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A GENERALIZED OPTIMUM REQUIREMENT SPANNING TREE PROBLEM WITH A MONGE-LIKE PROPERTY

Tsutomu Anazawa
Sapporo University

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Abstract We consider a generalized optimum requirement spanning tree problem (GORST problem) which is an extension of the problem studied by Hu. In the GORST problem, the degrees of vertices are restricted and the objective function is generalized. We will show that a particular tree T^*, which is obtained by a sort of greedy algorithm but is explicitly definable, is a solution to the GORST problem when a condition similar to the Monge property is satisfied. Also, we will define a problem of finding a tree network which minimizes the probability that a request of communication is not realized when the network has k failures (called a "k-failure problem"), and show that T^* is an explicit solution to the k-failure problem for any k when the maximum degree constraint is imposed and the Monge-like property is satisfied.

1. Introduction

We begin by introducing the *optimum requirement spanning tree problem* (ORST problem) studied by Hu [7], which has motivated our studies. Let $V = \{0, 1, \ldots, n-1\}$ be a set of n vertices, $\binom{V}{2}$ the set of all pairs of distinct vertices in V, and $\mathcal{T}$ the whole set of undirected spanning trees on V. A tree $T \in \mathcal{T}$ with an edge set E is denoted by $T = (V, E)$, and the edge $e \in E$ connecting two vertices $v, u \in V$ is denoted by $e = (v, u)$. Assume that a nonnegative value r_{vu} is given to each pair $\{v, u\} \in \binom{V}{2}$, where $r_{vu} = r_{uv}$ holds. Hu [7] defined an ORST as a tree $T \in \mathcal{T}$ which minimizes

$$f(T) = \sum_{\{v,u\} \in \binom{V}{2}} d(v, u; T) r_{vu},$$

where $d(v, u; T)$ is the length of the path between v and u on T. ORSTs can be regarded as communication networks of tree type with the minimum average cost when the cost of communication between v and u is proportional to $d(v, u; T)$ and r_{vu} denotes the frequency of communication between v and u. Hu [7] showed that a tree minimizing f is obtained by the Gomory-Hu algorithm [5] when the degrees of vertices are *not* restricted.

The author and his colleagues have extended the ORST problem in various manners. Anazawa, Kodera and Jimbo [2, 3] considered the problem of finding a tree $T \in \mathcal{T}$ which minimizes f under the constraint that, for each vertex v, the degree of v in T denoted by $\deg(v; T)$ cannot exceed a given integer l_v, that is,

$$\deg(v; T) \leq l_v \text{ holds for all } v \in V, \tag{1}$$

where

$$l_0 \geq l_1 \geq \cdots \geq l_{n-1} \geq 1 \quad \text{and} \quad \sum_{v=0}^{n-1} l_v \geq 2(n-1) \tag{2}$$

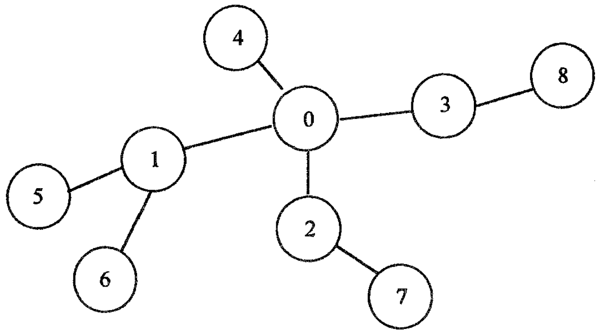

Figure 1: T^* for $n = 9$ and $l_0 = 4, l_1 = 3, l_2 = \cdots = l_8 = 2$.

are assumed. (The problem for $l_0 = l_1 = \cdots = l_{n-1} = n-1$ is equivalent to Hu's one.) And they showed that if

- a positive value p_v is assigned to each vertex $v \in V$,
- $p_0 \geq p_1 \geq \cdots \geq p_{n-1} > 0$ is satisfied (only strictly-decreasingness is assumed in [2]), and
- $r_{vu} = cp_v p_u$ holds for all $\{v,u\} \in \binom{V}{2}$ (where c is a positive constant),

then a particular tree $T^* = (V, E^*)$ is an explicit solution to the problem. The definition of T^* is as follows: Assuming that

$$l_v \geq 1 \text{ holds for all } v \in V \text{ and } \sum_{v=0}^{\nu-1} l_v > 2(\nu - 1) \text{ holds for all } \nu \in \{1, 2, \ldots, n-1\} \quad (3)$$

(condition (2) implies condition (3), which is proved in Appendix 1), we set $s_{-1} = 0$, $s_u = \sum_{v=0}^{u} l_v - u$ $(u = 0, 1, \ldots, n-1)$ and let N be the minimum integer satisfying $n - 1 \leq s_{N-1}$; also we define a function π on a set $\{1, 2, \ldots, n-1\}$ by

$$\pi(v) = \begin{cases} u & \text{if } s_{u-1} + 1 \leq v \leq s_u \text{ for } u = 0, 1, 2, \ldots, N-2 \\ N-1 & \text{if } s_{N-2} + 1 \leq v \leq n-1 \end{cases},$$

and let $E^* = \{e_1, e_2, \ldots, e_{n-1}\}$ where $e_v = (\pi(v), v)$ $(v = 1, 2, \ldots, n-1)$. Then we obtain $T^* = (V, E^*)$. Appendix 1 shows that if condition (3) is satisfied then π is definable and T^* surely is a tree. Roughly speaking, the tree T^* is constructed by the following procedure: First, to vertex 0, connect the remaining vertices by ascending order of vertex number as many as possible; secondly, to vertex 1, connect the remaining vertices by the same order as many as possible; and continue to connect the remaining vertices in the same manner until all n vertices are connected. This procedure can be regarded as a sort of "greedy algorithm". An example of T^* (for $n = 9$ and $l_0 = 4, l_1 = 3, l_2 = \cdots = l_8 = 2$) is shown by Figure 1. Anazawa et al. [2, 3] also gave another interpretation of ORSTs as follows: They minimize the probability that a request of communication is not realized when there is one failure on a vertex or an edge (the probability for k failures will be shown in Section 5).

Anazawa [1] considered the problem of minimizing f under constraint (1) satisfying $l_0 = l_1 = \cdots = l_{n-1} = L \geq 2$ (where L is a commonly-given integer), and showed that T^* is a unique explicit solution under the conditions that

- $r_{vu} > r_{vu'}$ holds for all $v, u, u' \in V$ $(u \neq v, u' \neq v, u < u')$, and
- $r_{vu} + r_{v'u'} > r_{vu'} + r_{v'u}$ holds for all $v, v', u, u' \in V$ $(v < v', u < u')$ such that r_{vu}, $r_{v'u'}$, $r_{vu'}$ and $r_{v'u}$ are all defined.

Since the above inequalities hold if $r_{vu} = cp_v p_u$ $(c > 0)$ and $p_0 > p_1 > \cdots > p_{n-1} > 0$ are satisfied, the conditions of $\{r_{vu}\}$ assumed by Anazawa [1] are more general than those considered by Anazawa et al. [2].

The aim of this paper is to generalize the problems and results discussed in the literatures [1, 2, 3]. Let $g(x)$ be an arbitrary real-valued function of real variable x such that it is monotone nondecreasing on $[0, n-1]$, and consider a problem of finding a tree $T \in \mathcal{T}$ which minimizes a function

$$f_g(T) = \sum_{\{v,u\}\in\binom{V}{2}} g(d(v,u;T))r_{vu}$$

subject to constraint (1) satisfying (2). We call this problem a *generalized optimum requirement spanning tree problem* (GORST problem), and a solution to this problem an *f_g-optimum tree*. Our main assertion on the GORST problem in this paper is that T^* is an f_g-optimum tree if $r_{vu} \geq r_{vu'}$ holds for all $v, u, u' \in V$ $(u \neq v, u' \neq v, u < u')$ and $r_{vu} + r_{v'u'} \geq r_{vu'} + r_{v'u}$ holds for all $v, v', u, u' \in V$ $(v < v', u < u')$ such that r_{vu}, $r_{v'u'}$, $r_{vu'}$ and $r_{v'u}$ are all defined. Further, by introducing *dummy vertices* $\{v|v \geq n\}$ and setting $r_{vu} = 0$ if v or u is dummy, the main assertion can be described more simply as follows:

Main Theorem *If $\{r_{vu}\}$ satisfies*

$$r_{vu} + r_{v'u'} \geq r_{vu'} + r_{v'u} \tag{4}$$

for all 4-tuple $\{v, v', u, u'\}$ $(v < v', u < u')$ such that r_{vu}, $r_{v'u'}$, $r_{vu'}$ and $r_{v'u}$ are all defined, then T^* defined above is f_g-optimum.

Remark (a) By setting $v' \geq n$ in inequality (4), we have $r_{vu} \geq r_{vu'}$. Also, we easily find that inequality (4) holds if $r_{vu} = cp_v p_u$ $(c > 0)$ and $p_0 \geq p_1 \geq \cdots \geq p_{n-1} \geq p_n = \cdots = 0$ are satisfied. Therefore, the condition of $\{r_{vu}\}$ in Main Theorem is more general than those in the literatures [1, 2, 3]. (b) It is easy to see that if $r_{vu} + r_{v'u'} > r_{vu'} + r_{v'u}$ holds for some 4-tuple $\{v, v', u, u'\}$ $(v < v', u < u')$ then both v and u are non-dummy. In fact, it follows from $r_{vu} + r_{v'u'} > 0$ and $r_{vu} \geq r_{vu'} \geq r_{v'u'} \geq 0$ that $r_{vu} > 0$ holds.

It is of interest that condition (4) is closely related to the Monge property. A $m \times n$ matrix $C = [c_{vu}]$ is called a *Monge matrix* if it satisfies the *Monge property*

$$c_{vu} + c_{v'u'} \leq c_{vu'} + c_{v'u} \text{ for all } 0 \leq v < v' \leq m-1,\ 0 \leq u < u' \leq n-1. \tag{5}$$

The property is named after the French mathematician Gaspard Monge, and is rediscovered by Hoffman [6] (compactly reviewed by Pferschy et al. [8] and Deineko et al. [4]). It is well-known that, in the classical Hitchcock transportation problem, if the cost matrix is Monge then a feasible solution obtained by the north-west-corner rule is optimum for any feasible demand and supply vectors. Also, Monge matrices make some NP-hard problems (ex. travelling salesman problem) efficiently solvable (see [8]). If C has unspecified elements and satisfies the first inequality in (5) for all 4-tuple $\{v, v', u, u'\}$ $(v < v', u < u')$ such that c_{vu}, $c_{v'u'}$, $c_{vu'}$ and $c_{v'u}$ are all specified, then C is called an *incomplete Monge matrix*. For $\{r_{vu}\}$ satisfying the condition in Main Theorem, if $n \times n$ matrix C is defined so as to satisfy $c_{vu} = ar_{n-v-1,n-u-1} + b$ (where a (< 0) and b are arbitrary constants) with diagonal elements unspecified, then C is a symmetric incomplete Monge matrix.

In this paper, after giving mathematical preliminaries in Section 2, we will show some properties of the tree T^* in Section 3. The proof of Main Theorem will be given in Section 4.

As an example of the GORST problem, we will formulate in Section 5 a "k-failure problem" which is an extension of the "one-failure problem" discussed by Anazawa et al. [2, 3], and show that T^* is an explicit solution to the k-failure problem for any k ($0 < k \leq 2n-1$) when constraint (1) with (2) is imposed and the condition in Main Theorem is satisfied.

2. Preliminaries

Throughout this paper, we use the following notation. For a graph $G = (V, E)$ and a subset $U \subset V$, a subgraph $G \cap U$ is defined by $G' = (U, E')$, where $E' = \{(v,u) \in E | v,u \in U\}$; while a subgraph $G \backslash U$ is defined by $G'' = (V \backslash U, E'')$, where $E'' = \{(v,u) \in E | v,u \in V \backslash U\}$.

For a rooted tree $T \in \mathcal{T}$, if vertex v lies on a path $(\text{root}, \ldots, u)$ of T, then v is called the $d(v,u;T)$-th *ancestor* of u (note that any vertex is the 0-th ancestor of itself); especially the first ancestor of u is called the *parent* of u. As to T^* defined in Section 1, $\pi(v)$ is the parent of v for $v = 1, \ldots, n-1$ if vertex 0 is the root. Also, let $\chi(v) = \{u | v \text{ is the parent of } u\}$. Further, the *level* of v is defined by $d(v, \text{root}; T)$.

For a path $P = (u_1, u_2, \ldots, u_k)$ of a tree $T = (V, E) \in \mathcal{T}$, let F be a forest defined by

$$F = (V, E \setminus \{(u_1,u_2), (u_2,u_3), \ldots, (u_{k-1},u_k)\}),$$

and $T(u_i) = (V(u_i), E(u_i))$ ($i = 1, \ldots, k$) the connected components of F each of which contains u_i.

An edge (v,u) such that v or u is a dummy vertex is called a *dummy edge*. For a tree $T = (V,E)$, we will sometimes construct another tree $\tilde{T} = (\tilde{V}, \tilde{E})$ satisfying $\tilde{V} = V \cup \{\text{dummy vertices}\}$ and $\tilde{E} = E \cup \{\text{dummy edges}\}$, i.e. $\tilde{T} \setminus \{\text{dummy vertices}\} = T$. Then it is obvious that $f_g(\tilde{T}) = f_g(T)$ holds. When constructing a dummies-added tree $\tilde{T} = (\tilde{V}, \tilde{E})$, we do not restrict the degrees of vertices in $\tilde{V}$.

For a tree $T = (V,E) \in \mathcal{T}$ satisfying (1) with (2) and a path $P = (u_1, \ldots, u_k)$ ($k = 2$ or 3) of T, we define an isomorphism σ_P. Let $T(u_i) = (V(u_i), E(u_i))$ be defined for P, and $\tilde{T}(u_i) = (\tilde{V}(u_i), \tilde{E}(u_i))$ ($i = 1, k$) be obtained by adding dummies to $T(u_i) = (V(u_i), E(u_i))$ ($i = 1, k$) so that $\tilde{T}(u_1)$ and $\tilde{T}(u_k)$ can be isomorphic and the underlying isomorphism $\sigma_P : \tilde{V}(u_1) \to \tilde{V}(u_k)$ can satisfy the following two:

(i) $\sigma_P(u_1) = u_k$.

(ii) For any $v \in \tilde{V}(u_1)$, if $\chi(v)$ has at least one dummy vertex, then $\chi(\sigma_P(v))$ has no dummy vertices, where we regard u_1 as the root of $\tilde{T}(u_1)$ and u_k as that of $\tilde{T}(u_k)$.

We call such an isomorphism σ_P a *forced isomorphism* for P. Appendix 2 shows that σ_P can be defined for any tree $T \in \mathcal{T}$ and any path $P = (u_1, \ldots, u_k)$ ($k = 2$ or 3) of T. Also, letting $\tilde{T} = (\tilde{V}, \tilde{E})$ where

$$\tilde{V} = \tilde{V}(u_1) \cup \left(\bigcup_{i=2}^{k-1} V(u_i) \right) \cup \tilde{V}(u_k) \text{ and } \tilde{E} = \tilde{E}(u_1) \cup \left(\bigcup_{i=2}^{k-1} E(u_i) \right) \cup \tilde{E}(u_k) \cup \left(\bigcup_{i=1}^{k-1} \{(u_i, u_{i+1})\} \right),$$

we consider the following transformation of $\tilde{T}$ which may reduce the f_g value: Let $V_C = \{v \in \tilde{V}(u_1) | v > \sigma_P(v)\}$, and exchange v and $\sigma_P(v)$ for all $v \in V_C$. We call such a transformation *biasing* with respect to σ_P. Further, let $\tilde{T}'$ be a tree obtained from $\tilde{T}$ by biasing and $T' = \tilde{T}' \setminus \{\text{dummy vertices}\}$. (An example of constructing T' from T is illustrated by Figure 2.) The following lemma assures us that T' is also a tree belonging to $\mathcal{T}$ and satisfies constraint (1).

Lemma 1 *For $\tilde{T}$ and σ_P defined above and for an arbitrary vertex $v \in \tilde{V}(u_1)$, let*

$$N = \{w \in \chi(v) | w \leq n-1\}, \quad N' = \{w' \in \chi(\sigma_P(v)) | w' \leq n-1\},$$

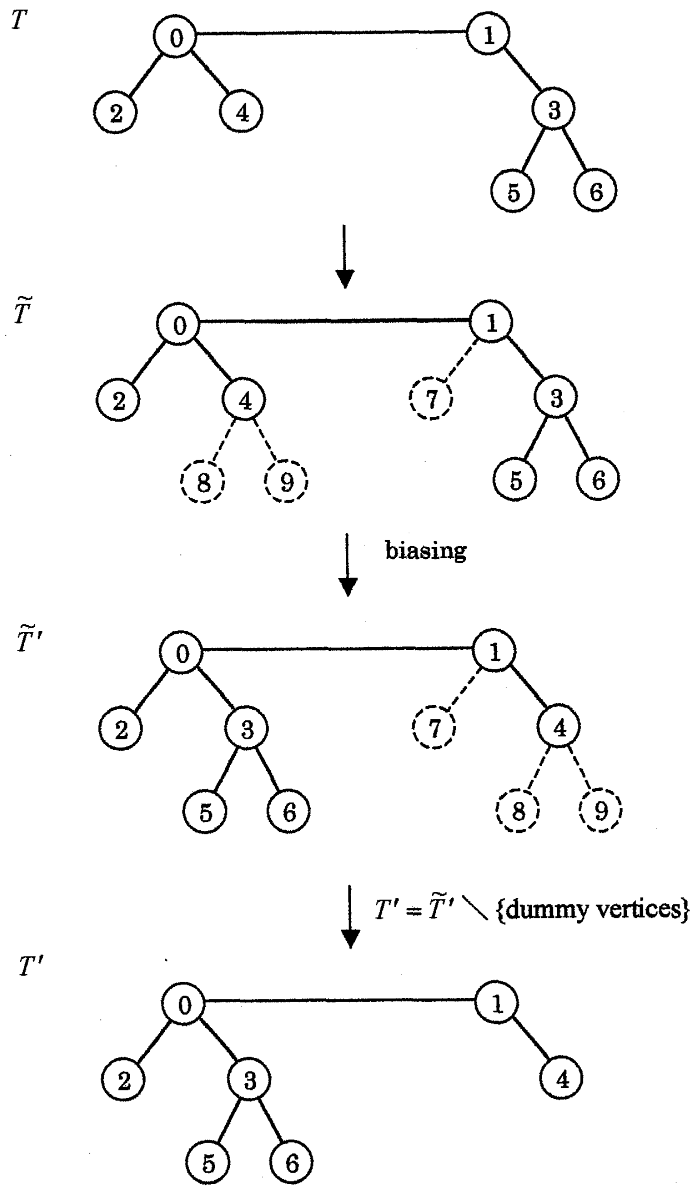

Figure 2: An example of constructing T' from T, where broken lines denote dummy vertices and edges. In T, we choose a path $P = (u_1, u_2)$ with $u_1 = 0$ and $u_2 = 1$, and define $T(u_1) = (V(u_1), E(u_1))$ and $T(u_2) = (V(u_2), E(u_2))$ for P where $V(u_1) = \{0, 2, 4\}$ and $V(u_2) = \{1, 3, 5, 6\}$. T has two isomorphic dummies-added trees $\tilde{T}(u_1) = (\tilde{V}(u_1), \tilde{E}(u_1))$ and $\tilde{T}(u_2) = (\tilde{V}(u_2), \tilde{E}(u_2))$ where $\tilde{V}(u_1) = \{0, 2, 4, 8, 9\}$ and $\tilde{V}(u_2) = \{1, 3, 5, 6, 7\}$. The underlying forced isomorphism σ_P is defined by setting $\sigma_P(0) = 1$, $\sigma_P(2) = 7$, $\sigma_P(4) = 3$, $\sigma_P(8) = 5$ and $\sigma_P(9) = 6$. Hence, we have $V_C = \{4, 8, 9\}$.

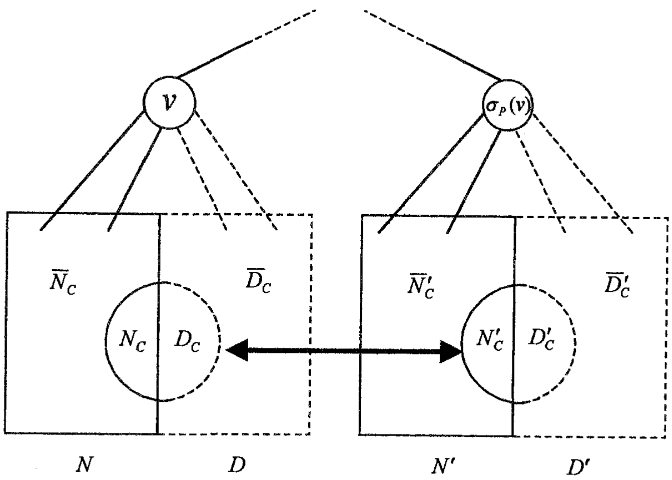

Figure 3: The inclusion relation of N, D, N', D' and their subsets. In this figure, the left-right arrow indicates that the elements of $N_C \cup D_C$ and those of $N'_C \cup D'_C$ are exchanged by biasing with respect to σ_P.

$$N_C = N \cap V_C, \bar{N}_C = N \setminus N_C, N'_C = N' \cap \sigma_P(V_C) \text{ and } \bar{N}'_C = N' \setminus N'_C,$$

where $\sigma_P(X) = \{\sigma_P(x) \in \tilde{V}(u_k) | x \in X\}$ *for* $X \subset \tilde{V}(u_1)$. *Then we find that*

(i) *if* $v < n \leq \sigma_P(v)$, *then* $|\bar{N}_C| + |N'_C| \leq l_v - 1$ *and* $|\bar{N}'_C| + |N_C| = 0$ *hold,*

(ii) *if* $\sigma_P(v) < n \leq v$, *then* $|\bar{N}_C| + |N'_C| \leq l_{\sigma_P(v)} - 1$ *and* $|\bar{N}'_C| + |N_C| = 0$ *hold,*

(iii) *if* $v < \sigma_P(v) < n$, *then* $|\bar{N}_C| + |N'_C| \leq l_v - 1$ *and* $|\bar{N}'_C| + |N_C| \leq l_{\sigma_P(v)} - 1$ *hold,*

(iv) *if* $\sigma_P(v) < v < n$, *then* $|\bar{N}_C| + |N'_C| \leq l_{\sigma_P(v)} - 1$ *and* $|\bar{N}'_C| + |N_C| \leq l_v - 1$ *hold.*

Proof Let $D = \chi(v) \setminus N$, $D' = \chi(\sigma_P(v)) \setminus N'$,

$$D_C = D \cap V_C, \bar{D}_C = D \setminus D_C, D'_C = D' \cap \sigma_P(V_C) \text{ and } \bar{D}'_C = D' \setminus D'_C.$$

The inclusion relation of N, D, N', D' and their subsets is illustrated by Figure 3. Note that the following five relations are always satisfied: $|N| + |D| = |N'| + |D'|$, $|N_C| + |D_C| = |N'_C| + |D'_C|$, $|\bar{N}_C| + |\bar{D}_C| = |\bar{N}'_C| + |\bar{D}'_C|$, $|N| \leq l_v - 1$ and $|N'| \leq l_{\sigma_P(v)} - 1$.

(i) Since $N = \bar{N}_C$ (i.e. $N_C = \emptyset$) and $N' = \bar{N}'_C \cup N'_C = \emptyset$, we have $|\bar{N}_C| + |N'_C| = |N| \leq l_v - 1$ and $|\bar{N}'_C| + |N_C| = 0$.

(ii) The proof is similar to that of (i).

(iii) In this case, $l_v \geq l_{\sigma_P(v)}$ holds and $D = \emptyset$ or $D' = \emptyset$ is satisfied. When $D = \emptyset$ holds, $D'_C = \emptyset$ is obvious. Then we have $|N_C| = |N'_C|$. Hence, we find that

$$|\bar{N}_C| + |N'_C| = |\bar{N}_C| + |N_C| = |N| \leq l_v - 1 \text{ and}$$

$$|\bar{N}'_C| + |N_C| = |\bar{N}'_C| + |N'_C| = |N'| \leq l_{\sigma_P(v)} - 1$$

are satisfied. When $D' = \emptyset$ holds, we have $|N| \leq |N'|$. Since it is obvious that $\bar{D}_C = \emptyset$ holds, we obtain $|\bar{N}_C| = |\bar{N}'_C|$. Hence, we find that

$$|\bar{N}_C| + |N'_C| = |\bar{N}'_C| + |N'_C| = |N'| \leq l_{\sigma_P(v)} - 1 \leq l_v - 1 \text{ and}$$

$$|\bar{N}'_C| + |N_C| = |\bar{N}_C| + |N_C| = |N| \leq |N'| \leq l_{\sigma_P(v)} - 1$$

are satisfied.

(iv) The proof is similar to that of (iii).

Lemma 2 *For a tree $T \in \mathcal{T}$ and a path $P = (u_1, \ldots, u_k)$ ($k = 2$ or 3) of T, let $\tilde{T}$ be a dummies-added tree on which a forced isomorphism σ_P is defined. Also, let $\tilde{T}'$ be a tree obtained from $\tilde{T}$ by biasing with respect to σ_P and $T' = \tilde{T}' \setminus \{$dummy vertices$\}$. If $\{r_{vu}\}$ satisfies the condition in Main Theorem, then $f_g(T') \le f_g(T)$ holds.*

Proof We have only to show that $f_g(\tilde{T}') \le f_g(\tilde{T})$ holds. Let $\bar{V}_C = \tilde{V}(u_1) \setminus V_C$ and $D_{vu} = \{g(d(v,u;\tilde{T}')) - g(d(v,u;\tilde{T}))\}r_{vu}$. First, we consider the case of $k = 2$. Then $f_g(\tilde{T}') - f_g(\tilde{T})$ is expressed by

$$\begin{aligned} &\sum_{v,u \in V_C} D_{vu} + \sum_{v \in V_C, u \in \sigma_P(V_C)} D_{vu} + \sum_{v,u \in \sigma_P(V_C)} D_{vu} \\ +&\sum_{v,u \in \bar{V}_C} D_{vu} + \sum_{v \in \bar{V}_C, u \in \sigma_P(\bar{V}_C)} D_{vu} + \sum_{v,u \in \sigma_P(\bar{V}_C)} D_{vu} \\ +&\sum_{v \in V_C, u \in \bar{V}_C} D_{vu} + \sum_{v \in V_C, u \in \sigma_P(\bar{V}_C)} D_{vu} + \sum_{v \in \sigma_P(V_C), u \in \bar{V}_C} D_{vu} + \sum_{v \in \sigma_P(V_C), u \in \sigma_P(\bar{V}_C)} D_{vu}. \quad (6) \end{aligned}$$

However, noting that the first six summations are all equal to zero, we have

$$f_g(\tilde{T}') - f_g(\tilde{T}) = \sum_{v \in V_C, u \in \bar{V}_C} (D_{vu} + D_{v\sigma_P(u)} + D_{\sigma_P(v)u} + D_{\sigma_P(v)\sigma_P(u)}).$$

For two vertices $v \in V_C$ and $u \in \bar{V}_C$, let $\delta_{vu} = d(v,u;\tilde{T})$ and $\Delta_{vu} = d(v,u;\tilde{T}')$. Then $\delta_{vu} < \Delta_{vu}$ is obvious. Also, noting that

$$\delta_{vu} = d(v,u;\tilde{T}) = d(\sigma_P(v), \sigma_P(u); \tilde{T}) = d(v, \sigma_P(u); \tilde{T}') = d(\sigma_P(v), u; \tilde{T}')$$

and

$$\Delta_{vu} = d(v,u;\tilde{T}') = d(\sigma_P(v), \sigma_P(u); \tilde{T}') = d(v, \sigma_P(u); \tilde{T}) = d(\sigma_P(v), u; \tilde{T}),$$

we obtain

$$f_g(\tilde{T}') - f_g(\tilde{T}) = -\sum_{v \in V_C, u \in \bar{V}_C} \{g(\Delta_{vu}) - g(\delta_{vu})\} \left(r_{\sigma_P(v)u} + r_{v\sigma_P(u)} - r_{\sigma_P(v)\sigma_P(u)} - r_{vu}\right). \quad (7)$$

Due to the assumption of $\{r_{vu}\}$, the second factor of the summand in (7) is always nonnegative, and if it is positive then we find from the remark of Main Theorem that both $\sigma_P(v)$ and u are non-dummy, which implies $\Delta_{vu} \le n - 1$. Hence, we find from the assumption of g that $f_g(\tilde{T}') - f_g(\tilde{T}) \le 0$ holds for $k = 2$.

In the case of $k = 3$, $f_g(\tilde{T}') - f_g(\tilde{T})$ is expressed by

$$\sum_{v,u \in V(u_2)} D_{vu} + \sum_{v \in \tilde{V}(u_1), u \in V(u_2)} D_{vu} + \sum_{v \in \tilde{V}(u_3), u \in V(u_2)} D_{vu} + (6).$$

However, the first three summations are all equal to zero. Hence, $f_g(\tilde{T}') - f_g(\tilde{T}) \le 0$ is similarly obtained. The proof is just completed.

3. Properties of T^*

Here, we show some properties of the tree $T^* = (V, E^*)$ defined in Section 1. Suppose that T^* satisfies (2). Let $V_\nu = \{0, 1, \ldots, \nu - 1\}$ ($1 \le \nu \le n$) and $T^*_\nu = T^* \cap V_\nu$. Note that T^*_ν ($1 \le \nu \le n$) are subtrees of T^*.

Lemma 3 *For each* T_ν^* $(\nu \geq 2)$, *let* $P = (u_1, u_2, \ldots, u_k)$ *be an arbitrary path of* T_ν^* *satisfying* $u_1 < u_k$, *and let* $m = \left\lfloor \frac{k}{2} \right\rfloor$ *where* $\lfloor x \rfloor$ *is the maximum integer not exceeding* x. *Then* $u_i < u_{k-i+1}$ *and* $\deg(u_i; T_\nu^*) \geq \deg(u_{k-i+1}; T_\nu^*)$ *hold for* $i = 1, 2, \ldots, m$.

Proof Let vertex 0 be the root of T_ν^*. For two vertices v and u on T_ν^*, we find from the procedure of constructing T^* that

(i) if v is the d-th $(d > 0)$ ancestor of u, then $v < u$ holds,

(ii) if $0 < v < u$, then $\pi(v) \leq \pi(u)$ holds, and

(iii) if $v < u$, then the level of v does not exceed that of u.

For the path P, if u_1 is an ancestor of u_k, then we find from (i) that $u_i < u_{k-i+1}$ holds for $i = 1, 2, \ldots, m$. Otherwise, continue to compare u_i with u_{k-i+1} by ascending order of i until $\pi(u_i)$ becomes an ancestor of $\pi(u_{k-i+1})$ (this stopping condition is valid due to (iii)). Then we find from (ii) that $u_i < u_{k-i+1}$ holds for $i = 1, 2, \ldots, m$.

The rest of the lemma is proved as follows. Since T^* satisfies $\deg(u; T^*) = l_u$ $(u = 0, 1, \ldots, N-2)$, $2 \leq \deg(N-1; T^*) \leq l_{N-1}$ and $\deg(u; T^*) = 1$ $(u = N, N+1, \ldots, n-1)$, it follows from condition (2) that T_n^* $(= T^*)$ satisfies

$$\deg(0; T_n^*) \geq \deg(1; T_n^*) \geq \cdots \geq \deg(n-1; T_n^*).$$

We also find that $\deg(\pi(n-1); T_n^*) \geq 2$ and $\deg(u; T_n^*) = 1$ $(u = \pi(n-1)+1, \ldots, n-1)$ hold. Hence, considering a tree $T_{n-1}^* = T_n^* \setminus \{n-1\}$, we have

$$\deg(u; T_{n-1}^*) = \begin{cases} \deg(u; T_n^*) - 1 & \text{if } u = \pi(n-1) \\ \deg(u; T_n^*) & \text{otherwise,} \end{cases}$$

which implies that

$$\deg(0; T_{n-1}^*) \geq \deg(1; T_{n-1}^*) \geq \cdots \geq \deg(n-2; T_{n-1}^*)$$

holds. Continuing to delete the last vertex, we finally obtain T_ν^* and find that

$$\deg(0; T_\nu^*) \geq \deg(1; T_\nu^*) \geq \cdots \geq \deg(\nu-1; T_\nu^*)$$

holds. Hence, we have $\deg(u_i; T_\nu^*) \geq \deg(u_{k-i+1}; T_\nu^*)$ $(i = 1, 2, \ldots, m)$.

Lemma 4 *Suppose that a tree* $T \in \mathcal{T}$ *satisfies (1) with (2) and contains a subtree* T_ν^* *(i.e.* $T \cap V_\nu = T_\nu^*$*). Let* $P = (u_1, \ldots, u_k)$ $(k = 2$ *or* $3)$ *be an arbitrary path of* T. *For the tree* T *and the path* P, *let* $\tilde{T}$ *be a dummies-added tree on which a forced isomorphism* σ_P *is defined,* $\tilde{T}'$ *a tree obtained from* $\tilde{T}$ *by biasing with respect to* σ_P, *and* $T' = \tilde{T}' \setminus \{$*dummy vertices*$\}$. *Then* T' *also contains* T_ν^*.

Proof We have only to show that $\tilde{T}'$ contains T_ν^*. Since the proof varies according as where T_ν^* lies on $\tilde{T}$, we should consider the following three cases:

(i) $k = 3$ and $V_\nu \subset V(u_2)$,

(ii) $k = 2$ or 3, $V_\nu \cap \tilde{V}(u_1) \neq \emptyset$ and $V_\nu \cap \tilde{V}(u_k) = \emptyset$,

(iii) $k = 2$ or 3, $V_\nu \cap \tilde{V}(u_1) \neq \emptyset$ and $V_\nu \cap \tilde{V}(u_k) \neq \emptyset$.

In case (i), it is obvious for $\tilde{T}'$ obtained by biasing to contain T_ν^*. In case (ii), since $v \geq \nu$ holds for all $v \in \tilde{V}(u_k)$, it follows that $V_\nu \cap \tilde{V}(u_1) \subset \bar{V}_C$. Hence, $\tilde{T}'$ also contains T_ν^*. In case (iii), suppose that $u_1 < u_k$ holds without loss of generality. Then we find from Lemma 3 that $V_\nu \cap \tilde{V}(u_k) \subset \sigma_P(V_\nu \cap \tilde{V}(u_1))$ holds. We also find that $v < \sigma_P(v)$ holds for any $v \in V_\nu \cap \tilde{V}(u_1)$. In fact, if $\sigma_P(v) \in V_\nu \cap \tilde{V}(u_k)$, then $v < \sigma_P(v)$ comes from Lemma 3; otherwise $v < \nu \leq \sigma_P(v)$ holds. Hence, we have $V_\nu \cap \tilde{V}(u_1) \subset \bar{V}_C$ and $V_\nu \cap \tilde{V}(u_k) \subset \sigma_P(\bar{V}_C)$, which imply that $\tilde{T}'$ contains T_ν^*.

4. Proof of Main Theorem

Let $T^* = (V, E^*) \in \mathcal{T}$ be the tree defined in Section 1, and suppose that T^* satisfies (2). For a tree $T = (V, E) \in \mathcal{T}$, let

$$v_T = \begin{cases} \min\{v > 0 \mid e_v = (\pi(v), v) \in E^*, e_v \notin E\} & \text{if } E \neq E^* \\ n & \text{if } E = E^*. \end{cases}$$

We will show that any f_g-optimum tree can be transformed into T^* with the f_g value unchanged.

Let T be an f_g-optimum tree with $v_T < n$. Note that $T \cap \{0, 1, \ldots, v_T - 1\} = T^*_{v_T}$ (a subtree of T^*). Also, let $v^* = \pi(v_T)$ in T^*. Since $v_T < n$, we can consider a path $P' = (v^*, \ldots, v_T)$ of T. Among the vertices on P', a vertex adjacent to v_T is denoted by v_1, and a vertex adjacent to v^* is denoted by v_2 (v_2 may coincide with v_1). Then we find that $v^* < v_1$ holds. In fact, if $v^* = v_1$, then we have $(v_1, v_T) = (\pi(v_T), v_T) \in E$, which contradicts the definition of v_T; else, if $v^* > v_1$, then a certain vertex v' in $\{v < v_T \mid \pi(v) = v_1 \text{ in } T^*\}$ is pushed out by v_T, that is, $(v_1, v') = (\pi(v'), v') \in E^*$ and $(\pi(v'), v') \notin E$ hold, which contradicts the minimality of v_T. Here, we consider the following two cases: (i) $v_2 < v_T$ and (ii) $v_2 > v_T$.

In case (i), let $P = (u_1, \ldots, u_k)$ be a subpath of P' satisfying

$$d(u_1, v^*; T) = d(u_k, v_1; T) = \left\lfloor \frac{d(v^*, v_1; T) - 1}{2} \right\rfloor \quad \text{(then } k = 2 \text{ or } 3\text{)}.$$

Let $\tilde{T}$ be a dummies-added tree on which we can define a forced isomorphism σ_P such that $\sigma_P(v^*) = v_1$ holds and a vertex v^{**} adjacent to v^* satisfies $\sigma_P(v^{**}) = v_T$ and $v^{**} > v_T$ (it is obvious that such v^{**} exists). Also, let $\tilde{T}'$ be the tree obtained from $\tilde{T}$ by biasing with respect to σ_P, and $T' = (V, E') = \tilde{T}' \setminus \{\text{dummy vertices}\}$. Then we find from Lemmas 2 and 4 that T' is also f_g-optimum and satisfies $T' \cap \{0, 1, \ldots, v_T - 1\} = T^*_{v_T}$ and $(v^*, v_T) = (\pi(v_T), v_T) \in E'$, that is, $v_{T'} > v_T$ holds.

In case (ii), let $P = (u_1, \ldots, u_k)$ be a subpath of P' satisfying

$$d(u_1, v_2; T) = d(u_k, v_T; T) = \left\lfloor \frac{d(v_2, v_T; T) - 1}{2} \right\rfloor \quad \text{(then } k = 2 \text{ or } 3\text{)}.$$

Similarly to (i), let $\tilde{T}$ be a dummies-added tree on which we can define a forced isomorphism σ_P satisfying $\sigma_P(v_2) = v_T$. Also, let $\tilde{T}'$ be the tree obtained from $\tilde{T}$ by biasing with respect to σ_P, and $T' = \tilde{T}' \setminus \{\text{dummy vertices}\}$. Then we obtain $v_{T'} > v_T$ in the same way with that of (i).

By continuing this process, we find that T^* is f_g-optimum.

5. An Example of the GORST Problem

Finally, we show an example of the GORST problem, which is to find a tree network minimizing the probability that a request of communication is not realized when the network has k failures (called a "k-failure problem"). Let vertices be regarded as network hosts, edges as network cables, and $\{r_{vu}\}$ as relative frequencies of communication. A failure can occur on a vertex or an edge; if even one failure has occurred on a path $(v, \ldots, u)$, then v and u cannot communicate with each other. Let us define the probability that a request of communication is not realized on a tree network $T = (V, E)$ with k failures, denoted by $p(T; k)$. Consider a time interval I in which the number of failures on T does not change and at most one request of communication occurs on T. Let F_T denote the number of failures on T in I, R_{vu} the event that a request of communication between v and u occurs in I, and R_T the event that a request of communication between a certain pair on T occurs in I. Then the relative frequency of communication between v and u is expressed by $r_{vu} = \Pr\{R_{vu}|R_T\}$. We assume that each vertex (host) has enough ability of processing and, hence, the frequency of failure on each vertex does not depend on the amount of traffic. Also, we observe that an edge failure results mostly from incomplete connection at the connector of a host (rarely from the snapping of a cable) and it occurs independently of a vertex failure. Hence, we assume that

(i) a failure occurs equally often on n vertices, and so does it on $n-1$ edges, where the probability of vertex failure is not necessarily equal to that of edge failure,

(ii) any two failures occur mutually independently.

From these assumptions, letting

$$\alpha_{ij} = \Pr\{i \text{ vertices and } j \text{ edges are broken down}|F_T = i+j\},$$

we can assume that α_{ij}'s do not depend on the structure of T. Let $v \leftrightarrow u$ be the state that the path $(v,\ldots,u)$ has no failure in I, and $\overline{v \leftrightarrow u}$ the state that $(v,\ldots,u)$ has at least one failure in I. It is easy to see that

$$\begin{aligned}\Pr\{\overline{v \leftrightarrow u}|F_T = k\} &= 1 - \Pr\{v \leftrightarrow u|F_T = k\}\\ &= 1 - \sum_{i+j=k} \frac{\binom{n-(d(v,u;T)+1)}{i}\binom{(n-1)-d(v,u;T)}{j}}{\binom{n}{i}\binom{n-1}{j}}\alpha_{ij}.\end{aligned}$$

Then the desired probability is expressed by

$$\begin{aligned}p(T;k) &= \sum_{\{v,u\}\in\binom{V}{2}} \Pr\{\overline{v \leftrightarrow u} \text{ and } R_{vu}|F_T = k \text{ and } R_T\}\\ &= \sum_{\{v,u\}\in\binom{V}{2}} \Pr\{\overline{v \leftrightarrow u}|F_T = k\}r_{vu}.\end{aligned}$$

The k-failure problem is to find a tree T minimizing $p(T;k)$ for each k.

For a fixed k $(0 < k \leq 2n-1)$, let

$$g(x) = 1 - \sum_{i+j=k} \frac{\binom{n-1-x}{i}\binom{n-1-x}{j}}{\binom{n}{i}\binom{n-1}{j}}\alpha_{ij}$$

which is obtained by replacing $d(v,u;T)$ in $\Pr\{\overline{v \leftrightarrow u}|F_T = k\}$ by x. Then it is easy to find that $g(x)$ is monotone increasing on $[0, n-1]$. Hence, the k-failure problem under constraint (1) satisfying (2) is a special case of the GORST problem. Further, if $\{r_{vu}\}$ satisfies the condition in Main Theorem, then T^* defined in Section 1 minimizes $p(T;k)$ for any k $(0 < k \leq 2n-1)$.

Appendix 1

First, we show that condition (2) implies condition (3). Note that condition (3) also satisfies $\sum_{v=0}^{n-1} l_v \geq 2(n-1)$. In fact, since $\sum_{v=0}^{n-2} l_v \geq 2(n-1-1)+1 = 2n-3$ is obtained from (3), it follows from $l_{n-1} \geq 1$ that

$$\sum_{v=0}^{n-1} l_v = \sum_{v=0}^{n-2} l_v + l_{n-1} \geq 2(n-1)$$

holds. Assume that $\{l_v\}$ with (2) also satisfies $\sum_{v=0}^{\nu-1} l_v \leq 2(\nu-1)$ for some ν $(< n)$. Since $\sum_{v=0}^{n-1} l_v = \sum_{v=0}^{\nu-1} l_v + \sum_{v=\nu}^{n-1} l_v \geq 2(n-1)$, we have $2(\nu-1) + \sum_{v=\nu}^{n-1} l_v \geq 2(n-1)$, that is, $\sum_{v=\nu}^{n-1} l_v \geq 2(n-\nu)$. Then it follows from the monotoneity of $\{l_v\}$ that $l_\nu \geq 2$ holds. However, we also find that $\sum_{v=0}^{\nu-1} l_v \geq 2\nu$ holds, which is contradiction.

Secondly, we show that if condition (3) is satisfied then π is definable and T^* surely is a tree. Since condition (3) implies that $\sum_{v=0}^{n-1} l_v \geq 2(n-1)$ holds, we have $s_{n-1} \geq n-1$ under (3). This means that N in the definition of π is surely determined. Also, it is easy to see that T^* is a tree if $\pi(v) < v$ holds for all $v \in \{1, 2, \ldots, n-1\}$. For any vertex v satisfying $\pi(v) = u$ $(0 \leq u \leq N-1 < n)$, we have $s_{u-1}+1 \leq v$. On the other hand, we find from condition (3) that

$$u < \sum_{v=0}^{u-1} l_v - (u-1) + 1 = s_{u-1} + 1$$

holds. Hence, we obtain $\pi(v) = u < v$.

Appendix 2

For any tree $T \in \mathcal{T}$ and any path $P = (u_1, \ldots, u_k)$ $(k = 2$ or $3)$ of T, we can simultaneously establish two dummies-added trees $\tilde{T}(u_i) = (\tilde{V}(u_i), \tilde{E}(u_i))$ $(i = 1, k)$ and a forced isomorphism σ_P stated in Section 2 by using the following algorithm, where $T(u_i) = (V(u_i), E(u_i))$ $(i = 1, \ldots, k)$ are defined for the path P, and $\tilde{V}(u_i, l)$ is defined by $\{w \in \tilde{V}(u_i) | \text{the level of } w \text{ is } l\}$.

Procedure Make-$\tilde{T}(u_1)$-$\tilde{T}(u_k)$-and-σ_P (T: tree; P: path);
begin
  $\tilde{V}(u_1) := V(u_1)$; $\tilde{V}(u_k) := V(u_k)$; $\tilde{E}(u_1) := E(u_1)$; $\tilde{E}(u_k) := E(u_k)$;
  set $\sigma_P(u_1) = u_k$;
  dv := n; { dv denotes the current number of dummy vertex }
  $l := 0$;
  while $\tilde{V}(u_1, l) \cup \tilde{V}(u_k, l)$ has a vertex v with $\chi(v) \neq \emptyset$ **do begin**
    for all $v \in \tilde{V}(u_1, l)$ **do begin**
      $d^* := \max\{\deg(v; \tilde{T}(u_1)), \deg(\sigma_P(v); \tilde{T}(u_k))\}$;
      { adding dummy vertices and edges so that
        $\deg(v; \tilde{T}(u_1)) = \deg(\sigma_P(v); \tilde{T}(u_k)) = d^*$ }
      if $\deg(v; \tilde{T}(u_1)) < d^*$ **then**
        for $i := 1$ **to** $d^* - \deg(v; \tilde{T}(u_1))$ **do begin**
          $\tilde{V}(u_1) := \tilde{V}(u_1) \cup \{\text{dv}\}$;
          $\tilde{E}(u_1) := \tilde{E}(u_1) \cup \{(v, \text{dv})\}$;
          dv := dv + 1;
        end
      else if $\deg(\sigma_P(v); \tilde{T}(u_k)) < d^*$ **then**

```
                    for i := 1 to d* − deg(σ_P(v); T̃(u_k)) do begin
                        Ṽ(u_k) := Ṽ(u_k) ∪ {dv};
                        Ẽ(u_k) := Ẽ(u_k) ∪ {(σ_P(v), dv)};
                        dv := dv + 1;
                    end;
                { making one-to-one mapping from χ(v) to χ(σ_P(v)) }
                for all w ∈ χ(v) do begin
                    choose w' ∈ χ(σ_P(v)) to which any vertex in χ(v)
                        is not mapped by σ_P;
                    set σ_P(w) = w';
                end;
            end;
            l := l + 1;
        end;
    end;
```

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References

[1] T. Anazawa: On a condition for obtaining an explicit solution of optimum requirement spanning tree. *Networks*, **34** (1999) 132-135.

[2] T. Anazawa, T. Kodera and M. Jimbo: An explicit solution of optimum requirement spanning tree with maximum degree conditions. *Congressus Numerantium*, **117** (1996) 43-52.

[3] ———: Optimum requirement spanning trees and reliability of tree networks. *Networks*, **34** (1999) 122-131.

[4] V. Deineko, R. Rudolf and G. J. Woeginger: On the recognition of permuted Supnick and incomplete Monge matrices. *Acta Informatica*, **33** (1996) 559-569.

[5] R. E. Gomory and T. C. Hu: Multi-terminal network flows. *SIAM Journal on Applied Mathematics*, **9** (1961) 551-570.

[6] A. J. Hoffman: On simple linear programming problems. In V. Klee(ed.): *Convexity. Proc. Symposia in Pure Mathematics.* **7** (American Mathematical Society, Providence, RI, 1963) 317-327.

[7] T. C. Hu: Optimum communication spanning trees. *SIAM Journal on Computing*, **3** (1974) 188-195.

[8] U. Pferschy, R. Rudolf and G. J. Woeginger: Monge matrices make maximization manageable. *Operations Research Letters*, **16** (1994) 245-254.

Tsutomu Anazawa
Department of Industry and Information
Sapporo University
Sapporo, Hokkaido 062-8520, Japan
E-mail: anazawa@sapporo-u.ac.jp