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NEWTON'S METHOD FOR ZERO POINTS OF A MATRIX FUNCTION AND ITS APPLICATIONS TO QUEUEING MODELS

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Abstract Let $R(z)$ be a matrix function. We propose modified Newton's method to calculate zero points of $\det R(z)$. By the modified method, we can obtain accurate zero points by simple iterations. We also extend this problem to a multivariable case. Applications to the spectral analysis of $M/G/1$ type Markov chains are discussed. Important characteristics of these chains, e.g., the boundary vector and the matrix G, can be derived from zero points of a matrix function and corresponding null vectors. Numerical results are shown.

1. Introduction

Let $R(z)$ be a matrix function. In this paper, we consider Newton's method to obtain zero points of $\det R(z)$. At first, we propose a modification of the direct Newton's method and its extension it to a multivariable case. Second, applications of this work to the spectral analysis of $M/G/1$ type Markov chains are discussed. Important characteristics of these chains, e.g., the boundary vector and the matrix G, can be derived from zero points of $\det R(z)$ and corresponding null vectors of $R(z)$. Finally, numerical results are shown.

An $M/G/1$ and a $G/M/1$ type Markov chains, introduced by Neuts [11] are generalizations of an $M/G/1$ and a $G/M/1$ queue. Because these models have many applications to telecommunication techniques, they have received investigation in the last decade. The state space of these Markov chains is two-dimensional: the first element of state being level ($n = 0, 1, \ldots$ which can be interpreted as the number of customers in the system) and the second element of the state; the phase ($i = 1, \ldots, M$). By introducing a phase, we may represent a state of a telecommunication system, i.e. a state of correlated inputs to the system or a state of service modes. The problem is to obtain the boundary vector of the stationary probability distributions from the transition probability matrix of a block Toeplitz form. The boundary vector is obtained by introducing the matrix G, the phase transition probability matrix for the first passage time from a level $n+1$ to a level n, and calculating the stationary probability vector of G (see Neuts [11] and Lucantoni [9]). The matrix G is given by the minimal nonnegative solution of a nonlinear equation. When we calculate it numerically by an iteration method, at some point a truncation of level ($n = 0, 1, \ldots$) is necessary.

The transform method for the boundary vector has been studied in series of papers by Gail *et al.* [5], [6] and [7]. The vector generating function of the stationary probability is represented by the boundary vector and the matrix function. If the process is ergodic, then by using zero points of determinant of the matrix function on the unit disk and corresponding null vectors, the boundary vector is uniquely determined by the system of linearly independent equations.

Recently in order to derive the boundary vector, several methods have been investigated. Algorithms for the calculation of the matrix G are obtained by using Newton's method in Latouche [8] and by using the cyclic reduction technique in Bini and Meini [2]. Under the assumption that $R(z)$ is a matrix polynomial, spectral analysis is discussed in Mitrani [10]. To obtain null vectors of the matrix G, the invariant subspace approach is introduced by Akar and Sohraby [1] if the matrix function is rational. The fast Fourier transform is an approximate method with a wide use. Its application to an $M/G/1$ type Markov chain is discussed (see Bini and Meini [2]). There is a trade-off between computation time and accuracy. If we want to obtain an accurate value of the boundary vector, we need a large amount of computational time and a large memory.

The motivation of this paper is how to calculate accurate zero points numerically. For this purpose, we propose the usage of Newton's method because we can easily obtain accurate values of zero points by simple iterations. Moreover, suppose that a roughly approximated value obtained by some method. We may set it as the initial value of Newton's method. In Theorem 1, we modify the direct Newton's method. The direct usage of Newton's method implies $M+1$ determinant calculations in each step of the iteration. By the modified method, however, it is executed by a sweeping-out method. The latter is accomplished by a smaller computational time than the former. In Theorem 2, the modified method is extended to a multivariable case. In Section 3, assuming that all eigenvalues are distinct, we get a simple proof that the boundary vector is uniquely determined by the system of equations in Proposition 3. And we also consider applications of Newton's method in a multivariable case. It is proved in Theorem 4 that the spectral equation for a $MAP/SM/1$ queue is divided into two equations of small size matrices. In Theorem 5, the similar result is obtained for a single server queue whose arrival process is a superposition of independent Markovian arrival processes. In Section 4, numerical calculations of eigenvalues of a $M/G/1$ type queue are shown. The computational time and the speed of the convergence are shown.

2. Newton's Method

2.1. A single variable case

Suppose that $R(z) = [r_{ij}(z)]$ $(i,j = 1,\dots,M)$ is an $M\times M$ matrix function of z. Let us consider the problem to solve the equation

$$\det R(z) = 0 \tag{1}$$

by using Newton's method.

Setting $z_0 \in \boldsymbol{C}$ as an initial value and applying Newton's method to (1), we get the sequence $\{z_k\}$ such that

$$z_{k+1} = z_k - \frac{\det R(z_k)}{\frac{d}{dz}\det R(z)|_{z=z_k}}. \tag{2}$$

Since

$$\begin{aligned}\frac{d}{dz}\det R(z) &= \begin{vmatrix} \frac{d}{dz}r_{11}(z) & r_{12}(z) & \cdots & r_{1M}(z) \\ \vdots & \vdots & & \vdots \\ \frac{d}{dz}r_{M1}(z) & r_{M2}(z) & \cdots & r_{MM}(z) \end{vmatrix} \\ &+ \begin{vmatrix} r_{11}(z) & \frac{d}{dz}r_{12}(z) & r_{13}(z) & \cdots & r_{1M}(z) \\ \vdots & \vdots & \vdots & & \vdots \\ r_{M1}(z) & \frac{d}{dz}r_{M2}(z) & r_{M3}(z) & \cdots & r_{MM}(z) \end{vmatrix}\end{aligned}$$

$$+ \cdots + \begin{vmatrix} r_{11}(z) & r_{12}(z) & \cdots & \frac{d}{dz}r_{1M}(z) \\ \vdots & \vdots & & \vdots \\ r_{M1}(z) & r_{M2}(z) & \cdots & \frac{d}{dz}r_{MM}(z) \end{vmatrix}, \tag{3}$$

calculations of $M+1$ determinants of $M \times M$ matrices appear in Equation (2). To reduce computational time of the direct Newton's method (2) to the problem (1), we propose the following theorem.

Theorem 1 *Suppose that a matrix $X(z_k)$ satisfies the linear matrix equation*

$$R(z_k)X(z_k) = \frac{d}{dz}R(z)|_{z=z_k}. \tag{4}$$

Equation (2) is rewritten as

$$z_{k+1} = z_k - \frac{1}{\mathrm{tr}X(z_k)}, \tag{5}$$

where tr$X$ *is the trace of a matrix* X.

Proof. Consider the $M \times M$ matrix $X(z) = [x_{ij}(z)]$ $(i,j = 1,\ldots,M)$ which satisfies

$$R(z)X(z) = \frac{d}{dz}R(z). \tag{6}$$

The jth column of $X(z)$ satisfies

$$\begin{bmatrix} r_{11}(z) & \cdots & r_{1M}(z) \\ \vdots & & \vdots \\ r_{M1}(z) & \cdots & r_{MM}(z) \end{bmatrix} \begin{bmatrix} x_{1j}(z) \\ \vdots \\ x_{Mj}(z) \end{bmatrix} = \begin{bmatrix} \frac{d}{dz}r_{1j}(z) \\ \vdots \\ \frac{d}{dz}r_{Mj}(z) \end{bmatrix}.$$

By Cramer's rule, $x_{jj}(z)$, the jth diagonal entry of $X(z)$, is given by

$$x_{jj}(z) = \frac{1}{\det R(z)} \begin{vmatrix} r_{11}(z) & \cdots & r_{1\,j-1}(z) & \frac{d}{dz}r_{1j}(z) & r_{1\,j+1}(z) & \cdots & r_{1M}(z) \\ \vdots & & \vdots & \vdots & \vdots & & \vdots \\ r_{M1}(z) & \cdots & r_{M\,j-1}(z) & \frac{d}{dz}r_{Mj}(z) & r_{M\,j+1}(z) & \cdots & r_{MM}(z) \end{vmatrix}.$$

Since tr$X(z)$ becomes

$$\begin{aligned} \mathrm{tr}X(z) &= x_{11}(z) + x_{22}(z) + \cdots + x_{MM}(z) \\ &= \frac{1}{\det R(z)} \left\{ \begin{vmatrix} \frac{d}{dz}r_{11}(z) & r_{12}(z) & \cdots & r_{1M}(z) \\ \vdots & \vdots & & \vdots \\ \frac{d}{dz}r_{M1}(z) & r_{M2}(z) & \cdots & r_{MM}(z) \end{vmatrix} + \cdots \right. \\ &\qquad \left. \cdots + \begin{vmatrix} r_{11}(z) & \cdots & r_{1M-1}(z) & \frac{d}{dz}r_{1M}(z) \\ \vdots & & \vdots & \vdots \\ r_{M1}(z) & \cdots & r_{MM-1}(z) & \frac{d}{dz}r_{MM}(z) \end{vmatrix} \right\}, \end{aligned}$$

we get

$$\frac{d}{dz}\det R(z) = \det R(z) \cdot \mathrm{tr}X(z). \tag{7}$$

Substituting (7) to (2) leads to the conclusion. □

In the iterative formula (5), $\mathrm{tr}X(z_k)$ can be calculated by a simple sweeping-out method as follows. Suppose that the matrix $R(z_k)$ and $\frac{d}{dz}R(z)|_{z=z_k}$ are transformed to an upper triangular matrix $[\tilde{r}_{ij}(z_k)]$ $(i,j = 1,\ldots,M)$ and $[\tilde{r}'_{ij}(z_k)]$ $(i,j = 1,\ldots,M)$ by the same elementary transformation, respectively. That is, (5) is written as

$$\begin{bmatrix} \tilde{r}_{11}(z_k) & \cdots & \tilde{r}_{1M}(z_k) \\ 0 & \cdots & \tilde{r}_{2M}(z_k) \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \tilde{r}_{MM}(z_k) \end{bmatrix} \begin{bmatrix} x_{11}(z_k) & \cdots & x_{1M}(z_k) \\ x_{21}(z_k) & \cdots & x_{2M}(z_k) \\ \vdots & & \vdots \\ x_{M1}(z_k) & \cdots & x_{MM}(z_k) \end{bmatrix} = \begin{bmatrix} \tilde{r}'_{11}(z_k) & \cdots & \tilde{r}'_{1M}(z_k) \\ \tilde{r}'_{21}(z_k) & \cdots & \tilde{r}'_{2M}(z_k) \\ \vdots & & \vdots \\ \tilde{r}'_{M1}(z_k) & \cdots & \tilde{r}'_{MM}(z_k) \end{bmatrix}.$$

Then $x_{jj}(z_k)$ $(j = 1,\ldots,M)$, the jth diagonal entry of $X(z_k)$, is obtained from the formula

$$\begin{aligned} x_{Mj}(z_k) &= \frac{1}{\tilde{r}_{MM}(z_k)}\tilde{r}'_{Mj}(z_k) \\ x_{ij}(z_k) &= \frac{1}{\tilde{r}_{ii}(z_k)}\left(\tilde{r}'_{ij}(z_k) - \sum_{l=0}^{M-1-i} \tilde{r}_{iM-l}(z_k)x_{M-lj}(z_k)\right) \quad (i = M-1,\ldots,j). \end{aligned}$$

To compute $[\tilde{r}_{ij}(z_k)]$ $(i,j = 1,\ldots,M)$ and $[\tilde{r}'_{ij}(z_k)]$ $(i,j = 1,\ldots,M)$ requires $\frac{5}{6}M^3 + O(M^2)$ multiplications in the elementary transformation. And $\frac{1}{6}M^3 + O(M^2)$ more multiplications are needed in the procedure to determine all diagonal entries of $X(z_k)$. So the total computational cost of each step by using (5) is $O(M^3)$. The other way, in the iterative formula (2), calculations of $M+1$ determinants of $M \times M$ matrices are needed. Since the cost of calculation for a determinant of an $M \times M$ matrix is $\frac{1}{3}M^3 + O(M^2)$, the cost in each step is $O(M^4)$. Thus the iterative formula (5) has an advantage over the formula (2) in the computational time. Moreover, the iterative formula (5) is simpler than (2). So it is well-suited for creating programs of numerical computations.

2.2. A multivariable case

In this subsection we consider multivariable Newton's method solving zero points for determinants of matrix functions. For $\xi = 1,\ldots,K$, let $R_\xi(z_1,\ldots,z_K) = [r_{ij,\xi}(z_1,\ldots,z_K)]$ $(i,j = 1,\ldots,M_\xi)$ be $M_\xi \times M_\xi$ matrix functions of K variables $z_1,\ldots,z_K$. We now consider the problem to solve the system of K equations

$$\begin{cases} \det R_1(z_1,\ldots,z_K) = 0 \\ \qquad\vdots \\ \det R_K(z_1,\ldots,z_K) = 0. \end{cases} \tag{8}$$

Setting $(z_1^{(0)},\ldots,z_K^{(0)})$ as a initial vector and applying Newton's method to the problem (8), we get the vector sequence $\{(z_1^{(k)},\ldots,z_K^{(k)})\}$ such that

$$\begin{pmatrix} z_1^{(k+1)} \\ \vdots \\ z_K^{(k+1)} \end{pmatrix} = \begin{pmatrix} z_1^{(k)} \\ \vdots \\ z_K^{(k)} \end{pmatrix} - J^{-1}(z_1^{(k)},\ldots,z_K^{(k)}) \begin{bmatrix} \det R_1(z_1^{(k)},\ldots,z_K^{(k)}) \\ \vdots \\ \det R_K(z_1^{(k)},\ldots,z_K^{(k)}) \end{bmatrix}, \tag{9}$$

where

$$J(z_1,\ldots,z_K) = \begin{bmatrix} \frac{\partial}{\partial z_1}\det R_1(z_1,\ldots,z_K) & \cdots & \frac{\partial}{\partial z_K}\det R_1(z_1,\ldots,z_K) \\ \vdots & & \vdots \\ \frac{\partial}{\partial z_1}\det R_K(z_1,\ldots,z_K) & \cdots & \frac{\partial}{\partial z_K}\det R_K(z_1,\ldots,z_K) \end{bmatrix}.$$

Put

$$z^{(k)} = (z_1^{(k)}, \ldots, z_K^{(k)})^{\boldsymbol{T}},$$

where the symbol $\boldsymbol{T}$ means the transposition of a vector. We can rewrite (9) as

$$z^{(k+1)} = z^{(k)} - J^{-1}(z^{(k)}) \begin{bmatrix} \det R_1(z^{(k)}) \\ \vdots \\ \det R_K(z^{(k)}) \end{bmatrix}. \tag{10}$$

Theorem 2 *For $\xi, \eta = 1, \ldots, K$, let $X_{(\xi\eta)}(z^{(k)})$ be the $M_\xi \times M_\xi$ matrix defined by*

$$R_\xi(z^{(k)}) X_{(\xi\eta)}(z^{(k)}) = \frac{\partial}{\partial z_\eta} R_\xi(z)|_{z=z^{(k)}}.$$

Put

$$T(z^{(k)}) = \begin{bmatrix} \mathrm{tr} X_{(11)}(z^{(k)}) & \cdots & \mathrm{tr} X_{(1K)}(z^{(k)}) \\ \vdots & & \vdots \\ \mathrm{tr} X_{(K1)}(z^{(k)}) & \cdots & \mathrm{tr} X_{(KK)}(z^{(k)}) \end{bmatrix}.$$

Then (10) can be rewritten as

$$z^{(k+1)} = z^{(k)} - T^{-1}(z^{(k)}) e_K, \tag{11}$$

where $e_K = (1, \ldots, 1)^{\boldsymbol{T}}$ is the 1's column vector of size K.

Proof. For $\xi, \eta = 1, \ldots, K$, consider the $M_\xi \times M_\xi$ matrix $X_{(\xi\eta)}(z) = \left[x_{ij(\xi\eta)}(z)\right]$ $(i, j = 1, \ldots, M_\xi)$ which satisfies

$$R_\xi(z) X_{(\xi\eta)}(z) = \frac{\partial}{\partial z_\eta} R_\xi(z). \tag{12}$$

For the jth column of $X_{(\xi\eta)}(z)$, we get

$$\begin{bmatrix} r_{11,\xi}(z) & \cdots & r_{1M_\xi,\xi}(z) \\ \vdots & & \vdots \\ r_{M_\xi 1,\xi}(z) & \cdots & r_{M_\xi M_\xi,\xi}(z) \end{bmatrix} \begin{bmatrix} x_{1j(\xi\eta)}(z) \\ \vdots \\ x_{M_\xi j(\xi\eta)}(z) \end{bmatrix} = \begin{bmatrix} \frac{\partial}{\partial z_\eta} r_{1j,\xi}(z) \\ \vdots \\ \frac{\partial}{\partial z_\eta} r_{M_\xi j,\xi}(z) \end{bmatrix}$$

By Cramer's rule, $x_{jj(\xi\eta)}(z)$, the jth diagonal entry of $X_{(\xi\eta)}(z)$, is given by

$$x_{jj(\xi\eta)}(z) = \frac{1}{\det R(z)} \begin{vmatrix} r_{11,\xi}(z) & \cdots & \frac{\partial}{\partial z_\eta} r_{1j,\xi}(z) & \cdots & r_{1M_\xi,\xi}(z) \\ \vdots & & \vdots & & \vdots \\ r_{M_\xi 1,\xi}(z) & \cdots & \frac{\partial}{\partial z_\eta} r_{M_\xi j,\xi}(z) & \cdots & r_{M_\xi M_\xi,\xi}(z) \end{vmatrix}.$$

Hence $\mathrm{tr} X_{(\xi\eta)}(z)$ becomes

$$\begin{aligned} \mathrm{tr} X_{(\xi\eta)}(z) &= x_{11(\xi\eta)}(z) + \cdots + x_{MM(\xi\eta)}(z) \\ &= \frac{1}{\det R_\xi(z)} \left\{ \begin{vmatrix} \frac{\partial}{\partial z_\eta} r_{11,\xi}(z) & r_{12,\xi}(z) & \cdots & r_{1M_\xi,\xi}(z) \\ \vdots & \vdots & & \vdots \\ \frac{\partial}{\partial z_\eta} r_{M_\xi 1,\xi}(z) & r_{M_\xi 2,\xi}(z) & \cdots & r_{M_\xi M_\xi,\xi}(z) \end{vmatrix} + \cdots \right. \\ &\quad \left. + \begin{vmatrix} r_{11,\xi}(z) & \cdots & r_{1M_\xi-1,\xi}(z) & \frac{\partial}{\partial z_\eta} r_{1M_\xi,\xi}(z) \\ \vdots & & \vdots & \vdots \\ r_{M_\xi 1,\xi}(z) & \cdots & r_{M_\xi M_\xi-1,\xi}(z) & \frac{\partial}{\partial z_\eta} r_{M_\xi M_\xi,\xi}(z) \end{vmatrix} \right\}. \end{aligned}$$

Since

$$
\frac{\partial}{\partial z_\eta}\det R_\xi(\boldsymbol{z}) = \begin{vmatrix} \frac{\partial}{\partial z_\eta} r_{11,\xi}(\boldsymbol{z}) & r_{12,\xi}(\boldsymbol{z}) & \cdots & r_{1M_\xi,\xi}(\boldsymbol{z}) \\ \vdots & \vdots & & \vdots \\ \frac{\partial}{\partial z_\eta} r_{M_\xi 1,\xi}(\boldsymbol{z}) & r_{M_\xi 2,\xi}(\boldsymbol{z}) & \cdots & r_{M_\xi M_\xi,\xi}(\boldsymbol{z}) \end{vmatrix} + \cdots
+ \begin{vmatrix} r_{11,\xi}(\boldsymbol{z}) & \cdots & r_{1M_\xi-1,\xi}(\boldsymbol{z}) & \frac{\partial}{\partial z_\eta} r_{1M_\xi,\xi}(\boldsymbol{z}) \\ \vdots & & \vdots & \vdots \\ r_{M_\xi 1,\xi}(\boldsymbol{z}) & \cdots & r_{M_\xi M_\xi-1,\xi}(\boldsymbol{z}) & \frac{\partial}{\partial z_\eta} r_{M_\xi M_\xi,\xi}(\boldsymbol{z}) \end{vmatrix},
$$

we get

$$
\frac{\partial}{\partial z_\eta}\det R_\xi(\boldsymbol{z}) = \det R_\xi(\boldsymbol{z}) \cdot \mathrm{tr} X(\boldsymbol{z}).
$$

Therefore,

$$
\begin{aligned}
J(\boldsymbol{z})^{-1} &= \begin{bmatrix} \det R_1(\boldsymbol{z}) \cdot \mathrm{tr}X_{(11)}(\boldsymbol{z}) & \cdots & \det R_1(\boldsymbol{z}) \cdot \mathrm{tr}X_{(1K)}(\boldsymbol{z}) \\ \vdots & & \vdots \\ \det R_K(\boldsymbol{z}) \cdot \mathrm{tr}X_{(K1)}(\boldsymbol{z}) & \cdots & \det R_K(\boldsymbol{z}) \cdot \mathrm{tr}X_{(KK)}(\boldsymbol{z}) \end{bmatrix}^{-1} \\
&= \left\{ \begin{bmatrix} \det R_1(\boldsymbol{z}) & & \mathrm{O} \\ & \ddots & \\ \mathrm{O} & & \det R_K(\boldsymbol{z}) \end{bmatrix} \begin{bmatrix} \mathrm{tr}X_{(11)}(\boldsymbol{z}) & \cdots & \cdots & \mathrm{tr}X_{(1K)}(\boldsymbol{z}) \\ \vdots & & & \vdots \\ \vdots & & & \vdots \\ \vdots & & & \vdots \\ \mathrm{tr}X_{(K1)}(\boldsymbol{z}) & \cdots & \cdots & \mathrm{tr}X_{(KK)}(\boldsymbol{z}) \end{bmatrix} \right\}^{-1} \\
&= T^{-1}(\boldsymbol{z}) \begin{bmatrix} \frac{1}{\det R_1(\boldsymbol{z})} & & \mathrm{O} \\ & \ddots & \\ \mathrm{O} & & \frac{1}{\det R_K(\boldsymbol{z})} \end{bmatrix}. \qquad (13)
\end{aligned}
$$

Substituting (13) to (10), we get the conclusion. □

3. Applications of Newton's Method for the Spectral Analysis

The first problem in analysis of an $M/G/1$ type Markov chain is to obtain the boundary vector of the stationary distribution. The boundary vector is calculated by using all zero points ($|z| \leq 1$) of a matrix generating function and corresponding right null vectors. In this spectral analysis, these zero points should be calculated with great accuracy. Newton's method in the previous section is efficient because for a appropriate initial value, the sequence $\{z_k\}$ converges rapidly to the zero point. In this section, we will argue applications of the spectral analysis to queueing systems.

3.1. An $M/G/1$ type Markov chain

We consider an irreducible discrete time Markov chain C with a state space $\{(n,i); n = 0,1,\ldots; i = 1,2,\ldots,M\}$, where n and i are the level of the chain and the phase of the arrival process, respectively. For the sake of simplicity, suppose that the chain C has one boundary level. That is, the transitions from level $n+1$ ($n \geq 0$) to level $n+m$ are governed by the $M \times M$ matrix a_m $(m \geq 0)$, whereas the transitions from boundary level 0 to level m are given by the $M \times M$ matrix b_m $(m \geq 0)$. Transition matrix P of the chain C takes the form

$$P = \begin{bmatrix} b_0 & b_1 & b_2 & \cdots \\ a_0 & a_1 & a_2 & \cdots \\ 0 & a_0 & a_1 & \cdots \\ \vdots & \ddots & \ddots & \\ & & & \end{bmatrix}.$$

Put

$$a(z) = \sum_{n=0}^{\infty} a_n z^n, \quad b(z) = \sum_{n=0}^{\infty} b_n z^n.$$

In this section we suppose that $a(z)$ is analytic in $|z| < 1$ and continuous in $|z| \leq 1$. Moreover, $a(1)$ is irreducible and aperiodic. A general Markov chain with a reducible $a(1)$ is studied in [6]. Let $\boldsymbol{\pi} = (\boldsymbol{\pi}_0, \boldsymbol{\pi}_1, \ldots)$ be the stationary probability vector satisfying

$$\boldsymbol{\pi} = \boldsymbol{\pi} P \quad , \quad \boldsymbol{\pi}\boldsymbol{e} = 1, \tag{14}$$

where the nth vector $\boldsymbol{\pi}_n$ is a $1 \times M$ row vector and $\boldsymbol{e} = (1, 1, \ldots)^T$.

From (14)

$$\boldsymbol{\pi}_n = \boldsymbol{\pi}_0 b_n + \sum_{m=1}^{n+1} \boldsymbol{\pi}_m a_{n+1-m}. \tag{15}$$

Define

$$\Pi(z) = \sum_{n=0}^{\infty} \boldsymbol{\pi}_n z^n.$$

Then from (15)

$$\Pi(z)(zI - a(z)) = \boldsymbol{\pi}_0(zb(z) - a(z)), \tag{16}$$

where I is the $M \times M$ identity matrix.

We assume that the chain C is ergodic. Denote $\boldsymbol{\pi}^*$ as the probability vector given by $\boldsymbol{\pi}^* a(1) = \boldsymbol{\pi}^*$ and $\boldsymbol{\pi}^* \boldsymbol{e}_M = 1$, where $\boldsymbol{e}_M$ is 1's column vector of size M. And consider the phase transition probability matrix G during the first passage time from level $n+1$ to level n. The matrix G is given by the unique minimal non-negative solution of the nonlinear equation

$$G = \sum_{n=0}^{\infty} a_n G^n. \tag{17}$$

It is proved in Neuts [11] that under the ergodicity, $\rho \equiv \boldsymbol{\pi}^* a'(1) \boldsymbol{e}_M < 1$, $b'(1) < +\infty$ and G is stochastic.

The spectral analysis to obtain the boundary vector $\boldsymbol{\pi}_0$ is introduced by Gail *et al.* [5], [6] and [7] as follows. The number (counting multiplicities) of zero points of $\det(zI - a(z))$ in the open unit disk is $M - 1$. And $\det(zI - a(z))$ has a single zero point at $z = 1$. The boundary vector $\boldsymbol{\pi}_0$ satisfies the system of linear equations constructed from these zero points and is uniquely determined. When zero points are not distinct, the system of equations for $\boldsymbol{\pi}_0$ becomes complicated. Assuming that there exist M distinct zero points in $|z| \leq 1$, we will show a simple proof of the uniqueness.

Let $z_1, \ldots, z_M$ $(z_1 = 1)$ and $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_M$ $(\boldsymbol{v}_1 = \boldsymbol{e}_M)$ be distinct zero points on the unit disk and corresponding right null vectors, respectively. From $(z_i I - a(z_i))\boldsymbol{v}_i = \mathbf{0}$, we get

$$z_i \boldsymbol{v}_i = \sum_{n=0}^{\infty} a_n z_i^n \boldsymbol{v}_i. \tag{18}$$

Since G is stochastic, G has exactly $M-1$ eigenvalues in the open unit disk and one eigenvalue at $z=1$. Comparing (18) with (17), we conclude that z_i and $\boldsymbol{v}_i$ $(i=1,\ldots,M)$ are also eigenvalues of G and corresponding eigenvectors, respectively. We can express the matrix G as

$$G = V \begin{bmatrix} z_1 & & \\ & \ddots & \\ & & z_M \end{bmatrix} V^{-1},$$

where $V \equiv [\boldsymbol{v}_1,\ldots,\boldsymbol{v}_M]$. From the property that the set of eigenvalues of G is the same as the set of zero points of $\det(zI-a(z))$ on the unit disk, we get the following proposition.

Proposition 1 *Suppose that the chain C is ergodic and zero points of* $\det(zI-a(z))$ *on the unit disk are distinct. The boundary vector $\boldsymbol{\pi}_0$ is uniquely determined by $M-1$ linearly independent homogeneous equations*

$$\boldsymbol{\pi}_0(b(z_i)-I)\boldsymbol{v}_i = 0 \qquad (i=2,\ldots,M), \tag{19}$$

and the non homogeneous equation

$$\begin{aligned}&\boldsymbol{\pi}_0\left[(b(1)-a(1))(I-a(1)+\boldsymbol{e}_M\boldsymbol{\pi}^*)^{-1}a'(1)+I+b'(1)-a'(1)\right]\boldsymbol{e}_M \\ &= 1-\boldsymbol{\pi}^*a'(1)\boldsymbol{e}_M.\end{aligned} \tag{20}$$

Proof. For $i=2,\ldots,M$ setting $z=z_i$ and multiplying (16) by $\boldsymbol{v}_i$, we can easily obtain

$$\begin{aligned}\boldsymbol{\pi}_0(z_ib(z_i)-a(z_i))\boldsymbol{v}_i &= z_i\boldsymbol{\pi}_0(b(z_i)\boldsymbol{v}_i-\boldsymbol{v}_i) \\ &= z_i\boldsymbol{\pi}_0(b(z_i)-I)\boldsymbol{v}_i \\ &= 0.\end{aligned}$$

If $z_i \neq 0$, we get

$$\boldsymbol{\pi}_0(b(z_i)-I)\boldsymbol{v}_i = 0.$$

If $z_i=0$, we have $b(z_i)=b_0$ and $a_0\boldsymbol{v}_i = a(z_i)\boldsymbol{v}_i = \mathbf{0}$. Then

$$\begin{aligned}\boldsymbol{\pi}_0(b(z_i)-I)\boldsymbol{v}_i &= \boldsymbol{\pi}_0(b_0-I)\boldsymbol{v}_i \\ &= \boldsymbol{\pi}_1a_0\boldsymbol{v}_i \\ &= 0,\end{aligned}$$

where the second equation comes from the fact that $\boldsymbol{\pi}_0 = \boldsymbol{\pi}_0b_0+\boldsymbol{\pi}_1a_0$ for $n=0$ in (15). Hence, the system of equations (19) is obtained.

For $i=1$, $\boldsymbol{\pi}_0(b(1)-a(1))\boldsymbol{e}_M = 0$ is not a constraint equation to $\boldsymbol{\pi}_0$ because of $(b(1)-a(1))\boldsymbol{e}_M = \mathbf{0}$. Therefore

$$\boldsymbol{\pi}_0(b(z_i)-I)\boldsymbol{v}_i = 0. \quad (i=1,\ldots,M) \tag{21}$$

is equivalent to the system of equations (19). Rewriting (21) as a matrix form, we get

$$\boldsymbol{\pi}_0[(b(z_1)-I)\boldsymbol{v}_1,(b(z_2)-I)\boldsymbol{v}_2,\ldots,(b(z_M)-I)\boldsymbol{v}_M] = \mathbf{0}.$$

Define the matrix in the brackets by $\hat{B}$ and represent it by G as

$$\begin{aligned}\hat{B} &\equiv [(b(z_1)-I)\boldsymbol{v}_1,(b(z_2)-I)\boldsymbol{v}_2,\ldots,(b(z_M)-I)\boldsymbol{v}_M] \\ &= [\sum_{n=0}^{\infty} b_n z_1^n \boldsymbol{v}_1,\ldots,\sum_{n=0}^{\infty} b_n z_M^n \boldsymbol{v}_M] - [\boldsymbol{v}_1,\ldots,\boldsymbol{v}_M] \\ &= [\sum_{n=0}^{\infty} b_n V \begin{bmatrix} z_1^n & & \\ & \ddots & \\ & & z_M^n \end{bmatrix} V^{-1} - I]V \\ &= [\sum_{n=0}^{\infty} b_n G^n - I]V.\end{aligned}$$

Since from the ergodic condition, $\hat{P} \equiv \sum_{n=0}^{\infty} b_n G^n$ is the transition probability matrix of the embedded Markov chain from level 0 to level 0 at service completion epochs, $\mathrm{Rank}(\hat{P}-I) = M-1$. Therefore $M-1$ homogeneous equations in (19) are linearly independent.

The last non homogeneous equation for $\boldsymbol{\pi}_0$ is obtained as follows. It is well known that $I - a(1) + \boldsymbol{e}_M\boldsymbol{\pi}^*$ is nonsingular and $\boldsymbol{\pi}^*(I-a(1)+\boldsymbol{e}_M\boldsymbol{\pi}^*) = \boldsymbol{\pi}^*$. From (16) at $z=1$ we get

$$\Pi(1) = \boldsymbol{\pi}^* + \boldsymbol{\pi}_0(b(1)-a(1))(I-a(1)+\boldsymbol{e}_M\boldsymbol{\pi}^*)^{-1}.$$

Differentiating (16) at $z=1$ and multiplying it by $\boldsymbol{e}_M$, we get

$$\Pi(1)(I-a'(1))\boldsymbol{e}_M = \boldsymbol{\pi}_0(b(1)+b'(1)-a'(1))\boldsymbol{e}_M.$$

Eliminating $\Pi(1)$ in the above two equations leads to the last non homogeneous equation. □

Thus using the spectral method, we can obtain the boundary vector $\boldsymbol{\pi}_0$ and the matrix G numerically. By Ramaswami [15] the nth vector $\boldsymbol{\pi}_n$ $(n = 1,\ldots)$ are recursively calculated as

$$\boldsymbol{\pi}_n = \left[\boldsymbol{\pi}_0\bar{b}_n + \sum_{m=1}^{n-1} \boldsymbol{\pi}_n \bar{a}_{n+1-m}\right](I-\bar{a}_1)^{-1} \quad i \geq 1,$$

where

$$\bar{a}_m = \sum_{n=m}^{\infty} a_n G^{n-m}\ , \quad \bar{b}_m = \sum_{n=m}^{\infty} b_n G^{n-m} \quad m \geq 0.$$

In order to obtain the zero points of $\det(zI - a(z))$ on the unit disk with great accuracy, Newton's method is efficient. It is, however, important how to select the initial values of Newton's method. Suppose that a roughly approximated value obtained by some method ([1], [2], [8], [11] and etc.). Setting it as the initial value of Newton's method, the accurate zero point is easily obtained. If there is no information about zero points, we use lattice points on the unit disk by a suitable interval as the initial values of Newton's method. In this case we take a large amount of computational time to obtain all zero points on the unit disk. In the next section we give a numerical example that it takes about 2 hours to calculate all zero points of a 15×15 matrix function (See Figure 3).

3.2. A $MAP/SM/1$ queue

We consider a $MAP/SM/1$ queue in [9]. The arrival process is a Markovian arrival process with $M \times M$ coefficient matrices $\{D_n, n \geq 0\}$. Put $D(z) = \sum_{n=0}^{\infty} D_n z^n$. And suppose that $D = D(1)$ is irreducible and aperiodic. The successive service times are formulated as a semi-Markov process with N service modes. Let $H_{pq}(t)$ be the transition probability that the service of mode p lasts up to t and the next service mode is q. Put $H(t) = [H_{pq}(t)]$. Then $H = H(\infty)$ represents the transition probability matrix of service mode. Let $A_{pq}^{(n)}$ be an $M \times M$ matrix whose (i,j) entry represents the transition probability that under the condition the service begins at mode p and the arrival phase is i, the next service mode is q, the arrival phase is j and the number of arriving customers during the service time is n.

Put

$$A_{pq}(z) = \sum_{n=0}^{\infty} A_{pq}^{(n)} z^n.$$

Then $A_{pq}(z)$ is given by

$$A_{pq}(z) = \int_0^{\infty} e^{D(z)t} dH_{pq}(t).$$

Let $A^{(n)}$ be the $MN \times MN$ matrix ordered A_{pq} lexicographically. Put

$$A(z) = \sum_{n=0}^{\infty} A^{(n)} z^n.$$

Then $A(z)$ is given by

$$A(z) = \int_0^{\infty} e^{D(z)t} \otimes dH(t), \tag{22}$$

where the symbol $\otimes$ denotes the Kronecker product form of matrices. Let $MN \times MN$ matrix $B(z)$ be the generating function for the arrivals during the idle period. Then $B(z)$ satisfies

$$B(z) = \frac{1}{z}\left[-D_0^{-1}[D(z) - D_0] \otimes I\right] A(z),$$

where I is the $N \times N$ identity matrix. Let $p(n,i,q)$ be the joint stationary probability that at service completion epochs, the arrival phase, the service mode and the number of customers are i, q and n, respectively. Put

$$p_{i,q}(z) = \sum_{n=0}^{\infty} p(n,i,q) z^n.$$

As its vector representation

$$P(z) = (p_{1,1}(z), \ldots, p_{1,N}(z), p_{2,1}(z), \ldots, p_{2,N}(z), \ldots, p_{M,N}(z)). \tag{23}$$

is MN row vector listed in lexicographical order. Then

$$P(z)[zI - A(z)] = P(0)[zB(z) - A(z)]. \tag{24}$$

Now, we consider how to obtain the boundary vector $P(0)$ from zero points of $\det[zI - A(z)]$. In (23), the arrange of service modes $p = 1, \ldots, N$ are blocked. For the latter discussion, it is convenient to use the order such that arrival phases $i = 1, \ldots, M$ are blocked. Let $MN \times MN$ matrix $\bar{A}^{(n)}$ take a form as

$$\bar{A}^{(n)} = \begin{pmatrix} A_{11}^{(n)} & \cdots & A_{1N}^{(n)} \\ \vdots & & \vdots \\ A_{N1}^{(n)} & \cdots & A_{NN}^{(n)} \end{pmatrix}.$$

Note that the order of states

$$((1,1), \ldots, (M,1), (1,2), \ldots, (M,2), \ldots, (1,N), \ldots, (M,N)) \tag{25}$$

is not lexicographical. Put

$$\bar{A}(z) = \sum_{n=0}^{\infty} \bar{A}^{(n)} z^n = \left[\bar{A}_{pq}(z)\right].$$

By the similar derivation to (22), $\bar{A}(z)$ satisfies

$$\bar{A}(z) = \int_0^{\infty} dH(t) \otimes e^{D(z)t}. \tag{26}$$

In the same way let $\bar{B}(z)$ be the reordered matrix of $B(z)$. And put

$$\bar{P}(z) = (p_{1,1}(z), \ldots, p_{M,1}(z), p_{1,2}(z), \ldots, p_{M,2}(z), \ldots, p_{M,N}(z)).$$

Then

$$\bar{B}(z) = \frac{1}{z}\left[-I \otimes D_0^{-1}[D(z) - D_0]\right] \bar{A}(z)$$

and

$$\bar{P}(z)[zI - \bar{A}(z)] = \bar{P}(0)[z\bar{B}(z) - \bar{A}(z)] \tag{27}$$

are derived by the similar way to the derivation of previous equations.

If we get zero points of $\det(zI - \bar{A}(z))$ on the unit disk, we can obtain the boundary vector $\bar{P}(0)$ from (27). Let $h_{pq}(s)$ be the moment generating function of $H_{pq}(t)$ defined as

$$h_{pq}(s) = \int_0^{\infty} dH_{pq}(t) e^{st}$$

and $h(s)$ be the $N \times N$ matrix which takes the form as

$$h(s) = \begin{bmatrix} h_{11}(s) & \cdots & h_{1N}(s) \\ \vdots & & \vdots \\ h_{N1}(s) & \cdots & h_{NN}(s) \end{bmatrix}.$$

In general, for an $M \times M$ matrix W, we use notations as

$$h_{pq}(W) = \int_0^{\infty} dH_{pq}(t) e^{Wt}$$

and

$$h(W) = \begin{bmatrix} h_{11}(W) & \cdots & h_{1N}(W) \\ \vdots & & \vdots \\ h_{N1}(W) & \cdots & h_{NN}(W) \end{bmatrix} = \int_0^{\infty} dH(t) \otimes e^{Wt},$$

where $h_{pq}(W)$ and $h(W)$ are $M \times M$ and $MN \times MN$ matrices, respectively. Then, from the rearrangement of the order in (25) and (26), $\bar{A}_{pq}(z)$ and $\bar{A}(z)$ are represented as

$$\bar{A}_{pq}(z) = h_{pq}(D(z)) \text{ and } \bar{A}(z) = h(D(z)).$$

Let us now consider the problem to solve the equation

$$\det(zI - \bar{A}(z)) = 0 \quad (|z| \leq 1). \tag{28}$$

The equation (28) for the matrix of size $M \times N$ is divided into two equations written by two matrices of sizes M and N as follows:

Theorem 3 *Let $\hat{z}$ ($|\hat{z}| \le 1$) be a solution of* $\det(zI - \bar{A}(z)) = 0$. *Then there exists $\hat{\alpha}$ ($|\hat{\alpha} + \bar{d}| \le \bar{d}$) such that the pair $(\hat{z}, \hat{\alpha})$ satisfies*

$$\begin{cases} \det(\hat{\alpha} I - D(\hat{z})) = 0 \\ \det(\hat{z} I - h(\hat{\alpha})) = 0 \ , \end{cases} \tag{29}$$

where $\bar{d}$ is the maximal value of diagonal elements of $-D_0$.

Proof. Let $\hat{z}$ be a solution of $\det(zI - \bar{A}(z)) = 0$ ($|z| \le 1$). Now, for every $\hat{z}$, we consider Jordan's canonical form J of $D(\hat{z})$ such that

$$D(\hat{z}) = QJQ^{-1},$$

where Q is $M \times M$ matrix. Then $\bar{A}_{pq}(\hat{z})$ is rewritten as

$$\begin{aligned} \bar{A}_{pq}(\hat{z}) &= \int_0^\infty dH_{pq}(t) e^{D(\hat{z})t} \\ &= Q h_{pq}(J) Q^{-1}. \end{aligned} \tag{30}$$

Now we only discuss the case $M = 2$ and $N = 2$ because we can extend this discussion to the general case easily. Let α_1 and α_2 be the diagonal entries of Jordan's canonical form J. Then α_ν ($\nu = 1, 2$) satisfy

$$\det(\alpha_\nu I - D(\hat{z})) = 0 \quad (\nu = 1, 2).$$

From (30) $\bar{A}(\hat{z})$ is written as

$$\bar{A}(\hat{z}) = \begin{bmatrix} Qh_{11}(J)Q^{-1} & Qh_{12}(J)Q^{-1} \\ Qh_{21}(J)Q^{-1} & Qh_{22}(J)Q^{-1} \end{bmatrix}.$$

Let $\boldsymbol{\nu} = \binom{\boldsymbol{\nu}_1}{\boldsymbol{\nu}_2}$ be a right null vector of $\hat{z}I - \bar{A}(\hat{z})$ and put $\boldsymbol{\nu}' = \binom{Q^{-1}\boldsymbol{\nu}_1}{Q^{-1}\boldsymbol{\nu}_2}$. Then

$$\begin{bmatrix} \hat{z}I - h_{11}(J) & -h_{12}(J) \\ -h_{21}(J) & \hat{z}I - h_{22}(J) \end{bmatrix} \boldsymbol{\nu}' = 0.$$

Note that $h_{pq}(J)$ is an upper triangular matrix. By replacing rows and columns, we get

$$\begin{aligned} 0 &= \det \begin{bmatrix} \hat{z}I - h_{11}(J) & -h_{12}(J) \\ -h_{21}(J) & \hat{z}I - h_{22}(J) \end{bmatrix} \\ &= \det \begin{bmatrix} \hat{z} - h_{11}(\alpha_1) & * & -h_{12}(\alpha_1) & * \\ 0 & \hat{z} - h_{11}(\alpha_2) & 0 & -h_{12}(\alpha_2) \\ -h_{21}(\alpha_1) & * & \hat{z} - h_{22}(\alpha_2) & * \\ 0 & -h_{21}(\alpha_2) & 0 & \hat{z} - h_{22}(\alpha_2) \end{bmatrix} \\ &= (-1)^2 \det \begin{bmatrix} \hat{z} - h_{11}(\alpha_1) & -h_{12}(\alpha_1) & * & * \\ -h_{21}(\alpha_1) & \hat{z} - h_{22}(\alpha_1) & * & * \\ 0 & 0 & \hat{z} - h_{11}(\alpha_2) & -h_{12}(\alpha_2) \\ 0 & 0 & -h_{21}(\alpha_2) & \hat{z} - h_{22}(\alpha_2) \end{bmatrix} \\ &= \det(\hat{z}I - h(\alpha_1)) \det(\hat{z}I - h(\alpha_2)). \end{aligned}$$

From this we see that for $\hat{z}$ there exists $\hat{\alpha}$ such that the pair $(\hat{z}, \hat{\alpha})$ satisfies (29). □

For the existence region of $\hat{\alpha}$, we can find as follows. Considering the equation $\det(\hat{\alpha}I - D(\hat{z})) = 0$, we get

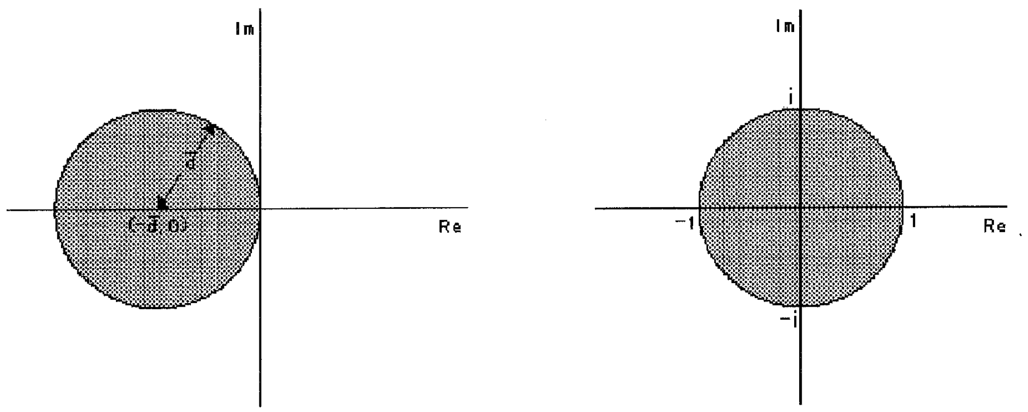
$$\begin{aligned} 0 &= \det(\hat{\alpha}I - D(\hat{z})) \\ &= \bar{d}^M \det\left(\frac{\hat{\alpha}}{\bar{d}}I - \frac{D(\hat{z})}{\bar{d}}\right) \\ &= \bar{d}^M \det\left(\left(\frac{\hat{\alpha}}{\bar{d}} + 1\right)I - \left(\frac{D(\hat{z})}{\bar{d}} + I\right)\right). \end{aligned}$$

Put $Q(\hat{z}) = [q_{ij}(\hat{z})] = \frac{D(\hat{z})}{\bar{d}} + I$. Then for $|\hat{z}| \leq 1$,

$$\left|\frac{\hat{\alpha}}{\bar{d}} + 1\right| \leq \max \sum_j |q_{ij}(\hat{z})| \leq 1.$$

Hence we can conclude that $\hat{\alpha}$ exists at least in $|\hat{\alpha} + \bar{d}| \leq \bar{d}$.

Since both existence regions of z and α (see Figure 1) are compact, we select a pair of lattice points by suitable intervals in both regions as the initial values of Newton's method to obtain all solutions of (29). A right null vector of $zI - \bar{A}(z)$ is represented by the Kronecker product of right null vectors of $\alpha I - D(z)$ and $zI - h(\alpha)$. It is proved that in those regions there are $M \times N$ pairs of $(\hat{z}, \hat{\alpha})$, if all zero points of $zI - \bar{A}(z)$ are distinct (see Nishimura and Jiang [13]).



The existence region of α. The existence region of z.

Figure 1: The existence region of the solutions of (29)

Remark

In general we can not obtain the explicit expression of $\bar{A}(z)$ because $\bar{A}(z)$ takes the form as (26). But considering the system of equation (29), we can get zero points of $\det(zI - \bar{A}(z))$. In a case of $N = 1$, a $MAP/SM/1$ queue reduces to a $MAP/G/1$ queue. Let $H(t)$ be the distribution function of service time with its moment generating function $h(\alpha)$. Then solutions of $\det(zI - a(z)) = 0$ $(|z| \leq 1)$ satisfy

$$\begin{cases} \det(\alpha I - D(z)) = 0 \\ z - h(\alpha) = 0 \end{cases} \quad (|z| \leq 1, |\alpha + \bar{d}| \leq \bar{d}),$$

where $a(z) = \int_0^\infty e^{D(z)t} dH(t)$. Therefore, if $h^{-1}(z)$, the inverse function of $z = h(\alpha)$, exists, we can obtain zero points of $\det(zI - a(z))$ on the unit disk from the equation

$$\det(h^{-1}(z)I - D(z)) = 0 \quad (|z| \leq 1). \tag{31}$$

In general, for a $MAP/G/1$ queue, solving the equation

$$\det(\alpha I - D(h(\alpha)) = 0 \quad (|\alpha + \bar{d}| \leq \bar{d}),$$

we can obtain all zero points on the unit disk. Especially, for a $MAP/D/1$ queue with a deterministic service time T, $h(\alpha) = e^{\alpha T}$. Although $h^{-1}(z) = \frac{1}{T}\log z$ is not unique, we may solve the equation

$$\det(\alpha I - D(e^{\alpha T})) = 0 \quad (|\alpha + \bar{d}| \leq \bar{d}), \tag{32}$$

and put $z = h(\alpha)$.

3.3. A Superposition of independent Markovian arrival processes

We consider a $MAP/G/1$ queue whose arrival process is a superposition of K independent Markovian arrival processes ([3] and [4]). The νth process ($\nu = 1, \ldots, K$) is formulated as the $M_\nu \times M_\nu$ coefficient matrices $\{F_{n,\nu}, n = 0, 1, \ldots\}$. Put $F_\nu(z) = \sum_{n=0}^{\infty} F_{n,\nu} z^n$ and suppose that $F_\nu = F_\nu(1)$ is irreducible and aperiodic. Then the generating function for the arrivals at service completion epochs is given by $a(z) = \int_0^\infty e^{D(z)t} dH(t)$, where $H(t)$ is the distribution function of service time and $D(z) \equiv F_1(z) \oplus F_2(z) \oplus \cdots \oplus F_K(z)$ is a matrix function for the arrival process. The size of the matrix function $D(z)$ is $M \times M$, where $M = \prod_{\nu=1}^{K} M_\nu$. The symbol $\oplus$ denotes the Kronecker sum of matrices. From the results in Section 3.2, in order to obtain zero points of $\det(zI - a(z))$ on the unit disk, we solve the equation

$$\det(\alpha I - D(h(\alpha))) = 0 \quad (|\alpha + \bar{d}| \leq \bar{d}), \tag{33}$$

where $\bar{d}$ is the maximal value of diagonal elements of $-D_0$ ($D_0 \equiv F_{0,1} \oplus \cdots \oplus F_{0,K}$), and put $z = h(\alpha)$. From the property of the Kronecker sum, we get the following theorem.

Theorem 4 *Let $\hat{\alpha}$ be a solution of* $\det(\alpha I - D(h(\alpha)) = 0$ ($|\alpha + \bar{d}| \leq \bar{d}$). *Then there exists $(\alpha_1, \ldots, \alpha_K)$ satisfying $\hat{\alpha} = \alpha_1 + \alpha_2 + \cdots + \alpha_K$ and*

$$\begin{cases} \det(\alpha_1 I - F_1(h(\hat{\alpha}))) = 0 \\ \qquad\vdots \\ \det(\alpha_k I - F_K(h(\hat{\alpha}))) = 0. \end{cases} \tag{34}$$

The existence region of α_ν ($\nu = 1, \ldots, K$) is given by $|\alpha_\nu + \bar{d}_\nu| \leq \bar{d}_\nu$, where $\bar{d}_\nu$ ($\nu = 1, \ldots, K$) is the maximal value of diagonal elements of $-F_{0,\nu}$.

Proof. For every ν ($\nu = 1, \ldots, K$), let $\beta_{i,\nu}$ ($i = 1, \ldots, M_\nu$) be eigenvalues of $F_\nu(h(\hat{\alpha}))$. According to the same discussion in Theorem 4, $\beta_{i,\nu}$ satisfies

$$\det(\beta_{i,\nu} I - F_\nu(h(\hat{\alpha}))) = 0 \qquad (i = 1, \ldots, M_\nu). \tag{35}$$

Considering Equation (35), we get

$$\begin{aligned} 0 &= \det(\beta_{i,\nu} I - F_\nu(h(\hat{\alpha}))) \\ &= \bar{d}_\nu^{M_\nu} \det\left(\left(\frac{\beta_{i,\nu}}{\bar{d}_\nu} + 1\right) I - \left(\frac{F_\nu(h(\hat{\alpha}))}{\bar{d}_\nu} + I\right)\right). \end{aligned}$$

Since $|h(\hat{\alpha})| \leq 1$, the maximum row sum of the absolute value of the entries of the matrix $\frac{F_\nu(h(\hat{\alpha}))}{\bar{d}_\nu} + I$ is less than or equal to 1. Therefore $|\frac{\beta_{i,\nu}}{\bar{d}_\nu} + 1| \leq 1$, that is $|\beta_{i,\nu} + \bar{d}_\nu| \leq \bar{d}_\nu$.

From the property of the Kronecker sum, $\hat{\alpha}$ (the eigenvalue of $D(h(\hat{\alpha})) = F_1(h(\hat{\alpha})) \oplus \cdots \oplus F_\nu(h(\hat{\alpha}))$) is the sum of eigenvalues of each $F_\nu(h(\hat{\alpha}))$ ($\nu = 1, \ldots, K$). So there exist $\alpha_\nu \in \{\beta_{1,\nu}, \ldots, \beta_{M_\nu,\nu}\}$ ($\nu = 1, \ldots, K$) such that $\hat{\alpha} = \alpha_1 + \cdots + \alpha_K$. □

We can apply Theorem 2 to the problem (34). Put $\boldsymbol{\alpha}^{(k)} = (\alpha_1^{(k)}, \ldots, \alpha_K^{(k)})$ and $\hat{\alpha}^{(k)} = \alpha_1^{(k)} + \cdots + \alpha_K^{(k)}$. For $\xi, \eta = 1, \ldots, K$, let $X_{(\xi\eta)}(\boldsymbol{\alpha}^{(k)})$ be the $M_\xi \times M_\xi$ matrix defined by

$$X_{(\xi\eta)}(\boldsymbol{\alpha}^{(k)}) = \begin{cases} -\left[\alpha_\xi^{(k)} I - F_\xi(h(\hat{\alpha}^{(k)}))\right]^{-1} F'_\xi(h(\hat{\alpha}^{(k)}))h'(\hat{\alpha}^{(k)}) & (\xi = \eta) \\ \left[\alpha_\xi^{(k)} I - F_\xi(h(\hat{\alpha}^{(k)}))\right]^{-1} \left[I - F'_\xi(h(\hat{\alpha}^{(k)}))h'(\hat{\alpha}^{(k)})\right] & (\xi \neq \eta), \end{cases}$$

where $F'_\xi(h(\hat{\alpha}^{(k)})) = \sum_{n=1}^{\infty} nF_{n,\xi}(h(\hat{\alpha}^{(k)}))^{n-1}$. Then setting

$$T(\boldsymbol{\alpha}^{(k)}) = \begin{bmatrix} \mathrm{tr}X_{(11)}(\boldsymbol{\alpha}^{(k)}) & \cdots & \mathrm{tr}X_{(1K)}(\boldsymbol{\alpha}^{(k)}) \\ \vdots & & \vdots \\ \mathrm{tr}X_{(K1)}(\boldsymbol{\alpha}^{(k)}) & \cdots & \mathrm{tr}X_{(KK)}(\boldsymbol{\alpha}^{(k)}) \end{bmatrix},$$

we get the Newton's iteration formula

$$\boldsymbol{\alpha}^{(k+1)} = \boldsymbol{\alpha}^{(k)} - T^{-1}(\boldsymbol{\alpha}^{(k)})\boldsymbol{e}_K, \tag{36}$$

where $\boldsymbol{e}_K$ is the 1's column vector of size K. The existence region of α_ν $(\nu = 1, \ldots, K)$ are given by $|\alpha_\nu + \bar{d}_\nu| \leq \bar{d}_\nu$.

4. Numerical Examples

In this section, considering some examples of queueing models, we show results of calculations for zero points by Newton's method and discuss their accuracy and computational times.

(1) A $MAP/M/1$ queue

For a $MAP/M/1$ queue, we now consider to obtain the zero points of $\det(zI - a(z))$ on the unit disk. Let $H_e(t)$ and $h_e(\alpha)$ be the exponential distribution of the service time with mean $\frac{1}{\mu}$ and its moment generating function, respectively, that is,

$$H_e(t) = 1 - e^{-\mu t} \ (t \geq 0), \quad h_e(\alpha) = \int_0^\infty e^{\alpha t} dH(t) = \frac{\mu}{\mu - \alpha}.$$

In order to obtain zero points of $\det(zI - a(z))$, from (31) we solve the equation

$$\det(h_e^{-1}(z)I - D(z)) = \det(\frac{\mu(z-1)}{z} I - D(z)) = 0 \quad (|z| \leq 1). \tag{37}$$

We consider two cases of $MAP/M/1$ queue. In the first case for Newton's method, selecting initial points on lattice points in the unit disk, we compare the computational time of several size matrices with complex zero points. In the second case, all zero points in the unit disk are real in $(0, 1]$. We compare the accuracy and computational time for Newton's method and those of Lucantoni's method.

The first case. Let $\mu = 2$ and

$$D(z) = \begin{bmatrix} -3+z & 2 & & & \\ & -3+z & 2 & & \\ & & \ddots & \ddots & \\ & & & -3+z & 2 \\ 2 & & & & -3+z \end{bmatrix}$$

be the matrix function of size M. Note that this MAP is the same as a Poisson process but this is not an essential assumption. When we use information of range where zero points are distributed, computational times become short. Our aim is to calculate all zero points in the unit disk by selecting all lattice points of the interval $\delta = 0.05$ as initial values of Newton's method.

In Table 1, the solutions and corresponding absolute values of $\det(h_e^{-1}(z)I - D(z))$ are shown when the matrix size of $D(z)$ is 7×7.

Table 1: The solutions of $\det(h_e^{-1}(z)I - D(z)) = 0$.

z	$\lvert\det(h_e^{-1}(z)I - D(z))\rvert$
1	0
$0.46545745851010 + 0.257899144795876\ i$	1.9333×10^{-12}
$0.46545745851010 - 0.257899144795876\ i$	1.9332×10^{-12}
$0.33546343391086 + 0.137010803145644\ i$	3.9816×10^{-12}
$0.33546343391086 - 0.137010803145644\ i$	3.9816×10^{-12}
$0.30176978238141 + 0.042247367116245\ i$	2.6022×10^{-12}
$0.30176978238141 - 0.042247367116245\ i$	2.6022×10^{-12}

In Figure 2, these zero points are plotted on the complex plane. In Table 2, the behavior of convergence to the solution $z = 0.465457458510102 + 0.257899144795876\ i$ is represented with $\lvert\det(h_e^{-1}(z_k)I - D(z_k))\rvert$ as evaluation of the accuracy. For $M = 2, \ldots, 15$, M distinct

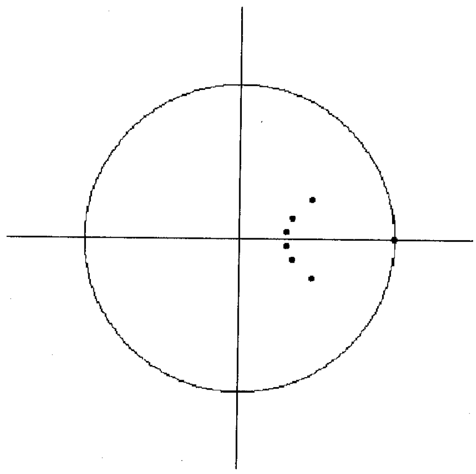


Figure 2: The distribution of the solutions.

zero points are obtained. In general, as the size of $D(z)$ becomes large, the computational time increases. In Figure 3, the computational time is plotted as a function of the size M (CPU: Pentium 166Mhz, Memory: 32MB, OS: Microsoft Windows 95). It takes about 2 hours to calculate all zero points in the case matrix size is 15×15 without using information about zero points. If we can use the knowledge of an region which contains all zero points in the unit disk, we may save the computational time.

The second case. Let $\{j = 0, \cdots, m\}$ be the state space of the underlying process. Suppose that $D(z)$ is an $(m+1) \times (m+1)$ matrix whose (i,j) element $(j,k = 0, \cdots, m)$ is given by

Table 2: The behavior of convergence to the solution.

k	z_k	$\lvert \det(h_e^{-1}(z_k)I - D(z_k)) \rvert$
0	$0.4 + 0.3i$	551.16
1	$0.453962570419582 + 0.26184698676777i$	39.947
2	$0.464050383948272 + 0.25766554894769i$	4.2392
3	$0.465450359403075 + 0.25787954348003i$	0.061518
4	$0.465457462871815 + 0.25789914573075i$	0.000013165
5	$0.465457458510102 + 0.25789914479587i$	1.9333×10^{-12}

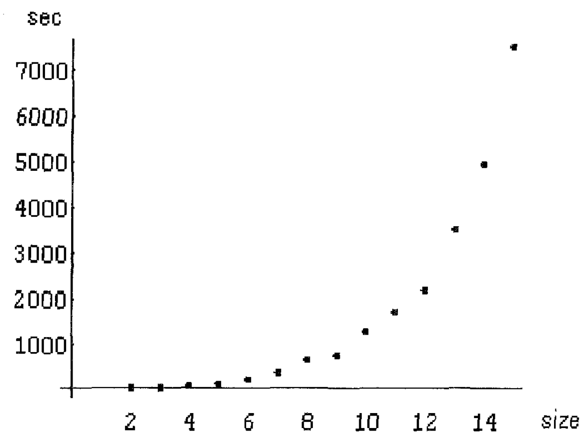


Figure 3: The computational time.

$$d_{j,k}(z) = \begin{cases} j(z-1)-2 & \text{if } j=k, \text{ and } j \neq 0, \quad m \\ j(z-1)-1 & \text{if } j=k, \text{ and } j=0, \quad m \\ z^j & \text{if } |j-k|=1 \\ 0 & \text{otherwise,} \end{cases}$$

where the traffic intensity is $\rho = 3(m+1)x/2$. Since arrival rate at state j is j, this numerical example possesses burstness of traffic. We will use the following steps. At first decide an interval where zero points are distributed. Next as initial points of Newton's method select equally spaced $5m$ points on the interval.

(i) Comparison of Newton's method and Lucantoni's method when $\rho = 0.5$. The results of Newton's method and Lucantoni's method is completely the same. Using both methods we can get accurate results. In Table 3, computational times of Newton's method and Lucantoni's method are given when $\rho = 0.5$ and $m = 5, 10, 15, 20$. Since in this case all zero points are real, Newton's method is very effective.

Table 3: Computational times (sec)

m	5	10	15	20
Newton's method	1.54	10.61	35.53	116.94
Lucantoni's method	160.32	838.16	2525.86	5519.25

(ii) Put $m = 20$ and $\rho = 0.99$. In authors' computations of the matrix G by Lucantoni's method, the calculated G is not stochastic because its row sums are not equal to 1. The lowest row sum of the matrix G is 0.959 with 5465.8 second computational time.

On the other hand, in our Newton's method, selecting 100 initial points in the interval [0.87,1], we get all 21 zero points. The smallest zero point is $0.8796392\cdots$. This calculation is executed during 119 seconds computational time. For Newton's method computational times do not depend on the traffic intensity ρ. In this heavy traffic case, the calculated matrix G contains negative numbers of order 10^{-6} by using double precision. Therefore, authors calculate the matrix G by using computer calculations of 1000 figures for Decimal BASIC (free ware) made by K. Shiraishi (kazuo.shiraishi@nifty.ne.jp). Then calculated G becomes stochastic.

(iii) When $m = 50$ and $\rho = 0.99$, selecting 250 initial points in the interval [0.93,1], we get all 51 zero points listed in Table 4. Accuracy of zero points is also checked because the calculated matrix G is stochastic. Authors also execute this check by using computer calculations of 1000 figures for Decimal BASIC.

Table 4: The list of zero points for $m = 50$ and $\rho = 0.99$.

k	z_k	k	z_k
0	1		$\cdots$
1	0.99938050663062681	47	0.94392538084580664
2	0.99805441835114526	48	0.94253106926812916
3	0.99616334925627522	49	0.94095250041013433
	$\cdots$	50	0.93897545256846729

(2) A $MAP/D/1$ queue

For a $MAP/D/1$ queue, let $H_D(t)$ and $h_D(\alpha)$ be the service time distribution (the service time is T) and its moment generating function, respectively. Then

$$H_D(t) = \begin{cases} 1 & (t \geq T) \\ 0 & (t < T) \end{cases}, \quad h_D(\alpha) = \int_0^\infty e^{\alpha t} dH_D(t) = e^{\alpha T}.$$

In order to obtain zero points of $\det(zI - a(z))$ on the unit disk, from (32) we solve the equation

$$\det(\alpha I - D(h_D(\alpha)) = \det(\alpha I - D(e^{\alpha T})) = 0 \quad (|\alpha + \bar{d}| \leq \bar{d}), \tag{38}$$

where $\bar{d}$ is the maximum value of diagonal entries of $-D_0$.

Put $T = \frac{1}{2}$ and

$$D(z) = \begin{bmatrix} -1 & z^2 & 0 & 0 \\ 0 & -1 & z^2 & 0 \\ 0 & 0 & -1 & z^2 \\ 0.05 & 0 & 0 & -0.55 + 0.5z \end{bmatrix}.$$

Then the solutions of (38) obtained by using Newton's method are 0, $-1.1256\cdots$, $-0.95107\cdots + 0.15089\cdots i$ and $-0.95107\cdots - 0.15089\cdots i$. Substituting these solutions into $z = h_D(\alpha)$, we get zero points. In Table 5, the solutions, their absolute values of $\det(\alpha I - D(e^{\alpha T}))$ and corresponding zero points are shown.

Table 5: The solutions of $\det(\alpha I - D(e^{\alpha T})) = 0$ and their corresponding values of z.

α / z	$\lvert\det(\alpha I - D(e^{\alpha T}))\rvert$
0	0
1	
$-1.1256637738136 - 6.24043216044599 \times 10^{-16} i$	2.96969×10^{-17}
$0.324437043056382 - 2.02462735752905 \times 10^{-16} i$	
$-0.95107027160039 + 0.150890530571235 i$	4.44089×10^{-16}
$0.381937723896332 + 0.058072184463382 i$	
$-0.95107027160039 - 0.150890530571235 i$	4.44089×10^{-16}
$0.381937723896332 - 0.058072184463382 i$	

(3) A $MAP/SM/1$ queue

For a $MAP/SM/1$ queue, we consider to solve the equation

$$\det(zI - \bar{A}(z)) = 0 \quad (|z| \le 1). \tag{39}$$

As an example, put

$$D(z) = \begin{bmatrix} -1.2 + z & 0.2 \\ 0.05 & -0.15 + 0.1z \end{bmatrix},$$

$$H(t) = \begin{bmatrix} 0.1(1 - e^{-2t}) & 0.9(1 - e^{-2t}) \\ 0.7(1 - e^{-t}) & 0.3(1 - e^{-t}) \end{bmatrix}.$$

To obtain the solution of (39), from Theorem 4 we solve the system of equations

$$\begin{cases} \det(\alpha I - D(z)) = 0 \\ \det(zI - h(\alpha)) = 0 \end{cases} \quad (|z| \le 1, \ |\alpha + \bar{d}| \le \bar{d}), \tag{40}$$

where

$$h(\alpha) = \begin{bmatrix} 0.1 \times \frac{2}{2-\alpha} & 0.9 \times \frac{2}{2-\alpha} \\ 0.7 \times \frac{1}{1-\alpha} & 0.3 \times \frac{1}{1-\alpha} \end{bmatrix}.$$

Table 6 gives the pair (z, α) which satisfies (40).

Table 6: The solutions of (40).

z / α	$\lvert\det(\alpha I - D(z))\rvert$ / $\lvert\det(zI - h(\alpha))\rvert$
1	0
0	0
$0.72219474048098 + 6.81050711456 \times 10^{-25} i$	2.77556×10^{-16}
$-0.501410751741313 + 1.426421813062 \times 10^{-24} i$	2.22045×10^{-16}
$-0.292221505080965 + 3.994082 \times 10^{-30} i$	3.9968×10^{-15}
$-1.49979398221541 + 3.537063 \times 10^{-30} i$	1.11022×10^{-16}
-0.526240073716001	0
-0.196088706216173	1.11022×10^{-16}

5. Conclusions

Comparing direct Newton's method with the method given in Theorem 1 and 2, the latter has a simpler structure and is executed with less computational time than the former. In a single variable case, the latter computational cost is $O(M^3)$ whereas the former is $O(M^4)$.

For a $MAP/G/1$ queueing system, calculation of the matrix G is first important problem. Lucantoni's method is very efficient. But heavy traffic systems, its convergence is slow. We propose the usage of Newton's method to calculate zero points of $\det(zI - a(z))$. There are two problems for usage of Newton's method. The first problem is the assumption that zero points in the unit disk are simple. We can not encounter multiple zero points if we do not make artificial examples. In our numerical examples, all zero points are simple. This assumption seems not to be so strong in applications. However, it seems that Gail *et al.*'s method [5] may be applied if there are zero points with multiplicity. Second problem is how to select initial points. To save the computational time the following steps are suggested. At the first step, find a restricted region which contains zero points. In authors' experience this step is not difficult because zero points are distributed in a small region. At the second step, select efficient number of initial points so as to find all zero points in the unit disk, since the number of zero points in the unit disk we need is equal to the size of $D(z)$. Another application of Newton's method is that when we have a roughly approximated matrix G by some method, we may set its eigenvalues as the initial values of Newton's method.

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