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ON A COMPETITIVE INVENTORY MODEL WITH A CUSTOMER'S CHOICE PROBABILITY

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Abstract In this paper we consider a model in which a customer chooses one of two retailers with a choice probability depending on the distance from the customer's position to the retailer's position over a line segment market. We study an optimal strategy for two retailers in a competitive inventory model by using an equilibrium point in the context of the game theory. We find an equilibrium point concerned with the optimal ordering quantity to minimize the total cost.

1. Introduction

In the studies concerned with inventory control problems there are many research works finding the optimal strategy for a single retailer. Studies on an equilibrium point for many retailers have been published in recent years. For instance Parlar [8] has proved the existance and uniqueness of the Nash solution for an inventory problem with two substitutable products having random demands. Lippman and McCardle [6] have examined the relation between equilibrium inventory levels and the splitting rule and provide conditions under which there is a unique equilibrium point for a competitive version of the classical newsboy problem. A model represented in this paper is considered as one of special cases of their problems.

We are interested in a problem finding the optimal strategy for many retailers such that they are related with something each other. In this paper we consider a model in which a customer chooses a retailer with a choice probability depending on the distance from the customer's position to the retailer's position over a line segment market and we discuss on the ordering policy including a state of customer's choice. The probability represented in Huff model [4] gives us more realistic model. However it is numerically complicated, so we use a very simple probability here. We study an optimal strategy for two retailers in a competitive inventory model by using an equilibrium point in the context of game theory. Our purpose is to find an equilibrium point concerned with the optimal ordering quantity to minimize the total cost, i.e. the sum of the ordering cost, the holding cost and the penalty cost.

The remainder of the paper is organized as follows. In Section 2 we describe the model in our study. In Section 3 we concretely provide the equilibrium point in this model formulated as a two-person nonzero-sum game. It is obvious that the results obtained in this paper differs from those for a single retailer. Section 4 deals with a numerical example. And finally, this paper ends with some conclusions in Section 5.

2. Model Description

We consider a single period model where customers go to buy a kind of product to two retailers with a probability. The problem discussed in this paper is formulated as a two-person nonzero-sum game on the inventory control. Two retailers, called Player I and Player II, begin to sell products at the same time. All customers are shared by two players. Player I locates his position at point 0 in the interval $[0,1]$ and Player II does at point 1. It is possible for the players to place an order only once at the beginning of the period. They receive products to sell without lead-time. If the products are in short supply, they are not backlogging after that. They purchase the products at an ordering cost per unit and sell them at a selling price per unit. If players have some stock to sell, then they are charged holding costs. On the other hand, if they do not have any stock and have occurred demand, then they are charged penalty costs.

The customers uniformly distribute in the interval $[0,1]$. The customer who stays at point u in $[0,1]$ first visits Player II with the probability u to go to buy one of products and visits Player I with the remaining probability $1-u$. If a player he has first visited does not have inventory to sell on his hand, he must visit another player. The customers start from their positions at the same time as their sales and transfer with the same speed. Then the arrival time taken from each of their positions to the firms is in proportion to the transference distance. Let t denote the unit transference time. We assume that their planning period is $2t$ if players are very kind and wait for them until that time the last possible customers may come to purchase it. For instance, if a customer stays at point 0 first visits Player II and he has nothing to sell, he will travel to Player I in order to satisfy his demand. We consider the planning period $2t$ as a single period and we deal with a single period inventory problem in this paper. The customers do not know inventory quantities which players have on hand at any time.

We assume that they know mutual unit ordering, holding and penalty costs and they are non-cooperative. The aim of each player is to minimize the personal total cost, i.e. the sum of the ordering cost, the holding cost and the penalty cost. Which strategies should they take to achieve their purposes? How much inventory quantities should they order at the beginning of the period? We study their strategies in a competitive inventory model using an equilibrium point in the context of game theory.

We make use of the following notations.

b	:	the number of customers on a market
x	:	the ordering quantity for Player I, which is a decision variable
y	:	the ordering quantity for Player II, which is a decision variable
c_i	:	the ordering cost per unit
r_i	:	the selling price per unit
h_i	:	the holding cost per unit
p_i	:	the penalty cost per unit
$Q_i(T)$	:	the inventory quantity at time T
I_1^i	:	the average quantity in inventory
I_2^i	:	the average shortage quantity in inventory
$C_j^i(x,y)$	:	the total cost per period

Here subscript i denotes the player's number and j denotes the number of situations described below. We give a natural assumption $r_i \geq c_i$ because of getting their rewards. The ordering quantities x and y are independently decided.

To begin with, we need consider some situations by the relation between the ordering quantities x, y and demand b. The total costs in each situation are calculated as follows. From these calculations we find that $C_j^i(x,y)$ are continuous in x and y on $[0,b]\times[0,b]$.

Situation 1. We consider the situation on $x \geq \frac{b}{2}$ and $y \geq \frac{b}{2}$ in the model. Players can supply the products to all customers when they first visit a player. Both of players do not yield the shortages in inventory. All demands of customers are satisfied by the time t. $Q_1(T)$, the inventory quantity for Player I, is represented by

$$\begin{aligned} Q_1(T) &= \begin{cases} x - \int_0^{T/t}(1-u)b\,du, & 0 \leq T < t \\ x - \int_0^1 (1-u)b\,du, & t \leq T \leq 2t \end{cases} \\ &= \begin{cases} x - \frac{T}{t}b + \frac{T^2}{2t^2}b, & 0 \leq T < t \\ x - \frac{1}{2}b, & t \leq T \leq 2t. \end{cases} \end{aligned} \tag{2.1}$$

Then the average quantity I_1^1, the average shortage quantity I_2^1 and the total cost $C_1^1(x,y)$ are calculated as follows:

$$\begin{aligned} I_1^1 &= \frac{1}{2t}\left[\int_0^t \left\{x - \frac{T}{t}b + \frac{T^2}{2t^2}b\right\} dT + \int_t^{2t}\left\{x - \frac{1}{2}b\right\} dT\right] = x - \frac{5}{12}b, \\ I_2^1 &= 0, \end{aligned}$$

and

$$\begin{aligned} C_1^1(x,y) &= c_1 \cdot x + h_1 \cdot I_1^1 + p_1 \cdot I_2^1 - r_1 \cdot \frac{1}{2}b \\ &= (c_1 + h_1)x - \left(\frac{5}{12}h_1 + \frac{1}{2}r_1\right) b. \end{aligned} \tag{2.2}$$

On the other hand $Q_2(T)$, the inventory quantity for Player II at time T, is represented by

$$\begin{aligned} Q_2(T) &= \begin{cases} y - \int_{1-T/t}^1 ub\,du, & 0 \leq T < t \\ y - \int_0^1 ub\,du, & t \leq T \leq 2t \end{cases} \\ &= \begin{cases} y - \frac{T}{t}b + \frac{T^2}{2t^2}b, & 0 \leq T < t \\ y - \frac{1}{2}b, & t \leq T \leq 2t. \end{cases} \end{aligned} \tag{2.3}$$

Calculating the total cost $C_1^2(x,y)$ for Player II as well as $C_1^1(x,y)$, we obtain

$$C_1^2(x,y) = (c_2 + h_2)y - \left(\frac{5}{12}h_2 + \frac{1}{2}r_2\right) b. \tag{2.4}$$

Situation 2. We next consider the situation on $0 \leq x < \frac{b}{2}, y \geq \frac{b}{2}$ and $x + y \geq b$. This situation supplies the products for all customers as well as Situation 1. It yields the shortages in inventory on Player I side, however they are satisfied by residual stock on hand of Player II side. The inventory quantity $Q_1(T)$ for Player I is given by Equation (2.1). Given a real number x, let t_1 denote the time T satisfying $0 \leq T < t$ and $x - \frac{T}{t}b + \frac{T^2}{2t^2}b = 0$. Then we regard t_1 as a function of x and let it write $t_1(x)$. Using the equation $x - \frac{t_1(x)}{t}b + \frac{\{t_1(x)\}^2}{2t^2}b = 0$ and calculating $C_2^1(x,y)$, we have the following equation:

$$C_2^1(x,y) = \left[c_1 + \frac{1}{6}h_1 - \frac{5}{6}p_1 - r_1\right] x + (h_1 + p_1)\left(\frac{t_1(x)}{3t}x - \frac{t_1(x)}{6t}b\right) + \frac{5}{12}p_1 b. \tag{2.5}$$

On the other hand, the inventory quantity $Q_2(T)$ is represented by

$$Q_2(T) = \begin{cases} y - \int_{1-T/t}^{1} ub\,du, & 0 \le T < t \\ y - \int_0^1 ub\,du, & t \le T \le t + t_1 \\ y - \int_0^1 ub\,du + Q_1(T-t), & t + t_1 < T \le 2t \end{cases}$$
$$= \begin{cases} y - \frac{T}{t}b + \frac{T^2}{2t^2}b, & 0 \le T < t \\ y - \frac{1}{2}b, & t \le T \le t + t_1 \\ x + y - \frac{2T}{t}b + \frac{T^2}{2t^2}b + b, & t + t_1 < T \le 2t. \end{cases} \tag{2.6}$$

The total cost $C_2^2(x,y)$ is given by

$$C_2^2(x,y) = [c_2 + h_2]y + \left[\frac{1}{3}h_2 + r_2\right]x - \left[\frac{7}{12}h_2 + r_2\right]b + h_2\left(\frac{t_1(x)}{6t}b - \frac{t_1(x)}{3t}x\right). \tag{2.7}$$

Situation 3. We consider the situation on $0 \le x < \frac{b}{2}, y \ge \frac{b}{2}$ and $x + y < b$. In this situation it yields the shortages in inventory on Player I side and not all the customers who have been not satisfied on Player I side are satisfied by stock on hand of Player II side. For Player I we obtain the similar results to those on Situation 2. On the other hand, the inventory quantity $Q_2(T)$ is given by Equation (2.6). Given real numbers x and y, let t_2 denote the time T satisfying $t + t_1 \le T < 2t$ and $x + y - \frac{2T}{t}b + \frac{T^2}{2t^2}b + b = 0$. Then we can regard t_2 as a function of x and y and let it write $t_2(x,y)$. Using the equations $x - \frac{t_1(x)}{t}b + \frac{\{t_1(x)\}^2}{2t^2}b = 0$ and $x + y - \frac{2t_2(x,y)}{t}b + \frac{\{t_2(x,y)\}^2}{2t^2}b + b = 0$ and arranging $C_3^2(x,y)$, we have the equation

$$C_3^2(x,y) = \left[c_2 + \frac{1}{3}h_2 - \frac{2}{3}p_2 - r_2\right]y - \left[\frac{1}{3}h_2 + \frac{2}{3}p_2\right]x + \left[\frac{1}{12}h_2 + \frac{2}{3}p_2\right]b$$
$$+h_2\left[\frac{t_1(x)}{6t}b - \frac{t_1(x)}{3t}x\right] + (h_2 + p_2)\frac{t_2(x,y)}{3t}(x + y - b). \tag{2.8}$$

Situation 4. We consider the situation on $0 \le x < \frac{b}{2}$ and $0 \le y < \frac{b}{2}$. In this situation only customers who have visited players earlier are satisfied their demands. If they are not satisfied by a player they have first visited, they are not satisfied after this. The inventory quantity $Q_1(T)$ is represented by

$$Q_1(T) = \begin{cases} x - \int_0^{T/t}(1-u)b\,du, & 0 \le T < t \\ x - \int_0^1 (1-u)b\,du, & t \le T \le t + t_3 \\ x - \int_0^1 (1-u)b\,du + Q_2(T-t), & t + t_3 < T \le 2t \end{cases}$$
$$= \begin{cases} x - \frac{T}{t}b + \frac{T^2}{2t^2}b, & 0 \le T < t \\ x - \frac{1}{2}b, & t \le T \le t + t_3 \\ x + y - \frac{2T}{t}b + \frac{T^2}{2t^2}b + b, & t + t_3 < T \le 2t. \end{cases} \tag{2.9}$$

$Q_2(T)$, the inventory quantity for Player II at time T, is given by Equation (2.6). Let t_3 denote the time T satisfying $0 \le T < t$ and $y - \frac{T}{t}b + \frac{T^2}{2t^2}b = 0$. Then we can regard t_3 as a function of y and let it write $t_3(y)$. Using the equations $x - \frac{t_1(x)}{t}b + \frac{\{t_1(x)\}^2}{2t^2}b = 0$ and $y - \frac{t_3(y)}{t}b + \frac{\{t_3(y)\}^2}{2t^2}b = 0$ and arranging $C_4^1(x,y)$, they are given by

$$C_4^1(x,y) = \left[c_1 + \frac{1}{6}h_1 - \frac{5}{6}p_1 - r_1\right]x + (h_1 + p_1)\left(\frac{t_1(x)}{3t}x - \frac{t_1(x)}{6t}b\right)$$
$$+p_1\left(-\frac{1}{3}y + \frac{7}{12}b + \frac{t_3(x)}{3t}y - \frac{t_3(y)}{6t}b\right). \tag{2.10}$$

Also we calculate $C_4^2(x,y)$ likewise and we have

$$C_4^2(x,y) = \left[c_2+\frac{1}{6}h_2-\frac{5}{6}p_2-r_2\right]y+(h_2+p_2)\left(\frac{t_3(y)}{3t}y-\frac{t_3(y)}{6t}b\right)$$
$$+p_2\left(-\frac{1}{3}x+\frac{7}{12}b+\frac{t_1(x)}{3t}x-\frac{t_1(x)}{6t}b\right). \quad (2.11)$$

Situation 5. We consider the situation on $x \geq \frac{b}{2}, 0 \leq y < \frac{b}{2}$ and $x+y \geq b$. This situation supplies the products for all customers as well as Situations 1 and 2. It yields the shortages in inventory on Player II side. However all the customers who have been not satisfied on Player II side are satisfied on Player I side. This case is the situation exchanged a role between Player I and II on Situation 2. Therefore the total costs $C_5^1(x,y)$ and $C_5^2(x,y)$ are given as follows:

$$C_5^1(x,y) = [c_1+h_1]x+\left[\frac{1}{3}h_1+r_1\right]y-\left[\frac{7}{12}h_1+r_1\right]b+h_1\left(\frac{t_3(y)}{6t}b-\frac{t_3(y)}{3t}y\right), \quad (2.12)$$
$$C_5^2(x,y) = \left[c_2+\frac{1}{6}h_2-\frac{5}{6}p_2-r_2\right]y+(h_2+p_2)\left(\frac{t_3(y)}{3t}y-\frac{t_3(y)}{6t}b\right)+\frac{5}{12}p_2b. \quad (2.13)$$

Situation 6. At last we consider the situation on $x \geq \frac{b}{2}$, $0 \leq y < \frac{b}{2}$ and $x+y<b$. In this situation it yields the shortages in inventory on Player II side and not all the customers who have been not satisfied on Player II side are satisfied them on Player I side. This is the situation exchanged a role between Player I and II on Situation 3. Therefore the total costs $C_6^1(x,y)$ and $C_6^2(x,y)$ are given as follows:

$$C_6^1(x,y) = \left[c_1+\frac{1}{3}h_1-\frac{2}{3}p_1-r_1\right]x-\left[\frac{1}{3}h_1+\frac{2}{3}p_1\right]y+\left[\frac{1}{12}h_1+\frac{2}{3}p_1\right]b$$
$$+h_1\left[\frac{t_3(y)}{6t}b-\frac{t_3(y)}{3t}y\right]+(h_1+p_1)\frac{t_4(x,y)}{3t}(x+y-b), \quad (2.14)$$
$$C_6^2(x,y) = \left[c_2+\frac{1}{6}h_2-\frac{5}{6}p_2-r_2\right]y+(h_2+p_2)\left(\frac{t_3(y)}{3t}y-\frac{t_3(y)}{6t}b\right)+\frac{5}{12}p_2b. \quad (2.15)$$

Here $t_4(x,y)$ denotes the time T satisfying $t+t_3 \leq T < 2t$ and $x+y-\frac{2T}{t}b+\frac{T^2}{2t^2}b+b=0$.

3. Equilibrium Point

In this section we formulate our model as a two-person non-zero-sum game and we find an equilibrium point in this model. Because a player becomes to have useless quantity in inventory if he orders products more than b, he never take such a behavior. Therefore we restrict his behavior to the interval $[0,b]$. Including all possibility in the range $[0,b]\times[0,b]$, we find an equilibrium point in the concepts of a mixed strategy with continuous strategies.

Let the *cdf* $F(x)$ consist of a mass part $\alpha_1 \geq 0$ at point 0, a density part $f(x)>0$ over an interval $(0,b)$ and a mass part $\alpha_2 \geq 0$ at point b; and let the *cdf* $G(y)$ consist of a mass part $\beta_1 \geq 0$ at point 0, a density part $g(y)>0$ over an interval $(0,b)$ and a mass part $\beta_2 \geq 0$ at point b. Here it must hold that $\alpha_1+\int_0^b f(x)dx+\alpha_2=1$ and $\beta_1+\int_0^b g(y)dy+\beta_2=1$. We suppose that Player II uses the mixed strategy $G(y)$ such that Player II orders y products with *cdf* $G(y)$ if Player I orders x products. The expected payoff kernel $M_1(x,G)$ for Player

I is given by

$$M_1(x,G) = \begin{cases} \beta_1 C_4^1(0,0) + \int_0^{b/2} C_4^1(0,y)g(y)dy + \int_{b/2}^{b} C_3^1(0,y)g(y)dy + \beta_2 C_3^1(0,b), \\ \qquad x=0 \\ \beta_1 C_4^1(x,0) + \int_0^{b/2} C_4^1(x,y)g(y)dy + \int_{b/2}^{b-x} C_3^1(x,y)g(y)dy \\ \quad + \int_{b-x}^{b} C_2^1(x,y)g(y)dy + \beta_2 C_2^1(x,b), \qquad 0 < x < \frac{b}{2} \\ \beta_1 C_6^1(x,0) + \int_0^{b-x} C_6^1(x,y)g(y)dy + \int_{b-x}^{b/2} C_5^1(x,y)g(y)dy \\ \quad + \int_{b/2}^{b} C_1^1(x,y)g(y)dy + \beta_2 C_1^1(x,b), \qquad \frac{b}{2} \le x < b \\ \beta_1 C_5^1(b,0) + \int_0^{b/2} C_5^1(b,y)g(y)dy + \int_{b/2}^{b} C_1^1(b,y)g(y)dy + \beta_2 C_1^1(b,b), \\ \qquad x=b \end{cases}$$

$$= \begin{cases} [c_1 + \frac{1}{6}h_1 - \frac{5}{6}p_1 - r_1]x + (h_1+p_1)(\frac{t_1(x)}{3t}x - \frac{t_1(x)}{6t}b) + \frac{5}{12}p_1 b \\ \quad + p_1 \int_0^{b/2} (-\frac{1}{3}y + \frac{1}{6}b + \frac{t_3(y)}{3t}y - \frac{t_3(y)}{6t}b) g(y)dy + \frac{1}{6}p_1\beta_1 b, \qquad 0 \le x < b/2 \\ [c_1 + h_1]x - (\frac{5}{12}h_1 + \frac{1}{2}r_1)b \\ \quad + \int_0^{b/2} \left\{ [\frac{1}{3}h_1 + r_1]y - [\frac{h_1}{6} + \frac{r_1}{2}]b + h_1 \left(\frac{t_3(y)}{6t}b - \frac{t_3(y)}{3t}y \right) \right\} g(y)dy \\ \quad + \int_0^{b-x} \left\{ [\frac{2}{3}h_1 + \frac{2}{3}p_1 + r_1](b-x-y) + (h_1+p_1)\frac{t_4(x,y)}{3t}(x+y-b) \right\} \\ \quad \cdot g(y)dy + \beta_1 \left\{ -[\frac{2}{3}h_1 + \frac{2}{3}p_1 + r_1]x + [\frac{1}{2}h_1 + \frac{2}{3}p_1 + \frac{1}{2}r_1]b \right. \\ \quad \left. + (h_1+p_1)\frac{t_4(x,0)}{3t}(x-b) \right\}, \qquad b/2 \le x \le b. \end{cases} \tag{3.1}$$

Also we suppose that Player I uses the mixed strategy $F(x)$ such that Player I orders x products with *cdf* $F(x)$ if Player II orders y products. The expected payoff kernel $M_2(F,y)$ for Player II is given by

$$M_2(F,y) = \begin{cases} \alpha_1 C_4^2(0,0) + \int_0^{b/2} C_4^2(x,0)f(x)dx + \int_{b/2}^{b} C_6^2(x,0)f(x)dx + \alpha_2 C_6^2(b,0), \\ \qquad y=0 \\ \alpha_1 C_4^2(0,y) + \int_0^{b/2} C_4^2(x,y)f(x)dx + \int_{b/2}^{b-y} C_6^2(x,y)f(x)dx \\ \quad + \int_{b-y}^{b} C_5^2(x,y)f(x)dx + \alpha_2 C_5^2(b,y), \qquad 0 < y < \frac{b}{2} \\ \alpha_1 C_3^2(0,y) + \int_0^{b-y} C_3^2(x,y)f(x)dx + \int_{b-y}^{b/2} C_2^2(x,y)f(x)dx \\ \quad + \int_{b/2}^{b} C_1^2(x,y)f(x)dx + \alpha_2 C_1^2(b,y), \qquad \frac{b}{2} \le y < b \\ \alpha_1 C_2^2(0,b) + \int_0^{b/2} C_2^2(x,b)f(x)dx + \int_{b/2}^{b} C_1^2(x,b)f(x)dx + \alpha_2 C_1^2(b,b), \\ \qquad y=b \end{cases}$$

$$= \begin{cases} [c_2 + \frac{1}{6}h_2 - \frac{5}{6}p_2 - r_2]y + (h_2+p_2)(\frac{t_3(y)}{3t}y - \frac{t_3(y)}{6t}b) + \frac{5}{12}p_2 b \\ \quad + p_2 \int_0^{b/2} (-\frac{1}{3}x + \frac{1}{6}b + \frac{t_1(x)}{3t}x - \frac{t_1(x)}{6t}b) f(x)dx + \frac{1}{6}p_2\alpha_1 b, \qquad 0 \le y < b/2 \\ [c_2 + h_2]y - (\frac{5}{12}h_2 + \frac{1}{2}r_2)b \\ \quad + \int_0^{b/2} \left\{ [\frac{1}{3}h_2 + r_2]x - [\frac{h_2}{6} + \frac{r_2}{2}]b + h_2 \left(\frac{t_1(x)}{6t}b - \frac{t_1(x)}{3t}x \right) \right\} f(x)dx \\ \quad + \int_0^{b-y} \left\{ [\frac{2}{3}h_2 + \frac{2}{3}p_2 + r_2](b-x-y) + (h_2+p_2)\frac{t_2(x,y)}{3t}(x+y-b) \right\} \\ \quad \cdot f(x)dx + \alpha_1 \left\{ -[\frac{2}{3}h_2 + \frac{2}{3}p_2 + r_2]y + [\frac{1}{2}h_2 + \frac{2}{3}p_2 + \frac{1}{2}r_2]b \right. \\ \quad \left. + (h_2+p_2)\frac{t_2(0,y)}{3t}(y-b) \right\}, \qquad b/2 \le y \le b. \end{cases} \tag{3.2}$$

To find the minimizer x we differentiate Equation (3.1) with respect to x. Using the equations

$$\frac{\partial t_1(x)}{\partial x} = \frac{t^2}{(t-t_1(x))b}; \qquad \frac{\partial t_4(x,y)}{\partial x} = \frac{t^2}{(2t-t_4(x,y))b}, \tag{3.3}$$

we have

$$\frac{\partial M_1(x,G)}{\partial x} = \begin{cases} c_1 - p_1 - r_1 + (h_1+p_1)\frac{t_1(x)}{2t}, & 0 \le x < b/2 \\ c_1 + h_1 - [h_1+p_1+r_1]\int_0^{b-x} g(y)dy + (h_1+p_1)\int_0^{b-x}\frac{t_4(x,y)}{2t}g(y)dy \\ \quad +\beta_1\{-[h_1+p_1+r_1] + (h_1+p_1)\frac{t_4(x,0)}{2t}\}, & b/2 \le x \le b, \end{cases} \tag{3.4}$$

and

$$\frac{\partial^2 M_1(x,G)}{\partial x^2} = \begin{cases} \frac{1}{2t}(h_1+p_1)\frac{\partial t_1(x)}{\partial x}, & 0 \le x < b/2 \\ r_1 g(b-x) + \frac{1}{2t}(h_1+p_1)\int_0^{b-x}\frac{\partial t_4(x,y)}{\partial x}g(y)dy \\ \quad +\beta_1\frac{1}{2t}(h_1+p_1)\frac{\partial t_4(x,0)}{\partial x}, & b/2 \le x \le b. \end{cases} \tag{3.5}$$

Therefore we see that $M_1(x,G)$ is a strictly convex function, so there uniquely exists the optimal strategy for Player I. Here we consider three cases.

- Case on $\lim_{x\to b/2-0}\frac{\partial M_1(x,G)}{\partial x} > 0$:
  Now we find the optimal ordering quantity x^* that is satisfied $\frac{\partial M_1(x,G)}{\partial x} = 0$. Setting the partial derivatives of $M_1(x,G)$ as zero, we obtain the value

$$t_1^* = \frac{2(r_1 - c_1 + p_1)}{h_1 + p_1}t. \tag{3.6}$$

  As it must follow $0 \le t_1^* < t$, we have a sufficient condition $0 \le r_1 - c_1 < \frac{h_1+p_1}{2}$. Substituting Equation (3.6) for $x^* = \frac{t_1^*}{t}b - \frac{(t_1^*)^2}{2t^2}b$, we have the optimal ordering quantity

$$x^* = \frac{2(c_1 - r_1 + h_1)(r_1 - c_1 + p_1)}{(h_1+p_1)^2}b. \tag{3.7}$$

- Case on $\lim_{x\to b/2-0}\frac{\partial M_1(x,G)}{\partial x} \le 0$ and $\lim_{x\to b/2+0}\frac{\partial M_1(x,G)}{\partial x} \ge 0$:
  It is easy to see that the optimal strategy x^* is equal to $\frac{b}{2}$ for Player I.
- Case on $\lim_{x\to b/2+0}\frac{\partial M_1(x,G)}{\partial x} < 0$:
  It must follow $0 \le y < b/2$ in this case. Using a uniqueness of the optimal strategy y^* for Player II under the similar analysis on $M_2(F,y)$, Equation (3.4) is rewritten as follows:

$$\frac{\partial M_1(x,G)}{\partial x} = \begin{cases} c_1 - p_1 - r_1 + (h_1+p_1)\frac{t_4(x,y^*)}{2t}, & 0 < y^* < b-x \\ c_1 + h_1, & y^* \ge b-x. \end{cases} \tag{3.8}$$

  Because $M_1(x,G)$ is an increasing linear function in x if y^* is greater than or equal to $b-x$, it is clear that he had better have less inventory quantity. Hence Player I chooses $x^* = b - y^*$ for his optimal strategy. Otherwise, we find the minimizer x^* on Equation (3.8). Setting the partial derivatives of $M_1(x,G)$ as zero, we have the value

$$t_4^* = \frac{2(r_1 - c_1 + p_1)}{h_1 + p_1}t. \tag{3.9}$$

  As it must follow $t + t_3^* \le t_4^* < 2t$, we have sufficient conditions $\frac{h_1-p_1}{2} + \frac{(h_1+p_1)(r_2-c_2+p_2)}{h_2+p_2} < r_1 - c_1 < h_1$ and $0 \le r_2 - c_2 < \frac{h_2-p_2}{2}$. Here we use the value $\frac{2(r_2-c_2+p_2)}{h_2+p_2}t$ for t_3^*. Substituting Equation (3.9) for $x^* = \frac{2t_4^*}{t}b - \frac{2(t_4^*)^2}{2t^2}b - b - y^*$, we obtain the optimal ordering quantity

$$x^* = \frac{b}{2} + \frac{(2r_2 - 2c_2 - h_2 + p_2)^2}{2(h_2+p_2)^2}b - \frac{2(r_1 - c_1 - h_1)^2}{(h_1+p_1)^2}b, \tag{3.10}$$

where we have made use of $y^* = \frac{2(c_2-r_2+h_2)(r_2-c_2+p_2)}{(h_2+p_2)^2}b$ with respect to y.

The above considerations lead us to the following theorem.

Theorem 1. *For nonzero-sum game $M_1(x,y), M_2(x,y), 0 \le x, y \le b$, there exist values v_1^0 and v_2^0 such that*

$$M_1(x,G) \ge M_1(x,G^*) = v_1^0, \qquad M_2(F,y) \ge M_2(F^*,y) = v_2^0, \tag{3.11}$$

where

$$F^*(x) = \begin{cases} 0, & 0 \le x < x^* \\ 1, & x^* \le x \le b, \end{cases} \qquad G^*(y) = \begin{cases} 0, & 0 \le y < y^* \\ 1, & y^* \le y \le b. \end{cases} \tag{3.12}$$

This pair (x^*, y^*) is an equilibrium point. Then we obtain the following equilibrium points as a result of this paper.

(i) *If it follows the sufficient conditions $0 \le r_i - c_i < \frac{h_i-p_i}{2}$, $i = 1,2$, then the equilibrium point is $\left(\frac{2(c_1-r_1+h_1)(r_1-c_1+p_1)}{(h_1+p_1)^2}b, \frac{2(c_2-r_2+h_2)(r_2-c_2+p_2)}{(h_2+p_2)^2}b\right)$;*

(ii) *If it follows the sufficient conditions $0 \le r_1 - c_1 < \frac{h_1-p_1}{2}$ and $\frac{h_2-p_2}{2} + \frac{(h_2+p_2)(r_1-c_1+p_1)}{h_1+p_1} \le r_2 - c_2 < h_2$, then the equilibrium point is $\left(\frac{2(c_1-r_1+h_1)(r_1-c_1+p_1)}{(h_1+p_1)^2}b, \frac{b}{2} + \frac{(2r_1-2c_1-h_1+p_1)^2}{2(h_1+p_1)^2}b - \frac{2(r_2-c_2-h_2)^2}{(h_2+p_2)^2}b\right)$;*

(iii) *If it follows the sufficient conditions $0 \le r_1 - c_1 < \frac{h_1-p_1}{2}$ and $\frac{h_2-p_2}{2} \le r_2 - c_2 < \frac{h_2-p_2}{2} + \frac{(h_2+p_2)(r_1-c_1+p_1)}{h_1+p_1}$, then the equilibrium point is $\left(\frac{2(c_1-r_1+h_1)(r_1-c_1+p_1)}{(h_1+p_1)^2}b, \frac{b}{2}\right)$;*

(iv) *If it follows the sufficient conditions $0 \le r_1 - c_1 < \frac{h_1-p_1}{2}$ and $r_2 - c_2 \ge h_2$, then the equilibrium point is $\left(\frac{2(c_1-r_1+h_1)(r_1-c_1+p_1)}{(h_1+p_1)^2}b, b - \frac{2(c_1-r_1+h_1)(r_1-c_1+p_1)}{(h_1+p_1)^2}b\right)$;*

(v) *If it follows the sufficient conditions $\frac{h_1-p_1}{2} + \frac{(h_1+p_1)(r_2-c_2+p_2)}{h_2+p_2} \le r_1 - c_1 < h_1$ and $0 \le r_2 - c_2 < \frac{h_2-p_2}{2}$, then the equilibrium point is $\left(\frac{b}{2} + \frac{(2r_2-2c_2-h_2+p_2)^2}{2(h_2+p_2)^2}b - \frac{2(r_1-c_1-h_1)^2}{(h_1+p_1)^2}b, \frac{2(c_2-r_2+h_2)(r_2-c_2+p_2)}{(h_2+p_2)^2}b\right)$;*

(vi) *If it follows the sufficient conditions $\frac{h_1-p_1}{2} \le r_1 - c_1 < \frac{h_1-p_1}{2} + \frac{(h_1+p_1)(r_2-c_2+p_2)}{h_2+p_2}$ and $0 \le r_2 - c_2 < \frac{h_2-p_2}{2}$, then the equilibrium point is $\left(\frac{b}{2}, \frac{2(c_2-r_2+h_2)(r_2-c_2+p_2)}{(h_2+p_2)^2}b\right)$;*

(vii) *If it follows the sufficient conditions $r_1 - c_1 \ge h_1$ and $0 \le r_2 - c_2 < \frac{h_2-p_2}{2}$, then the equilibrium point is $\left(b - \frac{2(c_2-r_2+h_2)(r_2-c_2+p_2)}{(h_2+p_2)^2}b, \frac{2(c_2-r_2+h_2)(r_2-c_2+p_2)}{(h_2+p_2)^2}b\right)$; and*

(viii) *If it follows the sufficient conditions $r_i - c_i \ge \frac{h_i-p_i}{2}$, $i = 1,2$, then the equilibrium point is $\left(\frac{b}{2}, \frac{b}{2}\right)$.*

4. A Numerical Example

In this section we give a numerical example of equilibrium points and their total costs for some fixed values h_i, p_i and $r_i - c_i$. The values $\frac{b}{2}, \frac{2(c_i-r_i+h_i)(r_i-c_i+p_i)}{(h_i+p_i)^2}b$ having obtained in the previous section yield the similar results to the ordering quantities that we obtain in the model when a player is not affected by another player at all. Hence we are interested in the cases which we cannot obtain in a single player's model, particularly Cases (ii) and (v). Since Player I and II play their symmetrical roles in this model, we deal with Case (ii) in a numerical example. We give the results on Table 1 through 4 by using the values $h_1 = 3, p_1 = 5, h_2 = 3, p_2 = 2, b = 1000$. We consider negative values of total costs as his rewards on Table 3.

Now we are interested in Player I side representing the values which we cannot obtain when a player does not have an influence on another player. For instance we see the total

Table 1: The optimal strategy x^* for Player I

r_2-c_2									
0.4								500.5	
0.3						500.4	502.0	502.9	
0.2					502.2	504.4	506.0	506.9	
0.1			501.6	505.0	507.8	510.0	511.5	512.5	
0.0	500.0	504.7	508.8	512.2	515.0	517.2	518.8	519.7	
	2.2	2.3	2.4	2.5	2.6	2.7	2.8	2.9	r_1-c_1

Table 2: The optimal strategy y^* for Player II

r_2-c_2									
0.4								499.2	
0.3						496.8	496.8	496.8	
0.2					492.8	492.8	492.8	492.8	
0.1			487.2	487.2	487.2	487.2	487.2	487.2	
0.0	480.0	480.0	480.0	480.0	480.0	480.0	480.0	480.0	
	2.2	2.3	2.4	2.5	2.6	2.7	2.8	2.9	r_1-c_1

Table 3: The total costs for Player I using x^*

r_2-c_2									
0.4								-1200.0	
0.3						-1099.6	-1149.7	-1199.9	
0.2					-1048.7	-1099.0	-1149.5	-1200.2	
0.1			-946.6	-997.0	-1047.6	-1098.5	-1149.6	-1200.8	
0.0	-843.3	-893.6	-944.3	-995.3	-1046.7	-1098.3	-1150.1	-1202.0	
	2.2	2.3	2.4	2.5	2.6	2.7	2.8	2.9	r_1-c_1

Table 4: The total costs for Player II using y^*

r_2-c_2									
0.4								50.0	
0.3						99.8	99.8	99.8	
0.2					149.3	149.3	149.3	149.3	
0.1			198.3	198.3	198.3	198.3	198.3	198.3	
0.0	246.7	246.7	246.7	246.7	246.7	246.7	246.7	246.7	
	2.2	2.3	2.4	2.5	2.6	2.7	2.8	2.9	r_1-c_1

costs for Player I have a few influence on the price of Player II on $r_1 - c_1 = 2.9$. On the other hand, Player II does not have an influence on Player I at all. Because they have the strategies with the other player under consideration, it is nothing that the sum of their strategies is over b.

5. Concluding Remarks

In this paper we deal with a single period competitive inventory model with the choice probability proportional to distance on a line segment market. Cases (ii), (iv), (v) and (vii) are new results having obtained in this work. This inventory problem was formulated as one of games with pure strategies of continuous cardinary. Consequently the mixed strategy was consistent with the pure strategy. Since all of costs were linear, the total cost became a strictly convex function, so we uniquely had the solution. We are also able to use this method in the non-linear case. Hence we will be able to find equilibrium points by means of the mixed strategy for more complicated models. Also this model will be able to be extended to a multi-period model and it will be compared with models changed assumptions. They are further research problems and we hope this paper becomes a stepstone of research on competitive inventory problems.

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