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A COMPOUND DEPENDABILITY MEASURE ARISING FROM SEMI-MARKOV RELIABILITY MODEL

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Abstract A compound dependability measure is proposed and analyzed under the Markovian assumption by Csenki (1996). We extend his analysis to the semi-Markov setting and obtain the corresponding closed form expression. The analysis is quite simple and transform-free. The resulting formula has a clear probabilistic interpretation. As a numerical example, we explore the behavior of a multi-mode system with periodic maintenance.

1. Introduction

In this paper, we study a compound dependability measure arising from a semi-Markov reliability model. Each state is classified as functional (up) or under repair (down). In addition there is one absorbing state corresponding to an irrecoverable or complete failure. A number of reliability/performance measures are explored in the literature. Analysis based on Markovian assumption can be found in Rubino and Sericola [9] and Sumita et al. [10] among others. Semi-Markovian reliability/performance models includes Ciardo et al. [2], Kulkarni et al. [6], Masuda [7] and Masuda and Sumita [8]. In a recent article [4], Csenki proposes an interesting dependability measure which incorporates both the cost and the benefit of the system. Specifically, the joint distribution of the cumulative up times and the number of repairs before the irrecoverable failure is examined. Underlying his model is the Markovian assumption.

There are two main points in this paper. First, we extend the analysis of Csenki [4], who derives a closed form expression of the compound dependability measure under the Markovian assumption. We relax the Markovian assumption and extend the model to the semi-Markov setting. Second, our analysis is transform-free. On the other hand, the analysis of Csenki [4] involves lengthy algebraic manipulations in the Laplace transform domain. Furthermore, our derivation is purely probabilistic and the correspondingly closed form formula has a clear probabilistic interpretation.

The analysis in Section 2 is quite simple once we construct an appropriate new semi-Markov process with an extended state space. It is shown that the compound dependability measure of the original process is merely the absorption probability of the new semi-Markov process. Thus, a closed form formula can be obtained by applying the standard analysis of Markov renewal process. In Section 3, we demonstrate that our formula can be numerically implemented using a simple example. Specifically, we numerically explore the dependability measure of multi-mode system with periodic preventive maintenance.

2. Model and Analysis

Let Y_t be the semi-Markov process governed by a semi-Markov matrix $\mathbf{A}(\mathbf{x}) = [\mathbf{A_{ij}}(\mathbf{x})]$ with finite state space N, which is partitioned into three non-empty subsets G, B, $\{\omega\}$. We interpret G (B) as the set of functional (repairable, respectively) states while ω as the state of irrecoverable failure. All states in $G \cup B$ communicate and are transient. ω is the unique absorbing state of Y_t.

Csenki (1996) defines a compound dependability measure in the following manner. Let T_G and N_B, respectively, be the cumulative time spent by Y_t in G and the cumulative number of transitions of Y_t from G to B. For notational convenience, let $P_i(\cdot) = P(\cdot|Y_0 = i)$ and define the conditional expectation $E_i(\cdot)$ similarly. Of interest is the joint distribution of these random variables

$$F_i(t,n) = P_i(T_G \le t, N_B \le n).$$

This captures the dependency between the cost factor (N_B) and the benefit factor (T_G) of the system, and reflects the dependability of the system.

To keep the analysis tidy, assume for a moment that the system is functional at time 0, i.e., $Y_0 \in G$. To evaluate $F_i(t,n)$ for a specific n, construct a new process $\hat{Y}_s$, $s \ge 0$, on the extended state space $(\{0,1,\cdots,n\} \times (G \cup B)) \cup \{\omega, \omega'\}$ in the following manner:

- X_t is same as Y_t except that all states in B are instantaneous, i.e., the dwell time of Y_t at each state in B is replaced by zero.
- J_t counts the total number of transitions of X_t from G to B up to time t.
- $\hat{Y}_t = \begin{cases} (J_t, X_t) & \text{if } J_t \le n \text{ and } X_t \ne \omega; \\ \omega & \text{if } J_t \le n \text{ and } X_t = \omega; \\ \omega' & \text{if } J_t > n. \end{cases}$

Since the dwell time in B is eliminated, $X_t = \omega$ if and only if $T_G \le t$. Thus, the desired probability is given as

$$F_i(t,n) = P_i(\hat{Y}_t = \omega).$$

It can be seen that $\hat{Y}_t$ is also a semi-Markov process. Then, the evaluation of $P_i(T_G \le t, N_B \le n)$, a seemingly nontrivial task, is reduced to the evaluation of the absorption probability of the new semi-Markov process. We note however that it is not a new idea to use an absorption probability for evaluating an entity of interest, see e.g. Keilson [5]. In the following, the semi-Markov matrix governing $\hat{Y}_t$ is identified and a closed form expression of $F_i(t,n)$ is derived.

For notational convenience we suppress the set B of instantaneous states, and redefine $\hat{Y}_t$ on $(\{0,1,\cdots,n\} \times G) \cup \{\omega,\omega'\}$. In the rest of the paper, $\hat{Y}_t$ always refers to the process defined on $(\{0,1,\cdots,n\} \times G) \cup \{\omega,\omega'\}$. Let $\mathbf{A_{GG}}(\mathbf{x}) = [\mathbf{A_{ij}}(\mathbf{x})]_{\mathbf{i,j \in G}}$. Similar notations will be used for other sub-matrices and vectors. From the construction of the new process, it can be seen that the behavior of $\hat{Y}_t$ in the set of transient states $\{0,1,\cdots,n\} \times G$ is described by the following defective semi-Markov matrix $\mathbf{C_n}(\mathbf{x})$ of order $(n+1)\,|G|$:

$$\mathbf{C_n}(\mathbf{x}) = \begin{pmatrix} \mathbf{A_{GG}}(\mathbf{x}) & \mathbf{B}^+(\mathbf{x}) & 0 & \cdots & 0 \\ 0 & \mathbf{A_{GG}}(\mathbf{x}) & \mathbf{B}^+(\mathbf{x}) & \ddots & \vdots \\ 0 & 0 & \mathbf{A_{GG}}(\mathbf{x}) & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & \mathbf{B}^+(\mathbf{x}) \\ 0 & \cdots & 0 & 0 & \mathbf{A_{GG}}(\mathbf{x}) \end{pmatrix} \quad (1)$$

where $\mathbf{B}^+(\mathbf{x})$ is defined by

$$\begin{aligned}\mathbf{B}^+(\mathbf{x}) &= \left[B^+_{ij}(x)\right]_{i,j\in G}\\ B^+_{ij}(x) &= P(\text{the first transition occurs to } (1,j) \text{ at}\\ &\qquad\qquad \text{some time } \tau \le x \mid (J_0, X_0) = (0,i)).\end{aligned}$$

The expression of $\mathbf{B}^+(\mathbf{x})$ in terms of $\mathbf{A}(\mathbf{x})$ can be found from the following argument. For (J_s, X_s) to reach $(1,j)$ within one transition at some time $\tau \le x$ given that $(J_0, X_0) = (0,i)$, the following sequence of events of Y_t must take place:

- the original process Y_t makes a transition from i to some $k \in B$ at some time $\tau \le x$,
- Y_t visits some state $l \in B$ after making m transitions within B for some $m \in \{0, 1, \cdots\}$, and
- a transition of Y_t takes place from l to $j \in G$.

Thus, it can be seen that

$$B^+_{ij}(x) = \sum_{m=0}^{\infty}\sum_{k\in B}\sum_{l\in B} A_{ik}(x)\tilde{A}^{(m)}_{BB:kl}\tilde{A}_{lj}$$

where $\tilde{A}_{ij} = \lim_{x\to\infty} A_{ij}(x)$ and $\tilde{A}^{(m)}_{BB:kl}$ is the (k,l)th component of $\left(\tilde{\mathbf{A}}_{\mathbf{BB}}\right)^m$. In the matrix notation,

$$\mathbf{B}^+(\mathbf{x}) = \mathbf{A}_{\mathbf{GB}}(\mathbf{x})\left(\mathbf{I} - \tilde{\mathbf{A}}_{\mathbf{BB}}\right)^{-1}\tilde{\mathbf{A}}_{\mathbf{BG}} \tag{2}$$

where $\mathbf{I}$ is an identity matrix of appropriate order.

The transition behavior of $\hat{Y}_t$ from $\{0, 1, \cdots n\} \times G$ to ω is described in terms of $\mathbf{A}(\mathbf{x})$ as follows. For $i \in G$ and $m = 0, 1, \cdots, n-1$, let

$$\begin{aligned}B_{i\omega}(x) &= P(\text{the dwell time of } \hat{Y}_s \text{ in } (m,i) \text{ is less than } x\\ &\qquad\qquad \text{and the following state is } \omega \mid J_s = m, X_s = i).\end{aligned}$$

This definition is meaningful since the right hand side clearly is independent of m for $m \le n-1$. $B_{i\omega}(x)$ consists of two components. The first component corresponds to the direct transition of the original process Y_t from i to ω, which is given by $A_{i\omega}(x)$. The second component corresponds to the transition of Y_t to ω through B. Thus, following the argument similar to the derivation of $B^+_{ij}(x)$ in (2), one has

$$B_{i\omega}(x) = A_{i\omega}(x) + \sum_{m=0}^{\infty}\sum_{k\in B}\sum_{l\in B} A_{ik}(x)\tilde{A}^{(m)}_{BB:kl}\tilde{A}_{l\omega}.$$

In the vector notation,

$$\mathbf{B}_{\mathbf{G}\omega}(\mathbf{x}) = \mathbf{A}_{\mathbf{G}\omega}(\mathbf{x}) + \mathbf{A}_{\mathbf{GB}}(\mathbf{x})\left(\mathbf{I} - \tilde{\mathbf{A}}_{\mathbf{BB}}\right)^{-1}\tilde{\mathbf{A}}_{\mathbf{B}\omega}. \tag{3}$$

When $(J_s, X_s) = (n,i)$, $\hat{Y}_s$ terminates in ω' if Y_t makes another transition to B. Thus,

$$\begin{aligned}&P(\text{the dwell time of } \hat{Y}_s \text{ in } (n,i) \text{ is less than } x \text{ and the}\\ &\qquad \text{following state is } \omega \mid J_s = n, X_s = i) = A_{i\omega}(x).\end{aligned} \tag{4}$$

Define the Markov renewal function associated with (J_s, X_s) by

$$\begin{aligned}\mathbf{R(t)} &= (\mathbf{R_{km}(t)})_{k,m=0}^{n}, \\ \mathbf{R_{km}(t)} &= (R_{km:ij}(t))_{i,j\in G}, \\ R_{km:ij}(t) &= E(\text{the number of visits by } (J_s, X_s) \\ &\quad \text{to } (m,j) \text{ in time interval } [0,t] \mid (J_0, X_0) = (k,i)).\end{aligned}$$

Note that $\mathbf{R_{km}(t)} = \mathbf{0}$ for $k > m$ and that only $\mathbf{R_{0m}(t)}$, $m = 0, 1, \cdots, n$, will be used in the following analysis. For $\hat{Y}_t = \omega$ to happen given $Y_0 = i \in G$, the process (J_s, X_s) has to visit (m, j) for some $m \in \{0, 1, \cdots, n\}$ and $j \in G$ at some time $\tau \le t$, followed by a transition from (m, j) to ω within $t - \tau$ time units. Thus, from (3) and (4), the desired probability is given by

$$F_i(t,n) = \sum_{m=0}^{n-1} \sum_{j\in G} \int_0^t \mathrm{d}R_{0m:ij}(\tau) B_{j\omega}(t-\tau) + \sum j \in G \int_0^t \mathrm{d}R_{0n:ij}(\tau) A_{j\omega}(t-\tau),$$

or

$$\mathbf{F_G(t,n)} = \sum_{\mathbf{m=0}}^{\mathbf{n-1}} \int_0^{\mathbf{t}} \mathbf{dR_{0m}}(\tau)\mathbf{B_{G\omega}(t-\tau)} + \int_0^{\mathbf{t}} \mathbf{dR_{0n}}(\tau)\mathbf{A_{G\omega}(t-\tau)}. \tag{5}$$

For the purpose of numerical evaluation, the following recursive formula of $\mathbf{F_G(t,n)}$ is useful.

$$\begin{aligned}\mathbf{F_G(t,n)} &= \mathbf{F_G(t,n-1)} + \int_0^{\mathbf{t}} \mathbf{dR_{0,n-1}}(\tau)\mathbf{A_{GB}(t-\tau)} \left(\mathbf{I} - \tilde{\mathbf{A}}_{\mathbf{BB}}\right)^{-1} \tilde{\mathbf{A}}_{\mathbf{B}\omega} \\ &\quad + \int_0^{t} \mathbf{dR_{0n}}(\tau)\mathbf{A_{G\omega}(t-\tau)}.\end{aligned} \tag{6}$$

It now remains to derive $\mathbf{R_{0m}(t)}$ for obtaining a closed form expression of $P_i(T_B \le t, N_B \le n)$, $i \in G$. Since $\mathbf{R(t)} = \sum_{\mathbf{m=0}}^{\infty} \mathbf{C_n^{(m)}(t)}$ (see e.g. Çinlar [3]) where $\mathbf{C_n^{(m)}}$ is the m-fold matrix convolution of $\mathbf{C_n}$ with itself, it can be seen from the structure of $\mathbf{C_n(t)}$ in (1) that

$$\mathbf{R_{0m}(t)} = \left(\mathbf{B_0} * \mathbf{B^+}\right)^{(\mathbf{m})} * \mathbf{B_0(t)}, \quad \mathbf{m} = \mathbf{0}, \mathbf{1}, \cdots, \mathbf{n}, \tag{7}$$

where

$$\mathbf{B_0(t)} = \sum_{\mathbf{k=0}}^{\infty} \mathbf{A_{GG}^{(k)}(t)}, \tag{8}$$

and $*$ represents a convolution operation.

We now turn our attention to the case where the original process starts from B. We first note that given $Y_0 = i \in B$

- the probability that Y_s terminates in ω before ever visiting G is given by the ith component of $\left(\mathbf{I} - \tilde{\mathbf{A}}_{\mathbf{BB}}\right)^{-1} \tilde{\mathbf{A}}_{\mathbf{B}\omega}$, and
- the probability that Y_s visits G for the first time at $j \in G$ is given by the (i, j)th component of $\left(\mathbf{I} - \tilde{\mathbf{A}}_{\mathbf{BB}}\right)^{-1} \tilde{\mathbf{A}}_{\mathbf{BG}}$.

As long as Y_s stays within B starting from $Y_0 = i \in B$, both the cumulative up time and the failure count do not increase. Thus $\mathbf{F_B(t,n)} = (\mathbf{F_i(t,n)})_{\mathbf{i\in B}}$ can be expressed in terms of $\mathbf{F_G(t,n)}$ as

$$\mathbf{F_B(t,n)} = \left(\mathbf{I} - \tilde{\mathbf{A}}_{\mathbf{BB}}\right)^{-1} \tilde{\mathbf{A}}_{\mathbf{B}\omega} + \left(\mathbf{I} - \tilde{\mathbf{A}}_{\mathbf{BB}}\right)^{-1} \tilde{\mathbf{A}}_{\mathbf{BG}}\mathbf{F_G(t,n)}. \tag{9}$$

In summary, the compound performability measure $F_i(t,n) = P_i(T_G \le t, N_B \le n)$ is given by (5) and (9) where $\mathbf{R_{0m}(t)}$ and $\mathbf{B_{G\omega}(t)}$ are given by (7) and (3), respectively.

3. Multi-mode System with Periodic Preventive Maintenance

In this section, we consider a multi-mode system with preventive maintenance modeled as a semi-Markov process. The system transitions are depicted in Figure 1. The system has two operational modes, normal (state 1) and degraded (state 2), one maintenance state (state 3) and one irrecoverable failure state (state 4). The behavior of the system is as follows.

- The system in the normal state fails with probability P_F or becomes degraded with probability $1 - P_F$ after X units of time where the distribution of X is Erlang with parameter (m, λ).
- The system in the degraded mode is on service and also under repair. The repair time is exponentially distributed with parameter μ_R. At the end of the repair period, the system is brought back to the normal mode. The system just after repair is as good as a new one. While the system is in the degraded state, it may fail with constant failure rate μ_F.
- The system in the normal mode is taken out of service for periodic preventive maintenance, which takes place as soon as it completes c units of time continuously in the normal mode. The maintenance takes d units of time. The system just after the preventive maintenance is as good as a new system.
- The irrecoverable failure is the unique absorbing state of the system.
- All the random variables are independent.

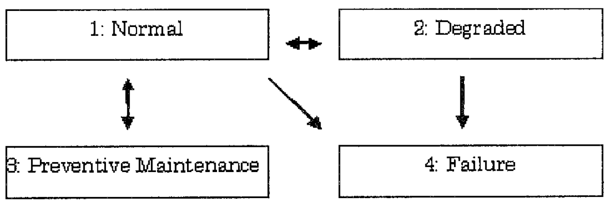

Figure 1: System Transitions

We note that, since Erlang distribution has an increasing failure rate, it is meaningful to perform preventive maintenance that makes the system as good as a new one.

Let G be the distribution function of X and $\bar{G}$ the corresponding survival function. Let $\mu = \mu_R + \mu_F$. Then, the semi-Markov matrix $\mathbf{A}(\mathbf{x})$ of Y_t is given by the following:

$$
\begin{aligned}
A_{12}(x) &= (1 - P_F)\left(G(x)U(c - x) + G(c)U(x - c)\right), \\
A_{13}(x) &= \bar{G}(c)U(x - c), \\
A_{14}(x) &= P_F\left(G(x)U(c - x) + G(c)U(x - c)\right), \\
A_{21}(x) &= \frac{\mu_R}{\mu}\left(1 - \exp(-\mu x)\right), \\
A_{24}(x) &= \frac{\mu_F}{\mu}\left(1 - \exp(-\mu x)\right), \\
A_{31}(x) &= U(x - d),
\end{aligned}
\tag{10}
$$

and all the other entries of $\mathbf{A}(\mathbf{x})$ are zero where $U(t)$ is the step function defined by $U(t) = 1$ if $t \geq 0$ and $U(t) = 0$ otherwise. Of interest is the joint distribution of the cumulative up time T_G and the number N_B of preventive maintenance before the irrecoverable failure. In other words, we set $G = \{1, 2\}$, $B = \{3\}$ and $\omega = 4$.

In the actual numerical experiment, the following parameters are used:

$$m = 2, \lambda = 10, P_F = 0.5, c = 20, d = 1, \mu_R = 1, \mu_F = 0.1. \qquad (11)$$

We assume that the system is new and in the normal mode at time zero, i.e., $Y_0 = 1$. The evaluation of the exact formulas (6), (7) and (3) is done symbolically in the Laplace transform domain. The numerical method employed here is quite straightforward, however, the recursive formula (6) makes a significant contribution toward the numerical efficiency. The resulting Laplace transform $\int_0^\infty e^{-st}F_1(n,t)dt$ is then inverted back to the real domain using the transform inversion method of Weeks [11] implemented by Cheng et al. [1]. All the procedures are implemented on Mathematica 2.2.

In Figures 2, $P_0(T_G \le t, N_B \le n)$ for $n = 0, 1, \cdots 5$ is depicted. To see the impact of N_B on T_G, the conditional distribution $P_0(T_G \le t \mid N_B = n)$ is also given, see Figure 3. Clearly, given $N_B = n$, $T_G \ge nc$ with probability 1, which is observed in Figure 3. Also, T_G given $N_B = n$ is stochastically increasing in n for the set of parameters (11), which would be consistent with our intuition.

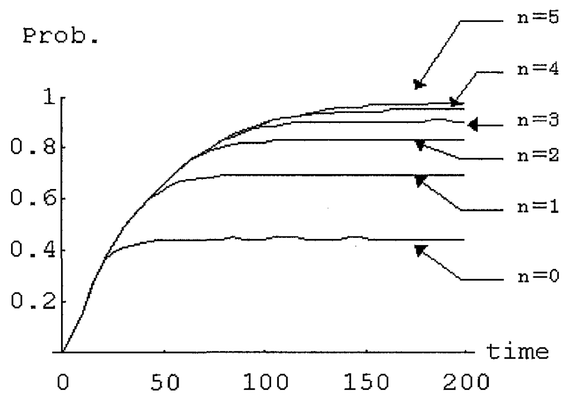

Figure 2: $P_0(T_G \le t, N_B \le n)$

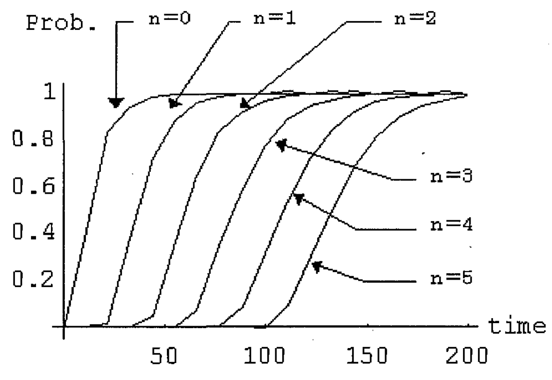

Figure 3: $P_0(T_G \le t \mid N_B = n)$

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