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THE VARIABLE LEAD TIME STOCHASTIC INVENTORY MODEL WITH A FUZZY BACKORDER RATE

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Abstract Cost and operation of inventory depends a great deal on what happens to demand when the system is out of stock. In real inventory systems, it is more reasonable to assume that part of the excess demand is backordered and the rest is lost. However, the amount of backorders (or lost sales) often incurs disturbance due to various uncertainties. To incorporate this reality, this article attempts to apply the fuzzy set concepts to deal with the uncertain backorders and lost sales. The purpose of this paper is to modify Moon and Choi's continuous review inventory model with variable lead time and partial backorders by fuzzifying the backorder rate (or equivalently, fuzzifying the lost sales rate). We first consider the case where the lost sales rate is treated as the triangular fuzzy number. Then, through the statistical method for establishing the interval estimation of the lost sales rate, we construct a new fuzzy number, namely statistic-fuzzy number. For each fuzzy case, we investigate a computing schema for the modified continuous review inventory model and develop an algorithm to find the optimal inventory strategy.

1. Introduction

Traditionally, the economic order quantity (EOQ) model dealing with continuous review inventory problems assumed that demand during the stockout period is either completely backordered or completely lost; the lead time is viewed as a prescribed constant or a random variable, which there is not subject to control [13, 18]. However, these are not quite practical. In real markets, we can often observe that, when the inventory system is out of stock, some of the customers are willing to wait for their demand, while others may fill their demand from another source. And hence, for inventory models in which shortages are allowed, it is more reasonable to assume that some of the excess demand is backordered and the rest is lost. In literature, several authors (e.g., Montgomery *et al.* [11], Kim and Park [8], Ouyang *et al.* [14], Moon and Choi [12] and Hariga and Ben-Daya [6]) have presented the inventory models with partial backorders, specifically, Montgomery *et al.* [11] is among the first who formulated and solved the continuous review, stochastic demand inventory problem.

On the other hand, as pointed out in Tersine [19], lead time usually consists of the following components: order preparation, order transit, supplier lead time, delivery time and setup time. In some cases, these components can be accomplished earlier than the regular time if one is willing to pay extra costs; in other words, lead time is controllable. For an example, one may adopt the special delivery (by air) instead of ordinary delivery (by water) to shorten the delivery time. Obviously, the air freight rate is higher than the water freight rate, and hence more money can be spent to shorten lead time. Also, through the Japanese successful experiences of using Just-In-Time (JIT) production, the advantages and benefits associated with efforts to reduce lead time have been evidenced. Lead time reduction has received a lot of interest in recent years. Liao and Shyu [10] first presented a probabilistic inventory model in which the order quantity is predetermined and lead time

is a unique variable. Ben-Daya and Raouf [1] extended Liao and Shyu's [10] model by considering both lead time and order quantity as decision variables. Ouyang *et al.* [14] generalized Ben-Daya and Raouf's [1] model by allowing shortages with partial backorders. Recently, Moon and Choi [12] and Hariga and Ben-Daya [6] further extended Ouyang *et al.*'s [14] model by considering the reorder point as one of the decision variables.

We note that the underlying assumption in above partial backorder models, no matter with lead time reduction [6, 12, 14] or not [8, 11], is that the fraction of excess demand backordered (or lost) is a fixed constant. However, in the real situation, when stockout occurs many potential factors such as properties of products and/or image of selling shop may affect customers' wills of backorders. In other words, the amount of lost demand caused by stockout probably has a little disturbance due to various uncertainties. Therefore, if we express the fuzzy backorder (or lost sales) rate as the neighborhood of the fixed backorder (or lost sales) rate, then it will more match with the real situation.

In fact, the application of fuzzy set concepts on EOQ inventory models have been proposed by many authors (e.g., Park [15], Chen *et al.* [3], Yao and Lee [20], Roy and Maiti [16], Chang *et al.* [2], Lee and Yao [9]). Specifically, Yao and Lee [20] used the extension principle to solve the inventory model with shortages by fuzzifying the order quantity, in which the shortage quantity is a real variable. Later, Chang *et al.* [2] fuzzified the shortage quantity in the backorder model, where the order quantity is a real variable. Inventory model without backorder is discussed by Lee and Yao [9], who fuzzify the order quantity to a fuzzy number, and solve the economic order quantity with the extension principle. However, these studies [2, 3, 9, 15, 16, 20] are almost concentrated on the simple EOQ forms so that there has few applications in the real inventory systems. The purpose of this paper is to present a more extensive EOQ model to modify Moon and Choi's [12] model by fuzzifying the lost sales rate and to solve this new inventory model in the fuzzy sense.

In this paper, we study the continuous review (Q, r) inventory models with partial backorders, where the lead time is viewed as a controllable variable and two fuzziness of lost sales rate are introduced. Firstly, we express the lost sales rate as one of the widely used fuzzy numbers, namely the triangular fuzzy number. Then, by employing the statistical method we construct a confidence interval for the lost sales rate, and through it to establish the corresponding fuzzy number called the statistic-fuzzy number. It is noted that the statistical technology has often been utilized to solve the problem with uncertainty in many research areas including Operations Research. Furthermore, this paper investigates a computing schema for each fuzzy case and develops an algorithm procedure to find the optimal inventory strategy. Two examples are given to illustrate the results derived and concluding remarks are made.

2. Membership Function of the Fuzzy Total Cost

First of all, the following notations and assumptions are employed thoughout this paper so as to develop the proposed models.

Notations :

Q = order quantity
A = ordering cost
r = reorder point
L = length of lead time
D = annual demand rate
h = annual inventory holding cost per unit

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π = fixed penalty cost per unit short
π_0 = marginal profit per unit
β = fraction of the shortage that will be backordered, $0 \leq \beta \leq 1$
X = the lead time demand which has a probability density function (*p.d.f.*) $f(\cdot)$ with finite mean $D \cdot L$ and standard deviation $\sigma \cdot \sqrt{L}$, where σ denotes the standard deviation of the demand per unit time
$B(r)$ = the expected demand shortage at the end of the cycle

Assumptions

1. The reorder point r = expected demand during lead time + safety stock (SS), and SS = $k \cdot$ (standard deviation of lead time demand), i.e., $r = DL + k\sigma\sqrt{L}$, where k is the safety factor.
2. Inventory is continuously reviewed and replenishments are made whenever the inventory level falls to the reorder point r.
3. The lead time L has n mutually independent components. The ith component has a minimum duration a_i and normal duration b_i, and a crashing cost per unit time c_i. Furthermore, for convenience, we rearrange c_i such that $c_1 \leq c_2 \leq \cdots \leq c_n$. Then, it is clear that the reduction of lead time should first occur on component 1 (because it has the minimum unit crashing cost), and then component 2, etc.
4. If we let $L_0 = \sum_{j=1}^{n} b_j$ and L_i be the length of lead time with components $1, 2, \cdots, i$ crashed to their minimum duration, then L_i can be expressed as $L_i = \sum_{j=1}^{n} b_j - \sum_{j=1}^{i}(b_j - a_j)$, $i = 1, 2, \cdots, n$; and the lead time crashing cost $R(L)$ per cycle for a given $L \in [L_i, L_{i-1}]$, is given by

$$R(L) = c_i(L_{i-1} - L) + \sum_{j=1}^{i-1} c_j(b_j - a_j) \quad \text{and} \quad R(L_0) = 0.$$

For the model in which the order quantity, Q, reorder point, r, and lead time, L, are treated as decision variables, we will closely follow Moon and Choi [12]. Specifically, by assumptions 1–4, the total expected annual cost, which is composed of ordering cost, inventory holding cost, stockout cost and lead time crashing cost, is expressed by

$$C(Q, r, L) = A\frac{D}{Q} + h\left[\frac{Q}{2} + r - DL + (1-\beta)B(r)\right] + \frac{D}{Q}[\pi + \pi_0(1-\beta)]B(r) + \frac{D}{Q}R(L). \quad (1)$$

Now we attempt to modify Moon and Choi's [12] model by fuzzifying the backorder rate (or equivalently, fuzzifying the lost sales rate). For convenience, we first let $\delta \equiv 1 - \beta$ denote the lost sales rate. Therefore, for any $Q > 0, r > 0$ and $L > 0$, we may rewrite the expected annual total cost function (1) as follows

$$\begin{aligned} C_{(Q,r,L)}(\delta) &\equiv C(Q, r, L) \\ &= [A + R(L) + \pi B(r)]\frac{D}{Q} + h\left(\frac{Q}{2} + r - DL\right) + \delta\left(h + \frac{\pi_0 D}{Q}\right)B(r). \end{aligned} \quad (2)$$

Note that in above model, the lost sales rate δ during the planning horizon is assumed to be a fixed constant. However, when the inventory planning is completed, due to various uncertainties the lost sales rate in practical problem may be not equal to δ but just close to it. This scenario can be expressed in fuzzy language as " $\tilde{\delta}$ = the real lost sales rate is

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around δ ". Therefore, we would like to replace the lost sales rate δ by the fuzzy number $\tilde{\delta}$, and consider it as the triangular fuzzy number, $\tilde{\delta} = (\delta - \Delta_1, \delta, \delta + \Delta_2)$, where $0 < \Delta_1 < \delta$ and $0 < \Delta_2 \leq 1 - \delta$, Δ_1 and Δ_2 are determined by the decision-makers. Also, here we describe the membership function of $\tilde{\delta}$ as follows:

$$\mu_{\tilde{\delta}}(x) = \begin{cases} \dfrac{x - \delta + \Delta_1}{\Delta_1}, & \text{if} \quad \delta - \Delta_1 \leq x \leq \delta; \\ \dfrac{\delta + \Delta_2 - x}{\Delta_2}, & \text{if} \quad \delta \leq x \leq \delta + \Delta_2; \\ 0, & \text{otherwise.} \end{cases} \tag{3}$$

The pictorial sees Figure 1. Then the centroid (see, e.g. [4,p.336]) for $\mu_{\tilde{\delta}}(x)$ is given by

$$\delta^* = \frac{\int_{-\infty}^{\infty} x\mu_{\tilde{\delta}}(x)dx}{\int_{-\infty}^{\infty} \mu_{\tilde{\delta}}(x)dx} = \delta + \frac{1}{3}(\Delta_2 - \Delta_1). \tag{4}$$

We regard this value as the estimate of lost sales rate in the fuzzy sense.

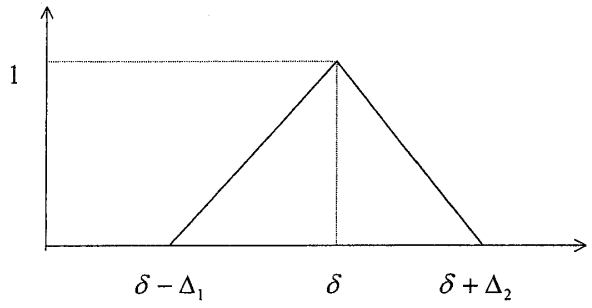

Figure 1: Triangular fuzzy number $\tilde{\delta}$

For any $Q > 0$, $r > 0$ and $L > 0$, we let $C_{(Q,r,L)}(x) = y\ (> 0)$. By extension principle [7, 21], the membership function of the fuzzy cost $C_{(Q,r,L)}(\tilde{\delta})$ is given by

$$\mu_{C_{(Q,r,L)}(\tilde{\delta})}(y) = \begin{cases} \sup\limits_{x \in C^{-1}_{(Q,r,L)}(y)} \mu_{\tilde{\delta}}(x), & \text{if} \quad C^{-1}_{(Q,r,L)}(y) \neq \emptyset; \\ 0, & \text{if} \quad C^{-1}_{(Q,r,L)}(y) = \emptyset. \end{cases} \tag{5}$$

From $C_{(Q,r,L)}(x) = y$ and equation (2), we get

$$[A + R(L) + \pi B(r)]\frac{D}{Q} + h\left(\frac{Q}{2} + r - DL\right) + x\left(h + \frac{\pi_0 D}{Q}\right)B(r) = y. \tag{6}$$

Consequently,

$$x = \frac{yQ - \left[WD + hQ\left(\dfrac{Q}{2} + r - DL\right)\right]}{(hQ + \pi_0 D)B(r)}, \tag{7}$$

where $W = A + R(L) + \pi B(r)$.

Therefore, from (3) and (7), the membership function of $C_{(Q,r,L)}(\tilde{\delta})$ can be written as

$$\mu_{C_{(Q,r,L)}(\tilde{\delta})}(y) = \begin{cases} \dfrac{yQ - \left[WQ + hQ\left(\dfrac{Q}{2} + r - DL\right)\right]}{(hQ + \pi_0 D)B(r)\Delta_1} - \dfrac{\delta - \Delta_1}{\Delta_1}, & y_1 \leq y \leq y_2; \\ \dfrac{\delta + \Delta_2}{\Delta_2} + \dfrac{\left[WQ + hQ\left(\dfrac{Q}{2} + r - DL\right)\right] - yQ}{(hQ + \pi_0 D)B(r)\Delta_2}, & y_2 \leq y \leq y_3; \\ 0, & \text{otherwise,} \end{cases} \tag{8}$$

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where

$$y_1 = W\frac{D}{Q} + h\left(\frac{Q}{2} + r - DL\right) + (\delta - \Delta_1)\left(h + \frac{\pi_0 D}{Q}\right)B(r),$$

$$y_2 = W\frac{D}{Q} + h\left(\frac{Q}{2} + r - DL\right) + \delta\left(h + \frac{\pi_0 D}{Q}\right)B(r),$$

and

$$y_3 = W\frac{D}{Q} + h\left(\frac{Q}{2} + r - DL\right) + (\delta + \Delta_2)\left(h + \frac{\pi_0 D}{Q}\right)B(r).$$

The pictorial of the membership function of $C_{(Q,r,L)}(\tilde{\delta})$ is shown in Figure 2.

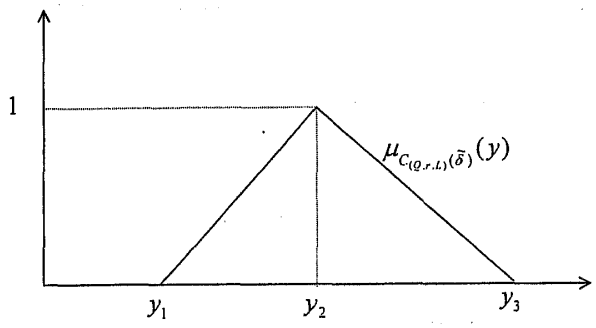

Figure 2: Triangular fuzzy number $C_{(Q,r,L)}(\tilde{\delta})$

We now derive the centroid of $\mu_{C_{(Q,r,L)}(\tilde{\delta})}(y)$ as follows:

$$M(Q,r,L) = \frac{\int_{-\infty}^{\infty} y\mu_{C_{(Q,r,L)}(\tilde{\delta})}(y)dy}{\int_{-\infty}^{\infty} \mu_{C_{(Q,r,L)}(\tilde{\delta})}(y)dy} = \frac{1}{3}(y_1 + y_2 + y_3)$$

$$= C(Q,r,L) + \frac{(\Delta_2 - \Delta_1)}{3}\left(h + \frac{\pi_0 D}{Q}\right)B(r),$$

which is an estimate of the expected annual total inventory cost in the fuzzy sense. Thus, we obtain the following property.

Property 1. For any $Q > 0, r > 0$ and $L > 0$, the estimate of the expected annual total inventory cost in the fuzzy sense is

$$M(Q,r,L) = C(Q,r,L) + \frac{(\Delta_2 - \Delta_1)}{3}\left(h + \frac{\pi_0 D}{Q}\right)B(r). \tag{9}$$

Moreover, if we let $S = \dfrac{(\Delta_2 - \Delta_1)}{3 \cdot C(Q,r,L)}\left(h + \dfrac{\pi_0 D}{Q}\right)B(r)$, then from (9) we obtain $\dfrac{M(Q,r,L) - C(Q,r,L)}{C(Q,r,L)} \times 100\% = S \times 100\%$, which implies

$$[M(Q,r,L) - C(Q,r,L)] \times 100\% = S \times C(Q,r,L) \times 100\%. \tag{10}$$

Remark 1

Case 1. If $\Delta_1 = \Delta_2$, then Figure 1 is an isosceles triangle and equation (9) reduces to $M(Q,r,L) = C(Q,r,L)$, this implies that the fuzzy case becomes the crisp case; i.e., the fixed lost sales rate inventory model is a special case of our new fuzzy lost sales rate inventory model.

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Case 2. If $\Delta_1 < \Delta_2$, then the triangle in Figure 1 is skewed to the right. In this case, $M(Q,r,L) > C(Q,r,L)$ and the increment of $M(Q,r,L)$ is $S\%$ of $C(Q,r,L)$ (from (10)).

Case 3. If $\Delta_1 > \Delta_2$, then the triangle in Figure 1 is skewed to the left. In this case, $M(Q,r,L) < C(Q,r,L)$ and the decrement of $M(Q,r,L)$ is $|S|\%$ of $C(Q,r,L)$ (from (10)).

3. Optimal Solution

This section investigates the optimal inventory strategy in the fuzzy sense for a situation where the lead time demand, X, follows a normal distribution with *p.d.f.* $f(\cdot)$, mean DL and standard deviation $\sigma\sqrt{L}$. We note that the reorder point $r = DL + k\sigma\sqrt{L}$ (assumption (i)) and the expected number of shortages at the end of the cycle $B(r) = \int_r^\infty (x-r)f(x)dx = \sigma\sqrt{L}\{\phi(k) - k[1-\Phi(k)]\} = \sigma\sqrt{L}\Psi(k)$, where $\phi(\cdot)$ and $\Phi(\cdot)$ denote the standard normal *p.d.f.* and *c.d.f.* (cumulative distribution function), respectively, and $\Psi(k) = \phi(k) - k[1-\Phi(k)]$.

Following the above result, we can allow the safety factor k as a decision variable instead of the reorder point r. Therefore, our problem of determining the optimal (Q,r,L) by minimizing (9) can be reduced to minimizing

$$\begin{aligned} M(Q,k,L) &= C(Q,k,L) + \frac{\Delta_2 - \Delta_1}{3}\left(h + \frac{\pi_0 D}{Q}\right)\sigma\sqrt{L}\Psi(k) \\ &= [A + R(L)]\frac{D}{Q} + h\left(\frac{Q}{2} + k\sigma\sqrt{L}\right) \\ &\quad + \left[\frac{\pi D}{Q} + \left(\delta + \frac{\Delta_2 - \Delta_1}{3}\right)\left(h + \frac{\pi_0 D}{Q}\right)\right]\sigma\sqrt{L}\Psi(k), \end{aligned} \tag{11}$$

over Q, k and L.

To solve this problem, we first note that $M(Q,k,L)$ is concave in $L \in [L_i, L_{i-1}]$ for fixed (Q,k) because

$$\frac{\partial^2 M(Q,k,L)}{\partial L^2} = -\frac{1}{4}hk\sigma L^{-\frac{3}{2}} - \frac{1}{4}\left[\frac{\pi D}{Q} + \left(\delta + \frac{\Delta_2 - \Delta_1}{3}\right)\left(h + \frac{\pi_0 D}{Q}\right)\right]\sigma L^{-\frac{3}{2}}\Psi(k) < 0.$$

Hence, for fixed (Q,k), the minimum expected annual total cost in fuzzy sense will occur at the end points of the interval $[L_i, L_{i-1}]$. On the other hand, it can be shown that $M(Q,k,L)$ is convex in (Q,k) for fixed $L \in [L_i, L_{i-1}]$ (see Appendix for the proof). Hence, for fixed $L \in [L_i, L_{i-1}]$, the minimum value of $M(Q,k,L)$ will occur at the point (Q,k), say (Q^*,k^*), which satisfying $\partial M(Q,k,L)/\partial Q = 0$ and $\partial M(Q,k,L)/\partial k = 0$, simultaneously. Solving these two equations result in

$$Q = \sqrt{\frac{2D}{h}\left\{A + R(L) + \left[\pi + \pi_0\left(\delta + \frac{\Delta_2 - \Delta_1}{3}\right)\right]\sigma\sqrt{L}\Psi(k)\right\}} \tag{12}$$

and

$$\Phi(k) = 1 - \frac{hQ}{\pi D + (hQ + \pi_0 D)\left(\delta + \dfrac{\Delta_2 - \Delta_1}{3}\right)}. \tag{13}$$

From equations (12) and (13), though it is difficult to find the closed-form solution of (Q^*,k^*), however, the optimal value of (Q^*,k^*) can be obtained using the iterative procedure

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(see, e.g. Hadley and Whitin [5]). Therefore, the following algorithm to find the optimal solutions for the order quantity, safety factor, and lead time can be developed.

Algorithm 1

Step 1. For given L_i, $i = 0, 1, 2, \cdots, n$, perform (i) to (iv).

(i) Start with $k_{i1} = 0$ and get $\Psi(k_{i1}) = 0.39894$ (which can be obtained by consulting the normal table $\phi(k_{i1}) = 0.39894$ and $\Phi(k_{i1}) = 0.5$).

(ii) Substituting $\Psi(k_{i1})$ into (12) evaluates Q_{i1}.

(iii) Utilizing Q_{i1} determines $\Phi(k_{i2})$ from (11), then finds k_{i2} by consulting the normal table, and hence $\Psi(k_{i2})$.

(iv) Repeat (ii) to (iii) until no change occurs in the values of Q_i and k_i. Denote the solution by (Q_i^*, k_i^*).

Step 2. For each (Q_i^*, k_i^*, L_i), $i = 0, 1, 2, \cdots, n$, calculate the corresponding fuzzy expected annual total inventory cost $M(Q_i^*, k_i^*, L_i)$ by utilizing (11).

Step 3. Find $\min_{i=0,1,2,\cdots,n} M(Q_i^*, k_i^*, L_i)$. If $M(Q_{\tilde{\delta}}, k_{\tilde{\delta}}, L_{\tilde{\delta}}) = \min_{i=0,1,2,\cdots,n} M(Q_i^*, k_i^*, L_i)$, then $(Q_{\tilde{\delta}}, k_{\tilde{\delta}}, L_{\tilde{\delta}})$ is the optimal solution in the fuzzy sense. Once $k_{\tilde{\delta}}$ and $L_{\tilde{\delta}}$ are obtained, the optimal reorder point $r_{\tilde{\delta}} = DL_{\tilde{\delta}} + k_{\tilde{\delta}}\sigma\sqrt{L_{\tilde{\delta}}}$ follows.

Example 1. In order to illustrate the above solution procedure, let us consider an inventory system with the data used in Moon and Choi ([12], which is the same as in Ouyang *et al.* [14]): $D = 600$ units per year, $A = \$200$ per order, $h = \$20$ per unit per year, $\pi = \$50$ per unit short, $\pi_0 = \$150$ per unit lost, $\sigma = 7$ units per week, and the lead time has three components with data shown in Table 1.

Table 1: Lead time data

Lead time component i	Normal duration b_i(days)	Minimum duration a_i(days)	Unit crashing cost c_i(\$ /day)
1	20	6	0.4
2	20	6	1.2
3	16	9	5.0

Here, we consider three cases: $(\Delta_1, \Delta_2) = (0.2, 0.2)$, $(\Delta_1, \Delta_2) = (0.1, 0.4)$, and $(\Delta_1, \Delta_2) = (0.4, 0.1)$. We solve each case for lost sales rate $\delta = 0.5$. The results of the solution procedure are summarized in Table 2.

From Table 2, when $\Delta_1 = \Delta_2 = 0.2$ (in this situation, the fuzzy case becomes the crisp case), by comparing $M(Q_i^*, r_i^*, L_i)$, $i = 0, 1, 2, 3$, we obtain the optimal solution $(Q_{\tilde{\delta}}, k_{\tilde{\delta}}, L_{\tilde{\delta}}) = (121, 72, 4)$ and the minimum expected annual total cost in fuzzy sense $M(Q_{\tilde{\delta}}, k_{\tilde{\delta}}, L_{\tilde{\delta}}) = \$2941.68$, which are the same as showed in Moon and Choi [12]. Moreover, when $\Delta_1 = 0.1$ and $\Delta_2 = 0.4$, i.e., the fuzzy number $\tilde{\delta} = (0.4, 0.5, 0.9)$, we have $(Q_{\tilde{\delta}}, r_{\tilde{\delta}}, L_{\tilde{\delta}}) = (121, 73, 4)$ and $M(Q_{\tilde{\delta}}, r_{\tilde{\delta}}, L_{\tilde{\delta}}) = \$2954.09$. Note that since $C(Q_s, r_s, L_s) = \$2941.68$ is the corresponding minimum expected annual total cost in the crisp case, and hence the absolute relative variation in the fuzzy sense for the minimum expected annual total cost is

$$R_M = \frac{|M(Q_{\tilde{\delta}}, r_{\tilde{\delta}}, L_{\tilde{\delta}}) - C(Q_s, r_s, L_s)|}{C(Q_s, r_s, L_s)} \times 100\% = \frac{|2954.09 - 2941.68|}{2941.68} \times 100\% = 0.42\%.$$

Similarly, for the case $\Delta_1 = 0.4$ and $\Delta_2 = 0.1$, i.e., the fuzzy number $\tilde{\delta} = (0.1, 0.5, 0.6)$, we have $(Q_{\tilde{\delta}}, r_{\tilde{\delta}}, L_{\tilde{\delta}}) = (121, 71, 4)$ and $M(Q_{\tilde{\delta}}, r_{\tilde{\delta}}, L_{\tilde{\delta}}) = \$2927.42$, and the absolute relative

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Table 2: Solution procedure of Algorithm 1 (L_i in weeks)

Δ_1	Δ_2	i	L_i	$R(L_i)$	Q_i^*	r_i^* (k_i^*)	$M(Q_i^*, r_i^*, L_i)$
0.2	0.2	0	8	0.0	117	129 (1.8689)	\$ 3090.09
		1	6	5.6	118	101 (1.8672)	2998.93
		2	4	22.4	121	72 (1.8555)	**2941.68**
		3	3	57.4	129	57 (1.8272)	3025.84
0.1	0.4	0	8	0.0	117	130 (1.9196)	3107.57
		1	6	5.6	118	102 (1.9179)	3014.07
		2	4	22.4	121	73 (1.9063)	**2954.09**
		3	3	57.4	129	57 (1.8786)	3036.69
0.4	0.1	0	8	0.0	118	128 (1.8104)	3070.01
		1	6	5.6	118	100 (1.8088)	2981.53
		2	4	22.4	121	71 (1.7969)	**2927.42**
		3	3	57.4	129	56 (1.7679)	3013.37

variation in the fuzzy sense for the minimum expected annual total cost is

$$R_M = \frac{|2927.42 - 2941.68|}{2941.68} \times 100\% = 0.48\%.$$

4. Using the Sample Data to Fuzzify the Lost Sales Rate

In general, the real lost sales rate δ is unknown in advance. In order to estimate the value of δ, intuitively, one may collect the random sample data of lost sales rate from past time, then compute the mean of the sample measurement (say $\bar{\delta}$) and use it as the estimate of δ. Such an issue belongs to the statistical problem. Moreover, though it can be shown that $\bar{\delta}$ is a good point estimator of δ, however, when the inventory planning is completed, the lost sales rate in practical problem may not equal to $\bar{\delta}$ but just close to it. This scenario can be described in fuzzy language as " $\tilde{\delta}^*$ = the real lost sales rate is around $\bar{\delta}$ ". Therefore, we need to combine the statistical and fuzzy technologies to deal with such an inventory problem. This section tackles this problem and the procedures are as follows.

Assume the actual lost sales rate δ (which can be regarded as the population mean of lost sales rate) is unknown, and suppose we have collected m random sample data of lost sales rate during past time, say $\delta_1, \delta_2, \cdots, \delta_m$, then the sample mean is $\bar{\delta} = \frac{1}{m}\sum_{i=1}^{m}\delta_i$, and the sample variance is $s^2 = \frac{1}{m-1}\sum_{i=1}^{m}(\delta_i - \bar{\delta})^2$. Furthermore, suppose the above sample data satisfy some certain statistical assumptions such as normality, then by the method for establishing the *interval estimation* of the parameter, we get the following $(1-\alpha)\times 100\%$ confidence interval for δ:

$$\left[\bar{\delta} - t_{m-1}(\alpha_1)\frac{s}{\sqrt{m}}, \bar{\delta} + t_{m-1}(\alpha_2)\frac{s}{\sqrt{m}}\right], \tag{14}$$

where $\alpha_1, \alpha_2 > 0$, $\alpha_1 + \alpha_2 = \alpha$, and $t_{m-1}(\alpha_i)$, $i = 1, 2$, is the tabulated upper α_i point of the t-distribution with $m-1$ degrees of freedom; that is, if T be a random variable distributed as t-distribution with $m-1$ degrees of freedom, then $t_{m-1}(\alpha_i)$ is the value that satisfies the following condition:

$$P\left[T > t_{m-1}(\alpha_i)\right] = \alpha_i, \quad i = 1, 2. \tag{15}$$

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Next, we take any point (denoted by $\bar{\delta}_0$) from the inside of above confidence interval (14). If $\bar{\delta}_0 = \bar{\delta}$, then the error of estimation $|\bar{\delta}_0 - \bar{\delta}| = 0$; in this case, the *confidence level* is viewed as 1. In contrast, the further the point $\bar{\delta}_0$ is from $\bar{\delta}$, the larger the error of estimation $|\bar{\delta}_0 - \bar{\delta}|$ to be, and hence, the smaller the *confidence level* will be given. If $\bar{\delta}_0$ is one of the end points of the confidence interval, then the error of estimation $|\bar{\delta}_0 - \bar{\delta}|$ is in the largest; in this case, the *confidence level* is viewed as 0. Thus, we can employ (14) to express the statistical-fuzzy lost sales rate $\tilde{\delta}^*$ as the following triangular fuzzy number:

$$\tilde{\delta}^* = \left(\bar{\delta} - t_{m-1}(\alpha_1)\frac{s}{\sqrt{m}}, \bar{\delta}, \bar{\delta} + t_{m-1}(\alpha_2)\frac{s}{\sqrt{m}}\right), \tag{16}$$

where $\alpha_1 + \alpha_2 = \alpha$. Note that the decision-makers can determine α_1 and α_2 so as to satisfy $\bar{\delta} - t_{m-1}(\alpha_1)\frac{s}{\sqrt{m}} > 0$.

Remark 2. We note that since the membership grade of (16) has the same property as the above *confidence level*, so using confidence level as membership grade to construct the statistic-fuzzy number (16) (corresponding to (14)) is feasible and validity.

The membership function of statistic-fuzzy lost sales rate $\tilde{\delta}^*$ is given by:

$$\mu_{\tilde{\delta}^*}(x) = \begin{cases} \dfrac{x - \bar{\delta} + t_{m-1}(\alpha_1)\frac{s}{\sqrt{m}}}{t_{m-1}(\alpha_1)\frac{s}{\sqrt{m}}}, & \text{if} \quad \bar{\delta} - t_{m-1}(\alpha_1)\dfrac{s}{\sqrt{m}} \le x \le \bar{\delta}; \\ \dfrac{\bar{\delta} + t_{m-1}(\alpha_2)\frac{s}{\sqrt{m}} - x}{t_{m-1}(\alpha_2)\frac{s}{\sqrt{m}}}, & \text{if} \quad \bar{\delta} \le x \le \bar{\delta} + t_{m-1}(\alpha_2)\dfrac{s}{\sqrt{m}}; \\ 0, & \text{otherwise.} \end{cases} \tag{17}$$

The pictorial sees Frgure 3. Then the centroid of $\mu_{\tilde{\delta}^*}(x)$ is

$$\delta^{**} = \bar{\delta} + \frac{1}{3}[t_{m-1}(\alpha_2) - t_{m-1}(\alpha_1)]. \tag{18}$$

We regard this value as the estimate of lost sales rate in the fuzzy sense. Obviously, $\delta^{**} > 0$ and δ^{**} belongs to the interval (14). For the special case $\alpha_1 = \alpha_2 = \alpha/2$, it gets $\delta^{**} = \bar{\delta}$.

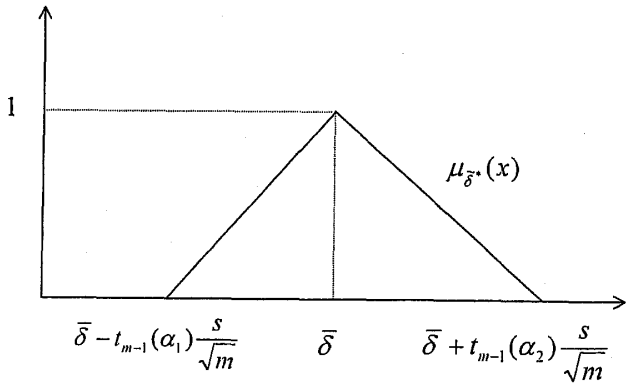

Figure 3: Triangular fuzzy number $\tilde{\delta}^*$

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We let $C_{(Q,r,L)}(x) = z$. By extension principle [7, 21], the membership function of the fuzzy cost $C_{(Q,r,L)}(\tilde{\delta}^*)$ is

$$\mu_{C_{(Q,r,L)}(\tilde{\delta}^*)}(z) = \begin{cases} \sup\limits_{x \in C^{-1}_{(Q,r,L)}(z)} \mu_{\tilde{\delta}^*}(x), & \text{if} \quad C^{-1}_{(Q,r,L)}(z) \neq \emptyset; \\ 0, & \text{if} \quad C^{-1}_{(Q,r,L)}(z) = \emptyset. \end{cases} \tag{19}$$

From $C_{(Q,r,L)}(x) = z$ and equations (2), (18) and (19), we obtain the membership function of $C_{(Q,r,L)}(\tilde{\delta}^*)$ as follows:

$$\mu_{C_{(Q,r,L)}(\tilde{\delta}^*)}(z) = \begin{cases} \dfrac{zQ - \left[WQ + hQ\left(\dfrac{Q}{2} + r - DL\right)\right]}{(hQ + \pi_0 D)B(r)t_{m-1}(\alpha_1)\dfrac{s}{\sqrt{m}}} - \dfrac{\bar{\delta} - t_{m-1}(\alpha_1)\dfrac{s}{\sqrt{m}}}{t_{m-1}(\alpha_1)\dfrac{s}{\sqrt{m}}}, & z_1 \leq z \leq z_2 \\ \dfrac{\bar{\delta} + t_{m-1}(\alpha_2)\dfrac{s}{\sqrt{m}}}{t_{m-1}(\alpha_2)\dfrac{s}{\sqrt{m}}} + \dfrac{\left[WQ + hQ\left(\dfrac{Q}{2} + r - DL\right)\right] - zQ}{(hQ + \pi_0 D)B(r)t_{m-1}(\alpha_2)\dfrac{s}{\sqrt{m}}}, & z_2 \leq z \leq z_3 \\ 0, & \text{otherwise,} \end{cases} \tag{20}$$

where

$$z_1 = W\frac{D}{Q} + h\left(\frac{Q}{2} + r - DL\right) + \left[\bar{\delta} - t_{m-1}(\alpha_1)\frac{s}{\sqrt{m}}\right]\left(h + \frac{\pi_0 D}{Q}\right)B(r),$$

$$z_2 = W\frac{D}{Q} + h\left(\frac{Q}{2} + r - DL\right) + \bar{\delta}\left(h + \frac{\pi_0 D}{Q}\right)B(r),$$

and

$$z_3 = W\frac{D}{Q} + h\left(\frac{Q}{2} + r - DL\right) + \left[\bar{\delta} + t_{m-1}(\alpha_2)\frac{s}{\sqrt{m}}\right]\left(h + \frac{\pi_0 D}{Q}\right)B(r).$$

Therefore, the centroid of $\mu_{C_{(Q,r,L)}(\tilde{\delta}^*)}(z)$ can be obtained and is given by

$$\begin{aligned} G(Q,r,L) &= \frac{\int_{-\infty}^{\infty} z\mu_{C_{(Q,r,L)}(\tilde{\delta}^*)}(z)dz}{\int_{-\infty}^{\infty} \mu_{C_{(Q,r,L)}(\tilde{\delta}^*)}(z)dz} = \frac{1}{3}(z_1 + z_2 + z_3) \\ &= C^*(Q,r,L) + \frac{1}{3}[t_{m-1}(\alpha_2) - t_{m-1}(\alpha_1)]\frac{s}{\sqrt{m}}\left(h + \frac{\pi_0 D}{Q}\right)B(r), \end{aligned}$$

where $C^*(Q,r,L) = [A + R(L) + \pi B(r)]\dfrac{D}{Q} + h\left(\dfrac{Q}{2} + r - DL\right) + \bar{\delta}\left(h + \dfrac{\pi_0 D}{Q}\right)B(r)$. Thus, we get the following property.

Property 2. For any $Q > 0$, $r > 0$ and $L > 0$, and given $\alpha_1 > 0, \alpha_2 > 0$ and $\alpha_1 + \alpha_2 = \alpha$, the estimate of the expected annual total inventory cost in the fuzzy sense is

$$G(Q,r,L) = EAC^*(Q,r,L) + \frac{1}{3}[t_{m-1}(\alpha_2) - t_{m-1}(\alpha_1)]\frac{s}{\sqrt{m}}\left(h + \frac{\pi_0 D}{Q}\right)B(r). \tag{21}$$

Note that the difference between $EAC^*(Q,r,L)$ and $EAC(Q,r,L)$ (defined in (2)) is that the lost sales rate δ in (2) is replaced by the sample mean $\bar{\delta}$.

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Remark 3. If we let $S^* = \dfrac{[t_{m-1}(\alpha_2) - t_{m-1}(\alpha_1)]}{3EAC^*(Q,r,L)} \dfrac{s}{\sqrt{m}} \left(h + \dfrac{\pi_0 D}{Q}\right) B(r)$, then from (21) we obtain $\dfrac{G(Q,r,L) - EAC^*(Q,r,L)}{EAC^*(Q,r,L)} \times 100\% = S^* \times 100\%$, which implies

$$[G(Q,r,L) - EAC^*(Q,r,L)] \times 100\% = S^* \times EAC^*(Q,r,L) \times 100\%. \tag{22}$$

Thus, we have the following results.

(i) If $\alpha_1 = \alpha_2 = \alpha/2$, then $t_{m-1}(\alpha_1) = t_{m-1}(\alpha_2)$, which implies $G(Q,r,L) = EAC^*(Q,r,L)$. That is, the total cost $EAC^*(Q,r,L)$ obtained by point estimate $\bar{\delta}$ is consistent with the total cost $G(Q,r,L)$ obtained by fuzzy number $\tilde{\delta}^*$ defined in (16).

(ii) If $0 < \alpha_2 < \alpha_1 < 1$, then $t_{m-1}(\alpha_1) < t_{m-1}(\alpha_2)$, which implies $G(Q,r,L) > EAC^*(Q,r,L)$, and the increment of $G(Q,r,L)$ is $S^*\%$ of $EAC^*(Q,r,L)$.

(iii) If $0 < \alpha_1 < \alpha_2 < 1$, then $t_{m-1}(\alpha_1) > t_{m-1}(\alpha_2)$, which implies $G(Q,r,L) < EAC^*(Q,r,L)$, and the decrement of $G(Q,r,L)$ is $|S^*|\%$ of $EAC^*(Q,r,L)$.

Now, we investigate the optimal inventory strategy in the fuzzy sense for the case where the lead time demand follows a normal distribution with mean DL and standard deviation $\sigma\sqrt{L}$. By the same arguments as in section 3, we obtain the expected annual total inventory cost $G(Q,r,L)$ in fuzzy sense as follows:

$$\begin{aligned} G(Q,k,L) = {} & [A + R(L)]\frac{D}{Q} + h\left(\frac{Q}{2} + k\sigma\sqrt{L}\right) \\ & + \left\{\frac{\pi D}{Q} + \left[\bar{\delta} + \frac{t_{m-1}(\alpha_2) - t_{m-1}(\alpha_1)}{3}\frac{s}{\sqrt{m}}\right] \right. \\ & \left. \times \left(h + \frac{\pi_0 D}{Q}\right)\right\} \sigma\sqrt{L}\Psi(k), \qquad L \in [L_i, L_{i-1}]. \end{aligned} \tag{23}$$

Now we seek to minimize $G(Q,k,L)$ by optimizing over Q,k and L. Once again, the approach employed in the previous section is utilized to solve this problem. We can show that $G(Q,k,L)$ is concave in $L \in [L_i, L_{i-1}]$ for fixed (Q,k). Hence, for fixed (Q,k), the minimum expected annual total cost in fuzzy sense will occur at the end points of the interval $[L_i, L_{i-1}]$. On the other hand, it can be shown that $G(Q,k,L)$ is convex in (Q,k) for fixed $L \in [L_i, L_{i-1}]$ (the proof is similar to that given in Appendix). Then upon solving $\partial G(Q,k,L)/\partial Q = 0$ and $\partial G(Q,k,L)/\partial k = 0$, we obtain

$$Q = \sqrt{\frac{2D}{h}\left\{A + R(L) + \left[\pi + \pi_0\left(\bar{\delta} + \frac{t_{m-1}(\alpha_2) - t_{m-1}(\alpha_1)}{3}\frac{s}{\sqrt{m}}\right)\right]\sigma\sqrt{L}\Psi(k)\right\}} \tag{24}$$

and

$$\Phi(k) = 1 - \frac{hQ}{\pi D + (hQ + \pi_0 D)\left(\bar{\delta} + \dfrac{t_{m-1}(\alpha_2) - t_{m-1}(\alpha_1)}{3}\dfrac{s}{\sqrt{m}}\right)}. \tag{25}$$

Thus, for given $\alpha_1 > 0, \alpha_2 > 0$ and $\alpha_1 + \alpha_2 = \alpha$, we can establish the following algorithm to find the optimal solutions for order quantity, safety factor and lead time. The convergence of the iterative procedure can be verified by using the graphical technique (see, e.g. Hadley and Whitin [5]).

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Algorithm 2

Step 1. Collected m sample data of lost sales rate, say $\delta_1, \delta_2, \cdots, \delta_m$, and then evaluate sample mean $\bar{\delta} = \frac{1}{m}\sum_{i=1}^{m}\delta_i$ and sample standard deviation $s = \sqrt{\frac{1}{m-1}\sum_{i=1}^{m}(\delta_i - \bar{\delta})^2}$. In addition, for given α_1 and α_2 ($\alpha_1 + \alpha_2 = \alpha$), consulting the t-distribution table to find the values of $t_{m-1}(\alpha_i)$ and $t_{m-1}(\alpha_2)$, where $t_{m-1}(\alpha_i)$ is the upper α_i point of the t-distribution with $m-1$ degrees of freedom, $i = 1, 2$.

Step 2. For given L_i, $i = 0, 1, 2, \cdots, n$, perform (i) to (iv).

(i) Start with $k_{i1} = 0$ and get $\Psi(k_{i1}) = 0.39894$.

(ii) Substituting $\Psi(k_{i1})$ into (24) evaluates Q_{i1}.

(iii) Utilizing Q_{i1} determines $\Phi(k_{i2})$ from (25), then finds k_{i2} by consulting the normal table, and hence $\Psi(k_{i2})$.

(iv) Repeat (ii) to (iii) until no change occurs in the values of Q_i and k_i. Denote the solution by $(\hat{Q}_i^*, \hat{k}_i^*)$.

Step 3. For each $(\hat{Q}_i^*, \hat{k}_i^*, L_i)$, $i = 0, 1, 2, \cdots, n$, calculate the corresponding fuzzy expected annual total cost $G(\hat{Q}_i^*, \hat{k}_i^*, L_i)$ by utilizing (23).

Step 4. Find $\min_{i=0,1,2,\cdots,n} G(\hat{Q}_i^*, \hat{k}_i^*, L_i)$.

If $G(Q_{\tilde{\delta}^*}, k_{\tilde{\delta}^*}, L_{\tilde{\delta}^*}) = \min_{i=0,1,2,\cdots,n} G(\hat{Q}_i^*, \hat{k}_i^*, L_i)$, then $(Q_{\tilde{\delta}^*}, k_{\tilde{\delta}^*}, L_{\tilde{\delta}^*})$ is the optimal solution in the fuzzy sense. When $k_{\tilde{\delta}^*}$ and $L_{\tilde{\delta}^*}$ are obtained, the optimal reorder point $r_{\tilde{\delta}^*} = DL_{\tilde{\delta}^*} + k_{\tilde{\delta}^*}\sigma\sqrt{L_{\tilde{\delta}^*}}$ is followed.

Example 2. We use the same data as in Example 1, but assume that the random sample of size 6 yields the sample mean of lost sales rate $\bar{\delta} = 0.5$ and sample standard deviation $s = 0.195$. We determine the optimal inventory strategy in fuzzy sense for the case where $\alpha_1 = 0.1$ and $\alpha_2 = 0.05$ (α_1 and α_2 are determined by the decision-makers, and here we take these two values to illustrate the results of proposed model). Consulting the t-distribution table, we find $t_5(0.1) = 1.476$ and $t_5(0.05) = 2.015$. The results of the solution procedure are summarized in Table 3.

Table 3: Solution procedure of Algorithm 2 (L_i in weeks)

i	L_i	$R(L_i)$	$\hat{Q}_i^*$	$\hat{r}_i^*$ $(\hat{k}_i^*)$	$G(\hat{Q}_i^*, \hat{k}_i^*, L_i)$
0	8	0	117	129 (1.8766)	\$ 3092.73
1	6	5.6	118	101 (1.8749)	3001.22
2	4	22.4	121	72 (1.8632)	**2943.56**
3	3	57.4	129	57 (1.8350)	3027.48

From Table 3, by comparing $G(\hat{Q}_i^*, \hat{k}_i^*, L_i)$, $i = 0, 1, 2, 3$, we find that the optimal strategy $(Q_{\tilde{\delta}^*}, k_{\tilde{\delta}^*}, L_{\tilde{\delta}^*}) = (121, 72, 4)$, which leads to the minimum expected annual total inventory cost \$ 2943.56.

5. Concluding Remarks

In this paper, we present the modified continuous review inventory model with partial backorders in the fuzzy sense to accommodate the practical situation. Two fuzziness of lost

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sales rates are introduced. In section 2, we discuss how to apply the fuzzy set concepts to deal with the problem in which no statistical data can be used. On the other hand, when there are available statistical data, we discuss how to combine the statistical and fuzzy technologies to deal with such a problem in section 4. We note that the optimal solution derived from the total cost function in [12] may not match the real situation, while using the optimal solution derived from the total cost through properties 1 and 2 in this article does.

This article assumes that the demand during lead time follows a normal distribution. In general, information about the distributional form of lead time demand is often limited. In future research, it would be interested to relax the normal demand assumption to consider the distribution free case where only the first two moments of lead time demand are known. The minimax distribution free approach as proposed by Scarf [17] can be utilized to solve such a problem.

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Appendix

The proof of $M(Q,k,L)$ is convex in (Q,k) for fixed $L \in [L_i, L_{i-1}]$.

For a given value of L, we first obtain the Hessian matrix $\mathbf{H}$ as follows

$$\mathbf{H} = \begin{bmatrix} \dfrac{\partial^2 M(Q,k,L)}{\partial Q^2} & \dfrac{\partial^2 M(Q,k,L)}{\partial Q \partial k} \\ \dfrac{\partial^2 M(Q,k,L)}{\partial k \partial Q} & \dfrac{\partial^2 M(Q,k,L)}{\partial k^2} \end{bmatrix}, \tag{A.1}$$

where

$$\frac{\partial^2 M(Q,k,L)}{\partial Q^2} = \frac{2D[A+R(L)]}{Q^3} + \frac{2D}{Q^3}\left[\pi + \pi_0\left(\delta + \frac{\Delta_2 - \Delta_1}{3}\right)\right]\sigma\sqrt{L}\Psi(k),$$

$$\frac{\partial^2 M(Q,k,L)}{\partial Q \partial k} = \frac{\partial^2 M(Q,k,L)}{\partial k \partial Q} = \frac{D}{Q^2}\left[\pi + \pi_0\left(\delta + \frac{\Delta_2 - \Delta_1}{3}\right)\right]\sigma\sqrt{L}[1 - \Phi(k)],$$

and

$$\frac{\partial^2 M(Q,k,L)}{\partial k^2} = \left[\frac{\pi D}{Q} + \left(\delta + \frac{\Delta_2 - \Delta_1}{3}\right)\left(h + \frac{\pi_0 D}{Q}\right)\right]\sigma\sqrt{L}\phi(k).$$

The first principal minor of $\mathbf{H}$ is

$$\begin{aligned} |H_{11}| &= \frac{\partial^2 M(Q,k,L)}{\partial Q^2} \\ &= \frac{2D[A+R(L)]}{Q^3} + \frac{2D}{Q^3}\left[\pi + \pi_0\left(\delta + \frac{\Delta_2 - \Delta_1}{3}\right)\right]\sigma\sqrt{L}\Psi(k) > 0, \end{aligned} \tag{A.2}$$

because the term $\delta + \dfrac{\Delta_2 - \Delta_1}{3}$ (which is the estimate of lost sales rate in the fuzzy sense (see equation (4) in text)) is a positive value.

The second principal minor of $\mathbf{H}$ is

$$|H_{22}| = \frac{\partial^2 M(Q,r,L)}{\partial Q^2} \times \frac{\partial^2 M(Q,r,L)}{\partial k^2} - \left[\frac{\partial^2 M(Q,r,L)}{\partial Q \partial k}\right]^2$$

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$$
\begin{aligned}
&= \left\{\frac{2D[A+R(L)]}{Q^3} + \frac{2D}{Q^3}\left[\pi + \pi_0\left(\delta + \frac{\Delta_2 - \Delta_1}{3}\right)\right]\sigma\sqrt{L}\Psi(k)\right\} \\
&\quad \times \left[\frac{\pi D}{Q} + \left(\delta + \frac{\Delta_2 - \Delta_1}{3}\right)\left(h + \frac{\pi_0 D}{Q}\right)\right]\sigma\sqrt{L}\phi(k) \\
&\quad - \left\{\frac{D}{Q^2}\left[\pi + \pi_0\left(\delta + \frac{\Delta_2 - \Delta_1}{3}\right)\right]\sigma\sqrt{L}[1 - \Phi(k)]\right\}^2 \\
&= \frac{2D}{Q^3}\left\{A + R(L) + \left[\pi + \pi_0\left(\delta + \frac{\Delta_2 - \Delta_1}{3}\right)\right]\sigma\sqrt{L}\Psi(k)\right\} \\
&\qquad \times h\left(\delta + \frac{\Delta_2 - \Delta_1}{3}\right)\sigma\sqrt{L}\phi(k) \\
&\quad + \frac{2D^2}{Q^4}[A + R(L)]\left[\pi + \pi_0\left(\delta + \frac{\Delta_2 - \Delta_1}{3}\right)\right]\sigma\sqrt{L}\phi(k) \\
&\quad + \left\{\frac{D}{Q^2}\left[\pi + \pi_0\left(\delta + \frac{\Delta_2 - \Delta_1}{3}\right)\right]\sigma\sqrt{L}\right\}^2 \{2\Psi(k)\phi(k) - [1 - \Phi(k)]^2\}. \qquad \text{(A.3)}
\end{aligned}
$$

From (A.3), we see that to prove $|H_{22}| > 0$ it only needs to prove the term in the last brace, $2\Psi(k)\phi(k) - [1 - \Phi(k)]^2$, is positive since all other terms are positive. Let $I(k) \equiv 2\Psi(k)\phi(k) - [1 - \Phi(k)]^2$. By taking the derivative of $I(k)$, we get $\dfrac{dI(k)}{dk} = -2k\Psi(k)\phi(k) < 0$, which means $I(k)$ is a decreasing functron of k. Moreover, by checking the normal table, we obtain $I(0) = 2 \cdot 0.3989 \cdot 0.3989 - (1 - 0.5)^2 = 0.0683$ and $\lim_{k\to\infty} I(k) = 0$. Therefore, $I(k) > 0$ for $k \in [0, \infty)$, the behavior of $I(k)$ sees Figure A-1. Thus, we have $|H_{22}| > 0$.

From the results: $|H_{11}| > 0$ and $|H_{22}| > 0$, it can be concluded that $M(Q, k, L)$ is convex in (Q, k) for fixed $L \in [L_i, L_{i-1}]$.

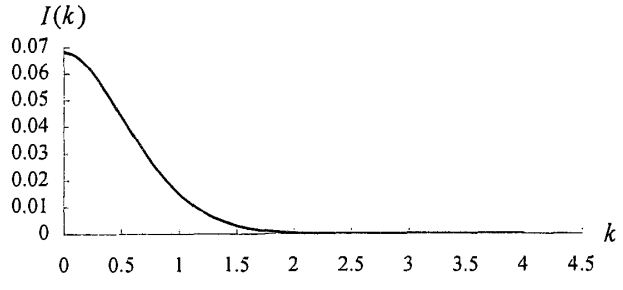

Figure A-1: Behavior of $I(k)$

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