cluster as follows:

Suppose that the l^{th} cluster is not the source cluster or that it does not consist of a single object. Then there exists a predecessor of the l^{th} cluster or we have $n(l) \geq 2$. Therefore we see from Lemma 13 that $M_i^l \geq 1$ for every $i = 1, \ldots, n(l)$. Suppose that $\hat{w}^k$ is the weight vector of the k^{th} cluster for every $k = 1, \ldots, l-1$. Let $\bar{b}_i^{lk}$ be the i^{th} row vector of $\bar{B}^{lk}$ for $k = 1, \ldots, l-1$ and let $\bar{a}_i^l$ be the i^{th} row vector of $\bar{A}^l$ for $i = 1, \ldots, n(l)$. The cluster-wise averaging principle for the l^{th} cluster is to solve

$$\min_{w>0} \max \left\{ \left| \frac{\sum_{k=1}^{l-1} \bar{b}_i^{lk} \hat{w}^k + \bar{a}_i^l w}{M_i^l w_i} - 1 \right| \,\middle|\, i = 1, \ldots, n(l) \right\} \tag{4.4}$$

and to define the optimal solution $\hat{w}$ of (4.4) as the weight vector $\hat{w}^l$ of the l^{th} cluster.

The first term $\sum_{k=1}^{l-1} \bar{b}_i^{lk}\hat{w}^k$ of the numerator in (4.4) is the sum of the external evaluation values of the i^{th} object in the l^{th} cluster from objects in the predecessors of the l^{th} cluster. The second term $\bar{a}_i^l w$ is the sum of the external evaluation values of the i^{th} object in the l^{th} cluster from the other objects in the l^{th} cluster. Therefore, $\left(\sum_{k=1}^{l-1} \bar{b}_i^{lk}\hat{w}^k + \bar{a}_i^l w\right)/M_i^l$ is the average of the external evaluation values of the the i^{th} object in the l^{th} cluster. Problem (4.4) means that all w_i's minimizing overall discrepancies between the self-evaluation value w_i and the average $\left(\sum_{k=1}^{l-1} \bar{b}_i^{lk}\hat{w}^k + \bar{a}_i^l w\right)/M_i^l$ of the external evaluation values for all $i = 1, \ldots, n(l)$ are the desirable estimation weights of objects in the l^{th} cluster. Hence, the meaning of Problem (4.4) is a natural extension of that of Problem (3.13).

We will discuss the positiveness of the weight vector by the cluster-wise averaging principle for all clusters except source clusters with a single object.

Firstly, we consider that the l^{th} cluster with $n(l) \geq 2$ is a source cluster. Then Problem (4.4) is reduced into

$$\min_{w>0} \max \left\{ \left| \frac{\bar{a}_i^l w}{M_i^l w_i} - 1 \right| \;\middle|\; i = 1, \ldots, n(l) \right\} \tag{4.5}$$

since $\bar{B}^{lk} = 0$ for $k = 1, \ldots, l-1$. Problem (4.5) is equivalent to Problem (3.13) whose a_i and M_i are replaced by $\bar{a}_i^l$ and M_i^l, respectively. Let

$$\hat{A}^l = \begin{bmatrix} \bar{a}_1^l / M_1^l \\ \vdots \\ \bar{a}_{n(l)}^l / M_{n(l)}^l \end{bmatrix} \tag{4.6}$$

for $l = 1, \ldots, L$, then we have the following lemma:

Lemma 14 *Suppose that the l^{th} cluster is a source cluster and that $n(l) \geq 2$. An optimal solution of Problem (4.5) for the l^{th} cluster is identical to a positive principal eigenvector of $\hat{A}^l$ defined by (4.6). Furthermore, it is identical to a positive principal eigenvector of*

$$\tilde{A}^l = \begin{bmatrix} a_1^l / M_1^l \\ \vdots \\ a_{n(l)}^l / M_{n(l)}^l \end{bmatrix}.$$

Proof: Since $n(l) \geq 2$, it follows from Lemma 10 that the standardized internal evaluation matrix $\bar{A}^l$ is irreducible. From Lemma 13 $\hat{A}^l$ is well defined and hence, it is also irreducible. Since $\hat{A}^l = \tilde{A}^l/\lambda^l$, it follows from the definitions of λ^l that a positive principal eigenvector of $\hat{A}^l$ is identical to that of $\tilde{A}^l$. It follows from Corollary 5 that an optimal solution of Problem (4.5) for the l^{th} cluster is identical to a positive principal eigenvector of $\hat{A}^l$ □

Lemma 14 means that the cluster-wise averaging principle for all source clusters with more than one object is to determine individually the weight vector of each cluster by the averaging principle stated in section 3.1. Hence, it follows from the definition of a source cluster that we can apply the averaging principle to each source cluster with more than one object before determining the weight vector of any cluster with a predecessor.

Secondarily, we consider that the l^{th} cluster with $n(l) \geq 2$ has a predecessor.

Lemma 15 *Let $\bar{A}^l$ be the standardized internal evaluation matrix of the l^{th} cluster. If there exists a predecessor of the l^{th} cluster and $n(l) \geq 2$, then the principal eigenvalue of $\hat{A}^l$ defined by (4.6) is less than 1.*

Proof: Since the l^{th} cluster has a predecessor, it follows from Lemma 12 that the submatrix $\left[\bar{B}^{l1},\ldots,\bar{B}^{ll-1}\right]$ of $\bar{A}$ is a nonnegative and nonzero matrix. Without loss of generality, we assume that the j^{th} row vector $[\bar{b}_j^{l1},\ldots,\bar{b}_j^{ll-1}]$ of $\left[\bar{B}^{l1},\ldots,\bar{B}^{ll-1}\right]$ is a nonzero vector. Since it follows from the irreducibility of $\bar{A}^l$ that $\bar{a}_j^l$ is also a nonzero vector, we have $M_j^l \geq 2$. This means from Lemmas 10 and 13 that for a positive principal eigenvector $\bar{w}$ of $\bar{A}^l$

$$\frac{\bar{a}_j^l\bar{w}}{M_j^l\bar{w}_j} < \frac{\bar{a}_j^l\bar{w}}{\bar{w}_j} = 1 \quad \text{and} \tag{4.7}$$

$$\frac{\bar{a}_i^l\bar{w}}{M_i^l\bar{w}_i} \leq \frac{\bar{a}_i^l\bar{w}}{\bar{w}_i} = 1 \quad \text{for } i \neq j. \tag{4.8}$$

It follows from (4.7), (4.8) and Theorem 4 that the principal eigenvalue of $\hat{A}^l$ is less than 1. □

Let

$$\hat{B}^{lk} = \begin{bmatrix} \bar{b}_1^{lk}/M_1^l \\ \vdots \\ \bar{b}_{n(l)}^{lk}/M_{n(l)}^l \end{bmatrix} \tag{4.9}$$

for $k = 1,\ldots,l-1$ and let

$$\hat{b}^l = \sum_{k=1}^{l-1} \hat{B}^{lk}\hat{w}^k \tag{4.10}$$

for the weight vectors $\hat{w}^1,\ldots,\hat{w}^{l-1}$ and $\hat{B}^{l1},\ldots,\hat{B}^{ll-1}$, then we have the following lemma:

Lemma 16 *Suppose that $\hat{w}^k$ is positive for every $k = 1,\ldots,l-1$ and that $n(l) \geq 2$. Then $\hat{b}^l$ defined by (4.10) is a nonnegative and nonzero vector if and only if there is a predecessor of the l^{th} cluster.*

Proof: Since $n(l) \geq 2$, it follows from Lemma 13 that $\left[\hat{B}^{l1},\ldots,\hat{B}^{ll-1}\right]$ is well defined. It follows from Lemma 12 that $\left[\hat{B}^{l1},\ldots,\hat{B}^{ll-1}\right]$ is a nonnegative and nonzero matrix if and only if there is a predecessor of the l^{th} cluster. Since $\hat{w}^k$ is positive for $k = 1,\ldots,l-1$, we have this assertion. □

Here, we introduce the key lemma of solving Problem (4.4) that is well-known in mathematical economics [8].

Lemma 17 *Let D be an irreducible nonnegative matrix of order n and let I be the identity matrix of order n. If ρ is more than the principal eigenvalue of D, then $\rho I - D$ is nonsingular and the inverse matrix of $\rho I - D$ is positive.*

The following lemmas imply that the cluster-wise averaging principle provides a positive weight vector $\hat{w}^l$ for the l^{th} cluster with a predecessor under the assumption of the positive weight vectors $\hat{w}^1,\ldots,\hat{w}^{l-1}$.

Lemma 18 *Assume that the weight vector $\hat{w}^k$ of the k^{th} cluster is positive for $k = 1,\ldots,l-1$. If there exists a predecessor of the l^{th} cluster and $n(l) \geq 2$, an optimal solution of Problem (4.4) is $\left(I - \hat{A}^l\right)^{-1}\hat{b}^l$, where $\hat{A}^l$ and $\hat{b}^l$ are defined by (4.6) and (4.10), respectively and I is the identity matrix of order $n(l)$.*

Proof: Let I be the identity matrix of order $n(l)$. Since the l^{th} cluster has a predecessor and $n(l) \geq 2$, it follows from Lemma 13 that $\hat{A}^l$ is well defined. Moreover, it follows from Lemmas 15 and 17 that $I - \hat{A}^l$ is nonsingular and that the inverse matrix $\left(I - \hat{A}^l\right)^{-1}$ is positive. From Lemma 16 and the assumption of the positive weight vector $\hat{w}^k$ for every $k = 1, \dots, l-1$, it follows that $\hat{b}^l$ defined by (4.10) is a nonnegative and nonzero vector. Let $\bar{w} = \left(I - \hat{A}^l\right)^{-1} \hat{b}^l$, then $\bar{w}$ is a positive vector and $\hat{A}^l \bar{w} + \hat{b}^l = \bar{w}$. Hence, it follows from the definitions of $\hat{A}^l$ and $\hat{b}^l$ that

$$0 = \frac{\bar{a}_1^l \bar{w} + \sum_{k=1}^{l-1} \bar{b}_1^{lk} \hat{w}^k}{M_1^l \bar{w}_1} - 1 = \cdots = \frac{\bar{a}_{n(l)}^l \bar{w} + \sum_{k=1}^{l-1} \bar{b}_{n(l)}^{lk} \hat{w}^k}{M_{n(l)}^l \bar{w}_{n(l)}} - 1.$$

This means that

$$\begin{aligned} 0 &= \max \left\{ \left| \frac{\bar{a}_i^l \bar{w} + \sum_{k=1}^{l-1} \bar{b}_i^{lk} \hat{w}^k}{M_i^l \bar{w}_i} - 1 \right| \middle| i = 1, \dots, n(l) \right\} \\ &\geq \min_{w>0} \max \left\{ \left| \frac{\bar{a}_i^l w + \sum_{k=1}^{l-1} \bar{b}_i^{lk} \hat{w}^k}{M_i^l w_i} - 1 \right| \middle| i = 1, \dots, n(l) \right\} \\ &\geq 0. \end{aligned}$$

Therefore, an optimal solution of Problem (4.4) is $\left(I - \hat{A}^l\right)^{-1} \hat{b}^l$. □

Finally, we consider that the l^{th} cluster with $n(1) = 1$ has a predecessor. Then the cluster-wise averaging principle provides a positive weight as follows:

Lemma 19 *Suppose that the weight vector $\hat{w}^k$ of the k^{th} cluster is positive for every $k = 1, \dots, l-1$. If there exists a predecessor of the l^{th} cluster and $n(l) = 1$, an optimal solution of Problem (4.4) is $\hat{b}^l$, where $\hat{b}^l$ is defined by (4.10).*

Proof: Since $n(l) = 1$, it follows from Lemma 11 that $\hat{A}^l = 0$. Since there exists a predecessor of the l^{th} cluster, it follows from Lemma 13 that $\left[\hat{B}^{l1}, \dots, \hat{B}^{ll-1}\right]$ is well defined and that Problem (4.4) is

$$\min_{w>0} \left| \frac{\hat{b}_1^l}{w} - 1 \right|.$$

From Lemma 12, $\hat{b}_1^l$ is positive. Therefore, an optimal solution of Problem (4.4) is $\hat{b}_1^l$. □

Lemmas 19 and 11 imply that $\hat{w}^l = \left(1 - \hat{A}^l\right)^{-1} \hat{b}^l$ when the l^{th} cluster with $n(l) = 1$ has a predecessor.

From Lemmas 14, 18 and 19 we show the positiveness and the uniqueness of all weight vectors by the cluster-wise averaging principle.

Theorem 20 *The cluster-wise averaging principle provides a unique positive weight vector of each cluster.*

Proof: We will prove this assertion by induction. The first cluster is a source cluster. In the case of $n(1) \geq 2$, it follows from Lemma 14 that the weight vector $\hat{w}^1$ of the first cluster by the cluster-wise averaging principle is positive and unique except for a scalar multiple. In the other case, that is $n(1) = 1$, it follows from the definition of the cluster-wise principle that $\hat{w}^1 = 1$.

Suppose that the k^{th} cluster has a unique positive weight vector $\hat{w}^k$ for $k = 1, \dots, l-1$. In the case that the l^{th} cluster has a predecessor, it follows from Lemmas 18 and 19 that the weight vector $\hat{w}^l$ of the l^{th} cluster by the cluster-wise averaging principle is positive and unique. In the other case, that is l^{th} cluster is a source cluster, it follows from Lemma 14 and the definition of the cluster-wise principle that $\hat{w}^l$ is positive and unique except for a scalar multiple. This has completed the induction. □

Notice that an eigenvalue of scalar 0 is 0 and that any positive scalar is an eigenvector corresponding to the eigenvalue 0. Let e and I be all ones vector and the identity matrix. We arrange the cluster-wise averaging principle algorithm as follows:

Step 0: Rearrange rows and columns of the evaluation matrix A appropriately and generate the following standardized evaluation matrix $\bar{A}$ by the cluster-wise standardization:

$$\bar{A} = \begin{bmatrix} \bar{A}^1 & 0 & \cdots & 0 \\ \bar{B}^{21} & \bar{A}^2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ \bar{B}^{L1} & \cdots & \bar{B}^{LL-1} & \bar{A}^L \end{bmatrix}.$$

Generate $\hat{A}^l$ and $\hat{B}^{lk}$ from $\bar{A}^l$ and $\bar{B}^{lk}$ by (4.6) and (4.9), respectively, for $k = 1, \ldots, l-1$ and $l = 1, \ldots, L$.

Step 1: Find a positive principal eigenvector $\bar{w}$ of $\hat{A}^1$ and define $\bar{w}/e^T\bar{w}$ as the weight vector $\hat{w}^1$ of the first cluster. If $L = 1$, then stop. Otherwise set $l = 2$ and go to step 2.

Step 2: Generate $\hat{b}^l$ from $\hat{w}^1, \ldots, \hat{w}^{l-1}$ by (4.10). If $\hat{b}^l = 0$, find a positive principal eigenvector $\bar{w}$ of $\hat{A}^l$ and define $\bar{w}/e^T\bar{w}$ as the weight vector $\hat{w}^l$ of the l^{th} cluster. Otherwise define $\left(I - \hat{A}^l\right)^{-1}\hat{b}^l$ as the weight vector $\hat{w}^l$ of the l^{th} cluster. Go to step 3.

Step 3: If $l = L$, then stop. Otherwise set $l = l + 1$ and go to step 2.

Notice that the above cluster-wise principle algorithm determines the weight vector of all clusters independent of the choice of the linear order corresponding to the topological order of clusters.

The cluster-wise averaging principle algorithm can be applied to the irreducible evaluation matrix A since $\bar{A} = \bar{A}^1$. Therefore, the cluster-wise averaging principle algorithm is simply called the averaging principle.

5. Illustrative Example

5.1. The example of externally weighted criteria

Here we take again Example 2 shown in Figure 2.2 whose supermatrix S in shown in (2.8). We assume the numerical values of U and W to be

$$U = \begin{bmatrix} 1 & 1 \\ 2 & 1/2 \\ 3 & 1/6 \end{bmatrix}, W = \begin{bmatrix} 1/4 & 3 & 2 \\ 1 & 1 & 1 \end{bmatrix} \tag{5.1}$$

and v_1 and v_2 are left to be variable parameters.

The structure of this example consists of 2 clusters; $\{G\}$ and $\{A, S, M, B, W\}$. (Note that our structuring is based on only graph theory so it might not coincide with the actual world structure. By the latter $\{A, S\}$ and $\{M, B, W\}$ should be different clusters.) And the total evaluation matrix is

$$A = \begin{bmatrix} A^1 & 0 \\ B^{12} & A^2 \end{bmatrix}, A^1 = [0], B^{12} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}, A^2 = \begin{bmatrix} 0 & W \\ U & 0 \end{bmatrix}. \tag{5.2}$$

Standardizing them, we have

$$\bar{A}^1 = A^1, \bar{B}^{12} = B^{12}, \bar{A}^2 = \frac{1}{3.65}A^2.$$

See section 4.1 for the standardizing method. The weight of G is 1 by our rule, so by the averaging principle in section 4.2 we have only to calculate

$$\hat{w}^2 = \left(I - \hat{A}^2\right)^{-1} \hat{b}^2, \tag{5.3}$$

where

$$\hat{b}^2 = \begin{bmatrix} v_1/4 \\ v_2/4 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \hat{A}^2 = \frac{1}{3.65}\begin{bmatrix} 0 & \frac{1}{4}W \\ \frac{1}{2}U & 0 \end{bmatrix}, \hat{w}^2 = \begin{bmatrix} x_1 \\ x_2 \\ y_1 \\ y_2 \\ y_3 \end{bmatrix} \tag{5.4}$$

to have $x = [x_1, x_2]^T$ and $y = [y_1, y_2, y_3]^T$;

x_1 is the weight of criterion A (advertising)

x_2 is the weight of criterion S (service)

y_1 is the weight of alternative M (McDonald)

y_2 is the weight of alternative B (Berger King)

y_3 is the weight of alternative W (Wendy).

The linear equation system (5.3) is decomposed into

$$\frac{1}{4}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} + \frac{1}{4 \times 3.65}Wy = x, \quad \frac{1}{2 \times 3.65}Ux = y \tag{5.5}$$

and eliminating y we have

$$\left(I - \frac{1}{4 \times 2 \times 3.65^2}WU\right)x = \frac{1}{4}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix}. \tag{5.6}$$

The solution x of (5.6) is

$$\begin{aligned} x_1 &= 0.2828v_1 + 0.0056v_2, \\ x_2 &= 0.0116v_1 + 0.2543v_2. \end{aligned} \tag{5.7}$$

We see from (5.7) that x_1(weight of A) receives height influence from v_1 and very low influence from v_2. This is the reasonable reflection of our structure (in Figure 2.2) where A receives the direct evaluation v_1 from G, but indirect (through other points) evaluation v_2. The weight x_2 of S also has the same reasonable property.

Further from (5.5) and (5.7) we have

$$\begin{aligned} y_1 &= 0.0410v_1 + 0.0356v_2, \\ y_2 &= 0.0786v_1 + 0.0190v_2, \\ y_3 &= 0.1166v_1 + 0.0081v_2, \end{aligned} \tag{5.8}$$

which also shows so reasonable inclination that y's have generally lower figures than that of x's, because x's have the direct but y's have the indirect influence of v's.

5.2. The example of externally weighted alternatives

This example is an extension of the one in section 2.1, and is based on the example shown in section 4.5 of [4]. We modify and simplify the latter and apply our method to it.

The set of alternatives is {McDonald, BurgerKing, Wendy} which is denoted by $\{C_1, C_2, C_3\}$ here. There are two kinds of sets of criteria. The first is the advertising set {creativity, promotion, frequency} which is denoted by $\{A_1, A_2, A_3\}$, and the second is the set of quality of food {nutrition, taste, portion} which is denoted by $\{Q_1, Q_2, Q_3\}$.

Further each company of $\{C_1, C_2, C_3\}$ has the general evaluations from international(R_1) and domestic (R_2) side. So $\{R_1, R_2\}$ is also another kind of criteria set. As a result, the set of all objects (including Goal (G)) is

$$\{G, R_1, R_2, C_1, C_2, C_3, A_1, A_2, A_3, Q_1, Q_2, Q_3, \}. \tag{5.9}$$

Through several investigation, we have the following total evaluation matrix A and its network structure shown in Figure 5.5, where the evaluations R_1 and R_2 by Goal are represented by variable parameters v_1 and v_2.

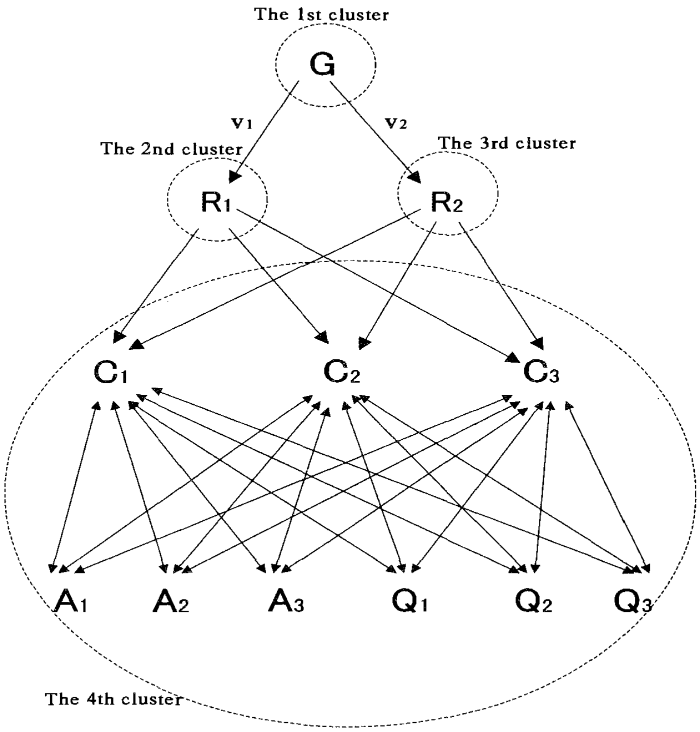

Figure 5.5: The evaluation structure of the example of externally weighted alternatives

This total evaluation matrix is given as follows:

$$A = \begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
v_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
v_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0.657 & 0.540 & 0 & 0 & 0 & 0.614 & 0.717 & 0.717 & 0.249 & 0.291 & 0.595 \\
0 & 0.196 & 0.297 & 0 & 0 & 0 & 0.268 & 0.195 & 0.195 & 0.157 & 0.105 & 0.128 \\
0 & 0.147 & 0.163 & 0 & 0 & 0 & 0.117 & 0.088 & 0.088 & 0.594 & 0.605 & 0.276 \\
0 & 0 & 0 & 0.104 & 0.089 & 0.134 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0.066 & 0.056 & 0.036 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0.330 & 0.355 & 0.325 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0.166 & 0.140 & 0.313 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0.070 & 0.036 & 0.140 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0.264 & 0.325 & 0.047 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}.$$

We have 4 clusters on strongly connected components $\{G\}$, $\{R_1\}$, $\{R_2\}$ and $\{C_1, C_2, C_3, A_1, A_2, A_3, Q_1, Q_2, Q_3\}$. By our notation shown in section 3, we have the following submatrices of A.

$$A^1 = 0, \quad B^{21} = v_1, \quad A^2 = 0, \quad B^{31} = v_2, \quad B^{32} = 0, \quad A^3 = 0.$$

$$B^{41} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, B^{42} = \begin{bmatrix} 0.657 \\ 0.196 \\ 0.147 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, B^{43} = \begin{bmatrix} 0.540 \\ 0.297 \\ 0.163 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix},$$

$$A^4 = \begin{bmatrix} 0 & 0 & 0 & 0.614 & 0.717 & 0.717 & 0.249 & 0.291 & 0.595 \\ 0 & 0 & 0 & 0.268 & 0.195 & 0.195 & 0.157 & 0.105 & 0.128 \\ 0 & 0 & 0 & 0.117 & 0.088 & 0.088 & 0.594 & 0.605 & 0.276 \\ 0.104 & 0.089 & 0.140 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0.066 & 0.056 & 0.036 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0.330 & 0.355 & 0.325 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0.166 & 0.140 & 0.313 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0.070 & 0.036 & 0.140 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0.264 & 0.325 & 0.047 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Standardizing these, we have

$$\bar{A}^1 = A^1, \bar{B}^{21} = B^{21}, \bar{A}^2 = A^2, \bar{B}^{31} = B^{31}, \bar{B}^{32} = B^{32}, \bar{A}^3 = A^3, \bar{A}^4 = A^4$$

and

$$\hat{A}^1 = \bar{A}^1, \hat{A}^2 = \bar{A}^2, \hat{A}^3 = \bar{A}^3.$$

The weight of G is 1. The weights $\hat{w}_i$ of $R_i (i = 1, 2)$ are given with $\hat{w}^2 = v_1$ and $\hat{w}^3 = v_2$ by the averaging principle. Further we have

$$\hat{B}^{41} = 1/8\bar{B}^{41}, \hat{B}^{42} = 1/8\bar{B}^{42}$$

and

$$\hat{b}^4 = \begin{bmatrix} (0.657v_1 + 0.540v_2)/8 \\ (0.196v_1 + 0.297v_2)/8 \\ (0.147v_1 + 0.163v_2)/8 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \hat{A}^4 = \begin{bmatrix} 1/8 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1/8 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1/8 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1/3 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1/3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1/3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1/3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1/3 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1/3 \end{bmatrix} \bar{A}^4.$$

Calculating

$$\hat{w}^4 = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ y_1 \\ y_2 \\ y_3 \\ z_1 \\ z_2 \\ z_3 \end{bmatrix} = \left(I - \hat{A}^4\right)^{-1} \hat{b}^4,$$

we have the final solutions

$$\begin{array}{rclcll}
x_1 & = & 0.085v_1 & + & 0.071v_2 & = \text{weight of } C_1, \\
x_2 & = & 0.026v_1 & + & 0.038v_2 & = \text{weight of } C_2, \\
x_3 & = & 0.020v_1 & + & 0.022v_2 & = \text{weight of } C_3, \\
y_1 & = & 0.004v_1 & + & 0.005v_2 & = \text{weight of } A_1, \\
y_2 & = & 0.003v_1 & + & 0.003v_2 & = \text{weight of } A_2, \\
y_3 & = & 0.015v_1 & + & 0.015v_2 & = \text{weight of } A_3, \\
z_1 & = & 0.008v_1 & + & 0.008v_2 & = \text{weight of } Q_1, \\
z_2 & = & 0.003v_1 & + & 0.003v_2 & = \text{weight of } Q_2, \\
z_3 & = & 0.011v_1 & + & 0.011v_2 & = \text{weight of } Q_3.
\end{array}$$

See the results, we can state that x's have greater coefficients of v's than those of y's and z's, because C's receive the direct influences from V's on the contrary A's and Q's are only indirectly influenced from v's.

6. Conclusion and Future Extensions

This article includes the following results;

i) Section 2 shows that the unified evaluation model deals with the matrix A of (2.10) such as the comparison matrix C in AHP and the super matrix S in ANP. This is more general without any restriction such as the reciprocality or the stochasticness.

ii) It is shown in section 3.2 that the principal eigenvalue of A is the solution of the evaluation problem when A is irreducible, that is the evaluation structure is strongly connected, and that this is based on Sekitani and Yamaki principle [5].

iii) It is shown in section 3.1 that the solution stated in ii) coincides with that of Saaty's method by the limiting process.

iv) We develop the general solving method of the evaluation problem with the reducible matrix A. This is very new method we have never seen before. The main idea of the method is to solve (4.5) that is equivalent to

$$\min_{w>0} \max \left\{ \frac{\sum_{k=1}^{l-1} \bar{b}_i^{lk} \hat{w}^k + \bar{a}_i^l w}{M_i^l w_i}, \frac{M_i^l w_i}{\sum_{k=1}^{l-1} \bar{b}_i^{lk} \hat{w}^k + \bar{a}_i^l w} \,\middle|\, i = 1, \ldots, n(l) \right\}. \tag{6.1}$$

An optimal solution $\left(I - \hat{A}^l\right)^{-1} \hat{b}^l$ of (4.5) and (6.1) can be rewritten as $\left(\sum_{\nu=1}^{\infty} \left(\hat{A}^l\right)^{\nu}\right) \hat{b}^l$. The interpretation of $\left(\sum_{\nu=1}^{\infty} \left(\hat{A}^l\right)^{\nu}\right) \hat{b}^l$ is left to one of the future extension of this research.

v) We should consider some variations of the averaging principle those are suitable to the specified case studies. For example one of the variations is the least squared principle that is based on solving $\min \sum_{i=1}^{n(l)} \left| \left(\sum_{k=1}^{l-1} \bar{b}_i^{lk} \hat{w}^k + \bar{a}_i^l w \right) / M_i^l - w_i \right|^2$, or the total principle[7] that is to solve (4.5) replaced with M_i^l by 1.

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