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ON A GENERALIZATION OF THE SECRETARY PROBLEM WITH UNCERTAIN SELECTION

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Abstract The secretary problem with uncertain selection, considered by Smith, is generalized to allow for the rejection probability to be rank-dependent. That is, if an offer of employment is given to the j-th best applicant, she rejects it with probability $q_j, 1 \leq j \leq n$ (n is the number of applicants to appear). The optimality equations can be derived with the objective of maximizing the probability of selecting the best applicant. Since giving general guidelines of the optimal policy is difficult, we focus our attention on the simplified problem, called the m-problem, where the probability sequence $\{q_j; 1 \leq j \leq n\}$ satisfies $q_{m+1} = q_{m+2} = \cdots = q_n$, $0 \leq m \leq n-1$. The value m plays a role that regulates the difficulty of the problem (the 0-problem is the Smith problem). We solve the 1- and 2-problems explicitly in both the finite and the asymptotic cases. The optimal policy of the 1-problem is shown to be a threshold rule, *i.e.*, it passes over some portion of the applicants and then makes an offer to relatively best applicants successively. As for the 2-problem, it can be shown that the optimal policy becomes a threshold rule if $q_2 \geq q_3$, while as n gets large there appears a time interval such that the optimal policy makes an offer alternately to relatively best applicants that appear on that interval if $q_2 < q_3$. We also present some numerical results for the 3-problem which demonstrate the complexity of the optimal policy. Our results give some affirmative evidences to the *conjecture* that the optimal policy remains a threshold rule so far as the sequence $\{q_j; 1 \leq j \leq n\}$ satisfies the monotone condition $q_1 \geq q_2 \geq \cdots \geq q_n$, which reflects the real world in the sense that the better the applicant is, the most likely it seems that she refuses an offer with the larger probability.

1. Introduction

Before discussing our problems, we review briefly the Smith[6] problem. A set of n rankable applicants (1 being the best and n the worst) appears before us one at a time in random order with all $n!$ permutations equally likely. Each time an applicant appears, we only observe the rank of the applicant relative to those preceding her and decide, based on the observed rank, whether to make an offer of selection to the current applicant or to pass over her and observe the next (if any). When an offer is given, the applicant accepts(rejects) it with a known fixed probability $p(q = 1-p)$, independent of the rank of the applicant and all else. If the applicant rejects an offer, we further observe the next. No recall of the previous applicants is allowed and the process continues until an offer is accepted(i.e., an applicant is selected) or the final stage is reached with no offer accepted. The objective is to find a policy that will maximize the probability of selecting the best overall. We shall abbreviate the event "selecting the best overall" to the single word *success*.

In this paper, we generalize the Smith problem to allow the probability of acceptance (rejection) to be rank-dependent. That is, for instance, if an offer is given to the j-th best applicant, she accepts(rejects) it with probability $p_j(q_j = 1 - p_j)$, $1 \leq j \leq n$, independent of all else. We assume $0 < q_j < 1$, unless otherwise specified and sometimes refer to the Smith problem as rank-independent problem for the sake of contrast.

In Section 2, we attempt to derive the optimality equations of our problems. When an applicant appears, the decision of either making an offer or not must be based not only on the relative rank of the current applicant but also on the sequence of the relative ranks of the applicants that refused an offer previously (if any), because in the rank-dependent case our knowledge about the true rank of the current applicant undergoes the Bayesian updating through these information. For any given sequence $\{q_j; 1 \le j \le n\}$, the optimality equations are given by Eq.(2.7). These equations can be solved in principle recursively to yield the optimal policy and the success probability, because $b_r(i_1, \cdots, i_k)$ is calculable from (2.9). Let us call an applicant a *candidate* for simplicity if she is best among those observed so far, i.e., relatively best applicant. Then Eq.(2.7) tells us that the optimal policy only makes an offer to candidate(s) (this is intuitive but not a priori clear).

Smith[6] showed that, in the rank-independent case, the optimal policy falls under the category of the threshold rule, when we call a policy *threshold rule* or more specifically r-threshold rule if the policy passes over the first $r-1$ applicants and then makes an offer successively to candidates that appear. However, giving general guidelines of the optimal policy seems difficult in the rank-dependent case. To have some meaningful results, we must restrict ourselves to some class of the sequences. Let us now define the problem as the m-problem if the sequence $\{q_j; 1 \le j \le n\}$ satisfies $q_{m+1} = q_{m+2} = \cdots = q_n, 0 \le m \le n-1$. We can regard the value m as a parameter that regulates the difficulty of the problem (as m gets larger, the problem becomes harder). As shown in Section 3, this restriction brings considerable simplification to the corresponding optimality equations. The 0-problem is, of course, the rank-independent problem treated by Smith[6]. The 1-problem is shown to be essentially equivalent to the 0-problem. Our main concerns in this paper are to solve the 2-problem. In this problem, the form of the optimal policy differs depending on which of the two values q_2 and $q_3 (= q_4 = \cdots = q_n)$ is larger. It can be shown that the optimal policy becomes a threshold rule if $q_2 \ge q_3$, while, when n is sufficiently large, there exists a time interval such that the optimal policy makes an offer alternately to candidates that appear on that interval if $q_2 < q_3$ (see Theorem 3.5). In Section 3.3, we solve the 3-problem numerically and observe that the optimal policy becomes very complicated for some set of values of q_2, q_3 and q_4 $(= q_5 = \cdots = q_n)$. The m-problems for $m \ge 4$ are left for a future study.

From the practical point of view, it is very important to consider the monotone sequence, i.e., $q_1 \ge q_2 \ge \cdots \ge q_n$, because, in the real world, the better the applicant is, the most likely it seems that she refuses an offer with the larger probability. In the light of the computational results from the 3-problem, in addition to the analytical results from the 0-, 1- and 2-problems, we'd like to make the following conjecture : If the sequence $\{q_j; 1 \le j \le n\}$ satisfies the condition $q_1 \ge q_2 \ge \cdots \ge q_n$, then the optimal policy becomes a threshold rule. The authours believe that this conjecture is a challenging problem.

For related works with uncertain selection, see Petruccelli[4],Tamaki[7],[8] and McNamara and Collins[2] (treated in the game theoretic approach). For a history and review of the secretary problem, the reader is referred to Ferguson[1] and Samuels[5].

2. Optimality Equations

Unless otherwise specified, lower case letters will denote integers. Let $N_0 = \phi$ and $N_r = \{1, \cdots, r\}, 1 \le r \le n$. Each time an applicant appears, we must decide either to make an offer (action a_1) or make no offer (action a_2), based on the number of the applicants observed so far and the ranks of both the current applicant and the previous applicants that rejected an offer (if any), relative to these observations. After the $(r-1)$-st decision, suppose that the ranks of the applicants that rejected an offer (if any) relative to the first $r-1$ constitute the set $(i_1,\cdots,i_k)\subset N_{r-1}$. This set $(i_1,\cdots,i_k)$ is referred to as *rejection history* and assumed to be arranged in ascending order, i.e., $i_1<\cdots<i_k$. If $r-1=n$, the trial terminates with no applicant selected and leads to a failure; otherwise the relative rank $X_r\in N_r$ of the r-th applicant is observed. The probability law of X_r depends on the rejection history and hence we denote by $p_r(i;i_1,\cdots,i_k)$ the conditional probability of $X_r=i$ given the rejection history $(i_1,\cdots,i_k)$ at time $r-1$. (Observe that, for example, $p_r(i;i_1,\cdots,i_k)\equiv 1/r$ for the rank-independent case). Assume now that the rejection history is $(i_1,\cdots,i_k)$ and $X_r=i$ has just been observed. If the r-th decision is a_1, then the process terminates if the offer is accepted; otherwise the process continues updating $(i_1,\cdots,i_k)$ to $(i_1,\cdots,i_k)^*i$, where

$$(i_1,\cdots,i_k)^*i=\begin{cases}(i,i_1+1,\cdots,i_k+1) & 1\le i\le i_1\\ (i_1,\cdots,i_{t-1},i,i_t+1,\cdots,i_k+1) & i_{t-1}<i\le i_t\ (2\le t\le k)\\ (i_1,\cdots,i_k,i) & i_k<i\le r.\end{cases}\tag{2.1}$$

If the r-th decision is a_2, then the process continues updating $(i_1,\cdots,i_k)$ to $(i_1,\cdots,i_k)^\circ i$, where

$$(i_1,\cdots,i_k)^\circ i=\begin{cases}(i_1+1,\cdots,i_k+1) & 1\le i\le i_1\\ (i_1,\cdots,i_{t-1},i_t+1,\cdots,i_k+1) & i_{t-1}<i\le i_t\ (2\le t\le k)\\ (i_1,\cdots,i_k) & i_k<i\le r.\end{cases}\tag{2.2}$$

Unless the process terminates thereby, the subsequent applicant with relative rank X_{r+1} is observed. X_{r+1} will take value j with probability $p_{r+1}(j;(i_1,\cdots,i_k)^*i)$ or $p_{r+1}(j;(i_1,\cdots,i_k)^\circ i)$ depending on whether the decision taken at time r is a_1 or a_2. If no rejection has occurred so far, rejection history $(i_1,\cdots,i_k)$ is to be interpreted as an empty set ϕ.

Formally then, we have a finite horizon Markov decision process with $S_r=\{(i;i_1,\cdots,i_k): i\in N_r,(i_1,\cdots,i_k)\subset N_{r-1}\}$ as the collection of states at time r ; $K=\{a_1,a_2\}$ as the action space ; and transition probabilities $p_r(i;i_1,\cdots,i_k), 1\le r\le n$ as described above. Denote by Π the set of (deterministic, Markov) policies $\pi(\pi_1,\cdots,\pi_n)$, where $\pi_r:S_r\to K, 1\le r\le n$. The value of the process is $v^*=\sup_{\pi\in\Pi}E_\pi[I_S]$, where I_S is the indicator function of the success event S, and E_π is the expectation operator under policy π. $\pi\in\Pi$ is optimal if $v^*=E_\pi[I_S]$. Π_0 will denote the set of optimal policies.

Let $s_r(i;i_1,\cdots,i_k)(c_r(i;i_1,\cdots,i_k))$ be the probability of success when we take action a_1 (a_2) in state $(i;i_1,\cdots,i_k)\in S_r$ and proceed optimally thereafter. Then max $\{s_r(i;i_1,\cdots,i_k), c_r(i;i_1,\cdots,i_k)\}$ represents the maximum probability of success given that the process is in state $(i;i_1,\cdots,i_k)\in S_r$. Write, for $0\le r-1<n$,

$$v_{r-1}(i_1,\cdots,i_k)=\sum_{i=1}^{r}p_r(i;i_1,\cdots,i_k)\max\{s_r(i;i_1,\cdots,i_k),c_r(i;i_1,\cdots,i_k)\},\tag{2.3}$$

with $v_n(i_1,\cdots,i_k)\equiv 0$. Then $v^*=v_0(\phi)=E_{\pi^*}[I_S]$ for the policy $\pi^*\in\Pi$ such that

$$\pi_r^*(i;i_1,\cdots,i_k)=a_1(a_2)\qquad if\qquad s_r(i;i_1,\cdots,i_k)\ge(<)c_r(i;i_1,\cdots,i_k)$$

for $(i;i_1,\cdots,i_k)\in S_r, 1\le r\le n$; thus $\pi^*\in\Pi_0$.

To make Eq.(2.3) work as a recursive formula, we must describe $s_r(i;i_1,\cdots,i_k)$ and $c_r(i;i_1,\cdots,i_k)$ in terms of $v_r(.)$ and derive the calculable form for $p_r(i;i_1,\cdots,i_k)$. To do so, we introduce some quantities. Let $a_r(i_1,\cdots,i_k)$ for $(i_1,\cdots,i_k)\subset N_r, 1\le r\le n$ be defined as the probability that all offers will be rejected, provided that k offers were made respectively to i_1-th best, $\cdots$, and to i_k-th best among the first r applicants. More specifically, if we denote by $C(r,i;n), 1\le i\le r, 1\le r\le n$ the i-th best among the first r applicants when the total number of applicants is n and denote by $A(r,i;n)$ the absolute rank of $C(r,i;n)$, i.e., $A(r,i;n)=j$ if $C(r,i;n)$ is $C(n,j;n)$, then we can write

$$a_r(i_1,\cdots,i_k)=E\left[\prod_{t=1}^{k} q_{A(r,i_t;n)}\right], \tag{2.4}$$

where E represents the expectation with respect to random variables $\{A(r,i_t;n), 1\le t\le k\}$. For an empty set we assume $a_r(\phi)=1$. Whereas $a_r(i_1,\cdots,i_k)$ is defined for the entire population of size n having sequence $\{q_j; 1\le j\le n\}$, similar quantity $b_r(i_1,\cdots,i_k)$ can be defined for the subpopulation of size $n-1$, which is constructed by taking out the best applicant from the entire population. Thus, for $(i_1,\cdots,i_k)\subset N_r, 1\le r\le n-1$

$$b_r(i_1,\cdots,i_k)=E\left[\prod_{t=1}^{k} q_{A(r,i_t;n-1)}\right]. \tag{2.5}$$

Note that, for this subpopulation, the j-th best rejects an offer with probability $q_{j+1}, 1\le j\le n-1$.

We now have the following lemmas.

Lemma 2.1 For $(i;i_1,\cdots,i_k)\in S_r, 1\le r\le n$,

$$p_r(i;i_1,\cdots,i_k)=\frac{1}{r}\left[\frac{a_r((i_1,\cdots,i_k)^\circ i)}{a_{r-1}(i_1,\cdots,i_k)}\right]$$

Proof. See Appendix A.1.

Lemma 2.2 For $(i;i_1,\cdots,i_k)\in S_r, 1\le r\le n$,

$$s_r(i;i_1,\cdots,i_k) = \begin{cases} p_1\left(\frac{r}{n}\right)\left[\frac{b_{r-1}(i_1,\cdots,i_k)}{a_r((i_1,\cdots,i_k)^\circ 1)}\right]+v_r((i_1,\cdots,i_k)^*1)\left[\frac{a_r((i_1,\cdots,i_k)^*1)}{a_r((i_1,\cdots,i_k)^\circ 1)}\right], & i=1 \\ v_r((i_1,\cdots,i_k)^*i)\left[\frac{a_r((i_1,\cdots,i_k)^*i)}{a_r((i_1,\cdots,i_k)^\circ i)}\right], & 2\le i\le r \end{cases}$$

$$c_r(i;i_1,\cdots,i_k)=v_r((i_1,\cdots,i_k)^\circ i), \qquad 1\le i\le r$$

Proof. See Appendix A.2.

Now define

$$V_r(i_1,\cdots,i_k)=a_r(i_1,\cdots,i_k)v_r(i_1,\cdots,i_k)/p_1.$$

Then applying Lemmas 2.1 and 2.2 to Eq.(2.3), we have the following form of the optimality equation

$$V_{r-1}(i_1,\cdots,i_k) = \frac{1}{r}\max\left\{\frac{r}{n}b_{r-1}(i_1,\cdots,i_k)+V_r((i_1,\cdots,i_k)^*1), V_r((i_1,\cdots,i_k)^\circ 1)\right\}$$

$$+\frac{1}{r}\sum_{i=2}^{r}\max\{V_r((i_1,\cdots,i_k)^*i), V_r((i_1,\cdots,i_k)^\circ i)\} \tag{2.6}$$

with $V_n(i_1,\cdots,i_k)\equiv 0$. Eq.(2.6) immediately shows that the optimal policy is independent of p_1 and so is $V^*=V_0(\phi)$, which is interpreted as the probability of making an offer to the best applicant. As a performance measure we use V^* instead of $v^*(=p_1V^*)$ to save parameter. Eq.(2.6) can be further simplified to

$$\begin{aligned}V_{r-1}(i_1,\cdots,i_k) &= \frac{1}{r}\max\left\{\frac{r}{n}b_{r-1}(i_1,\cdots,i_k)+V_r((i_1,\cdots,i_k)^*1), V_r((i_1,\cdots,i_k)^\circ 1)\right\}\\ &\quad +\frac{1}{r}\sum_{i=2}^{r}V_r((i_1,\cdots,i_k)^\circ i),\end{aligned} \tag{2.7}$$

because the optimal policy is shown to make an offer only to candidates (this property of the optimal policy is intuitive but not a priori clear). To prove this, it suffices to show that $V_r(i_1,\cdots,i_k)$ does not increase with additional rejection. Before showing this, we examine some properties of $a_r(i_1,\cdots,i_k)$ and $b_r(i_1,\cdots,i_k)$.

Lemma 2.3 $a_r(i_1,\cdots,i_k)$ and $b_r(i_1,\cdots,i_k)$ have the following properties (properties (ii) and (iii) are only described in terms of $a_r(i_1,\cdots,i_k)$, because $b_r(i_1,\cdots,i_k)$ has apparently the same nature as $a_r(i_1,\cdots,i_k)$ from its construction).

(i) $a_r(i_1,\cdots,i_k)$and $b_r(i_1,\cdots,i_k)$ satisfy the following recursive relations respectively

$$a_{r-1}(i_1,\cdots,i_k)=\frac{1}{r}\sum_{i=1}^{r}a_r((i_1,\cdots,i_k)^\circ i), \qquad 1<r\le n \tag{2.8}$$

with the boundary condition $a_n(i_1,\cdots,i_k)=\prod_{t=1}^{k}q_{i_t}$.

$$b_{r-1}(i_1,\cdots,i_k)=\frac{1}{r}\sum_{i=1}^{r}b_r((i_1,\cdots,i_k)^\circ i), \qquad 1<r\le n-1 \tag{2.9}$$

with the boundary condition $b_{n-1}(i_1,\cdots,i_k)=\prod_{t=1}^{k}q_{i_t+1}$.

(ii) $a_r(i_1,\cdots,i_k)$ does not increase with an additional rejection. That is, for $(i_1,\cdots,i_k)\subset N_{r-1}$ and $i\in N_r$,

$$a_r((i_1,\cdots,i_k)^\circ i)\ge a_r((i_1,\cdots,i_k)^*i).$$

(iii) Assume that $\{q_j; 1\le j\le n\}$ be a non-increasing(non-decreasing) sequence of j. Then

(a) $a_r(i_1,\cdots,i_k)$ is non-increasing(non-decreasing) in $i_t(1\le t\le k)$.
(b) $a_r(i_1,\cdots,i_k)$ is non-decreasing(non-increasing) in r.

Proof. See Appendix A.3.

Remarks (2.1)

1. $a_r(i_1,\cdots,i_k)$ can be given in a closed form for a particular sequence $\{q_j; 1\le j\le n\}$. For example, if $q_j\equiv q, j\ge 2$, then

$$a_r(i_1,\cdots,i_k)=\begin{cases} q^{k-1}\left[\frac{r}{n}q_1+\left(1-\frac{r}{n}\right)q\right], & i_1=1\\ q^k, & i_1\ge 2\end{cases}$$

2. If $\{q_j; 1 \le j \le n\}$ is a non-increasing(non-decreasing) sequence of j, then $p_r(i; i_1, \cdots, i_k)$ is non-decreasing(non-increasing) in i, which agrees with our intuition.

3. To solve Eq.(2.7) recursively, $b_r(i_1, \cdots, i_k)$ must be at hand. (2.9) gives an efficient way for calculating $b_r(i_1, \cdots, i_k)$.

Eq.(2.7) is now an immediate consequence of the following lemma which states that the optimal value function $V_r(i_1, \cdots, i_k)$ inherits some properties of $a_r(i_1, \cdots, i_k)$ (or $b_r(i_1, \cdots, i_k)$).

Lemma 2.4
(i) $V_r(i_1, \cdots, i_k)$ does not increase with an additional rejection. That is, for $(i_1, \cdots, i_k) \subset N_{r-1}$ and $i \in N_r$,

$$V_r((i_1, \cdots, i_k)^\circ i) \ge V_r((i_1, \cdots, i_k)^* i). \tag{2.10}$$

(ii) Assume that $\{q_j; 2 \le j \le n\}$ be a non-increasing(non-decreasing) sequence of j. Then $V_r(i_1, \cdots, i_k)$ is non-increasing(non-decreasing) in $i_t (1 \le t \le k)$.

Proof. See Appendix A.4.

We conclude this section with a simple example.

EXAMPLE (n=4)
A bit of calculation from Eq.(2.7) shows that, when n=4, the optimal policy becomes a threshold rule (more specifically 1- or 2-threshold rule) for any sequence $\{q_j; 1 \le j \le 4\}$. Let q_4 be fixed and denote by $Q_i(q_4)$ the set of values (q_3, q_2) for which the optimal policy remains i-threshold rule,$i = 1, 2$. Then, the entire set $\{(q_3, q_2) : 0 < q_j < 1, j = 2, 3\}$ is partitioned into $Q_1(q_4)$ and $Q_2(q_4)$ and the indifference curve $q_2 = f(q_3 \mid q_4)$(boundary between $Q_1(q_4)$ and $Q_2(q_4)$) is given by

$$f(q_3 \mid q_4) = \frac{6}{(1 + 2q_4) + (2 + q_4)q_3} - 1.$$

Moreover we have

$$V^* = \begin{cases} \frac{1}{24}(6 + 6q_2 + 3q_3 + 2q_4 + 3q_2q_3 + 2q_2q_4 + q_3q_4 + q_2q_3q_4), & (q_3, q_2) \in Q_1(q_4) \\ \frac{1}{24}(11 + 5q_2 + q_3 + q_2q_3), & (q_3, q_2) \in Q_2(q_4). \end{cases}$$

3. Optimal Policy of the m-problem

In this section, we restrict ourselves to the m-problem,$0 \le m \le n-1$. To make explicit the dependence on the same probability $q(= q_{m+1} = q_{m+2} = \cdots = q_n)$, the m-problem is sometimes written as the (m, q)-problem. Let us start with the following lemmas, which is intuitively clear from the definitions of $a_r(i_1, \cdots, i_k)$ and $b_r(i_1, \cdots, i_k)$ (The proof can be made rigorous by induction on r from Lemma 2.3(i)).

Lemma 3.1 For the (m, q)-problem($m \ge 1$), we have the followings.
(i) If $i_k \ge m + 1$ $(k \ge 2)$, then

$$a_r(i_1, \cdots, i_k) = \begin{cases} q^k, & i_1 \ge m + 1 \\ q^{k-l} a_r(i_1, \cdots, i_l), & i_1 \le m, \end{cases}$$

where $l = \max\{t : i_t \le m, 1 \le t \le k\}$. When $k = 1, a_r(i_1) = q, i_1 \ge m + 1$.

(ii) If $i_k \geq m$ $(k \geq 2)$, then

$$b_r(i_1, \cdots, i_k) = \begin{cases} q^k, & i_1 \geq m \\ q^{k-l} b_r(i_1, \cdots, i_l), & i_1 \leq m-1, \end{cases}$$

where $l = \max\{t : i_t \leq m-1, 1 \leq t \leq k\}$. When $k = 1, b_r(i_1) = q, i_1 \geq m$.

Lemma 2.2 shows that, in state $(1; i_1, \cdots, i_k) \in S_r$, the current candidate is judged to be the overall best with probability

$$\frac{r}{n}\left[\frac{b_{r-1}(i_1, \cdots, i_k)}{a_r((i_1, \cdots, i_k)^\circ 1)}\right].$$

For the m-problem, this probability is reduced, from Lemma 3.1, to

$$\begin{cases} \frac{r}{n}, & i_1 \geq m,\ k \geq 1 \\ \frac{r}{n}\left[\frac{b_{r-1}(i_1, \cdots, i_l)}{a_r((i_1, \cdots, i_l)^\circ 1)}\right], & i_1 < m \leq i_k,\ k \geq 2, \end{cases}$$

which implies that, concerning this probability, rejection history $(i_1, \cdots, i_k), i_k \geq m$, is identified with (ϕ) or $(i_1, \cdots, i_l)$ depending on whether $i_1 \geq m, k \geq 1$ or $i_1 < m \leq i_k, k \geq 2$, where $l = \max\{t : i_t \leq m-1, 1 \leq t \leq k\}$. This fact suggests that, concerning the optimal decision to be made, rejection history $(i_1, \cdots, i_k), i_k \geq m$, is also identified with (ϕ) or $(i_1, \cdots, i_l)$ depending on whether $i_1 \geq m, k \geq 1$ or $i_1 < m \leq i_k, k \geq 2$. The following lemma justifies this suggestion and brings consequent simplification of the optimality equation.

Lemma 3.2 For the (m, q)-problem $(m \geq 1)$, we have for $i_k \geq m (k \geq 2)$,

$$V_r(i_1, \cdots, i_k) = \begin{cases} q^k V_r(\phi), & i_1 \geq m \\ q^{k-l} V_r(i_1, \cdots, i_l), & i_1 \leq m-1, \end{cases}$$

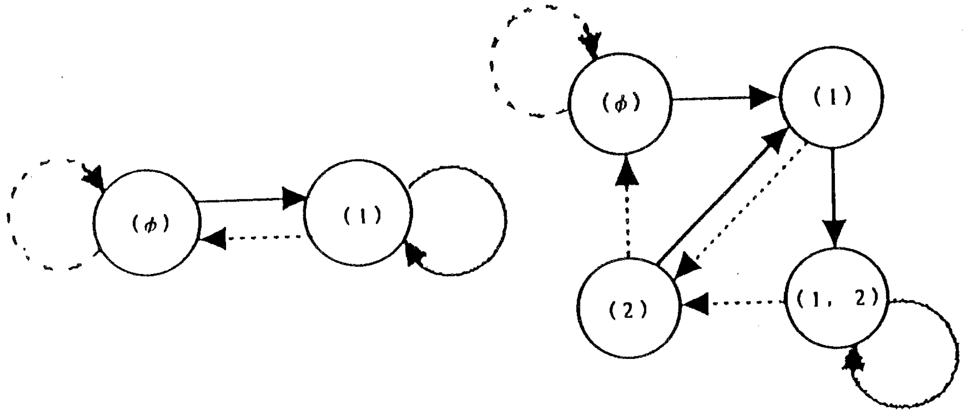
where $l = \max\{t : i_t \leq m-1, 1 \leq t \leq k\}$. When $k = 1, V_r(i_1) = qV_r(\phi), i_1 \geq m$.

Proof. See Appendix B.1.

From Lemma 3.2, the element in the rejection history becomes immaterial if it turns out to be none of m bests. Thus, to describe the evolution of the process, it suffices to consider only 2^{m-1} *basic rejection histories* $(\phi), (i_1)$ with $1 \leq i_1 \leq m-1$,and $(i_1, \cdots, i_l)$ with $i_l \leq m-1, l \geq 2$(note that, when $m = 1$, Lemma 3.2 states that $V_r(i_1, \cdots, i_k) = q^k V_r(\phi), k \geq 1$ and the only basic rejection history is (ϕ)). Table 1 gives a list of the basic rejection histories for m=1,2,3 and 4. Figure 1 illustrates the transition of the basic rejection history for m=2

Table 1
List of the basic rejection histories for m=1, 2, 3 and 4.

m	basic rejection histories
1	(ϕ)
2	(ϕ),(1)
3	(ϕ),(1),(2),(1,2)
4	(ϕ),(1),(2),(3),(1,2),(1,3),(2,3),(1,2,3)



(a) 2-problem (b) 3-problem

Figure 1
Transition diagram of the basic rejection histories for the 2-problem and the 3-problem : real (dotted) line corresponds to making an offer (no offer)

and 3 respectively. Though, for $m \geq 2$, the optimality equations corresponding to the basic rejection histories $(\phi), (i_1)$, with $1 \leq i_1 < m-1$ and $(i_1, \cdots, i_l)$ with $i_l < m-1, l \geq 2$, are covered by Eq.(2.7), those corresponding to $(m-1)$ and $(i_1, \cdots, i_l)$ with $i_l = m-1, l \geq 2$ are given, from Lemma 3.2, as follows :

$$\begin{aligned} V_{r-1}(m-1) &= \frac{1}{r}\max\left\{\frac{r}{n}b_{r-1}(m-1) + qV_r(1), qV_r(\phi)\right\} + \frac{m-2}{r}qV_r(\phi) \\ &+\frac{r-m+1}{r}V_r(m-1) \end{aligned} \tag{3.1}$$

$$\begin{aligned} V_{r-1}(i_1, \cdots, i_l) &= \frac{1}{r}\max\left\{\frac{r}{n}b_{r-1}(i_1, \cdots, i_l) + qV_r((i_1, \cdots, i_{l-1})^*1), qV_r((i_1, \cdots, i_{l-1})^\circ 1)\right\} \\ &+\frac{q}{r}\sum_{i=2}^{i_l} V_r((i_1, \cdots, i_{l-1})^\circ i) + \frac{r-i_l}{r}V_r(i_1, \cdots, i_l). \end{aligned} \tag{3.2}$$

When $m = 1$, the unique optimality equation corresponding to (ϕ) is given by

$$V_{r-1}(\phi) = \frac{1}{r}\max\left\{\frac{r}{n} + qV_r(\phi), V_r(\phi)\right\} + \frac{r-1}{r}V_r(\phi), \tag{3.3}$$

which is immediate from Eq.(2.7) combined with $V_r(1) = qV_r(\phi)$ given in Lemma 3.2.

The following lemma gives the differential form of Eq.(2.7), which can be used to derive the asymptotic solution of the problem.

Lemma 3.3 Let n and r tend to infinity with $r/n = x$, then, if $V_r(i_1, \cdots, i_k)$ converges to $V(x; i_1, \cdots, i_k)$, then $V(x; i_1, \cdots, i_k)$ satisfies the following differential equation(differential forms corresponding to Eqs.(3.1) - (3.3) can be obtained similarly),where $a^+ = \max\{a, 0\}$.

$$\begin{aligned} &\frac{d}{dx}V(x; i_1, \cdots, i_k) \\ &= -x^{-1}\left[\sum_{i=1}^{i_k}\{V(x; (i_1, \cdots, i_k)^\circ i) - V(x; i_1, \cdots, i_k)\}\right. \\ &+ \left.\{xb(x; i_1, \cdots, i_k) + V(x; (i_1, \cdots, i_k)^*1) - V(x; (i_1, \cdots, i_k)^\circ 1)\}^+\right], \end{aligned} \tag{3.4}$$

where

$$b(x;i_1,i_2,\cdots,i_k) = \sum_{j_1=i_1}^{\infty}\sum_{j_2=j_1+i_2-i_1}^{\infty}\cdots\sum_{j_k=j_{k-1}+i_k-i_{k-1}}^{\infty}\left(\prod_{s=1}^{k} q_{j_s+1}\right) \times \binom{j_1-1}{i_1-1}\binom{j_2-j_1-1}{i_2-i_1-1}\cdots\binom{j_k-j_{k-1}-1}{i_k-i_{k-1}-1}x^{i_k}(1-x)^{j_k-i_k}. \quad (3.5)$$

Proof. We give here an intuitive derivation (see Mucci[3] for more detail). (3.5) is immediate from (A.3), combined with (A.1), because, for fixed k and l,

$$\binom{n-l}{r-k}\Big/\binom{n}{r} \longrightarrow x^k(1-x)^{l-k}, \qquad (as\ n,r\to\infty\ with\ r/n\to x).$$

Write Eq.(2.7) as

$$\begin{aligned} &V_r(i_1,\cdots,i_k)-V_{r-1}(i_1,\cdots,i_k)\\ &= -\frac{1}{r}\left[\sum_{i=1}^{i_k}\{V_r((i_1,\cdots,i_k)^\circ i)-V_r(i_1,\cdots,i_k)\}\right.\\ &\quad + \left.\left\{\frac{r}{n}b_{r-1}(i_1,\cdots,i_k)+V_r((i_1,\cdots,i_k)^*1)-V_r((i_1,\cdots,i_k)^\circ 1)\right\}^+\right]. \end{aligned}$$

Dividing both the left and the right sides of this equation by 1/n and then letting $n,r\to\infty$ in such a way that $r/n\to x$, we obtain (3.4) if we put

$$\frac{d}{dx}V(x;i_1,\cdots,i_k) = \lim_{n,r\to\infty}[V_r(i_1,\cdots,i_k)-V_{r-1}(i_1,\cdots,i_k)]\big/n^{-1}.$$

3.1. 1-problem

The only basic rejection history is (ϕ) and the corresponding optimality equation is given by Eq.(3.3), which is essentially the same as is derived by Smith[6] (compare this with Eq.(9) in page 622) and can be solved to yield

Lemma 3.4 The optimal policy of the $(1,q)$-problem is r_0-threshold rule, where

$$r_0 = \min\left\{r : \prod_{j=r}^{n-1}\left(1+\frac{q}{j}\right) \le \frac{1}{p}\right\}.$$

The optimal value is given by

$$V^* = \frac{r_0-1}{qn}\left[\prod_{j=r_0-1}^{n-1}\left(1+\frac{q}{j}\right)-1\right].$$

The asymptotic results are as follows.

$$\lim_{n\to\infty}\frac{r_0}{n} = p^{\frac{1}{q}}, \qquad \lim_{n\to\infty}V^* = p^{\frac{p}{q}}.$$

3.2. 2-problem

The basic rejection histories of the 2-problem are (ϕ) and (1), and the corresponding optimality equations are given by

$$V_{r-1}(\phi) = \frac{1}{r}\max\left\{\frac{r}{n}+V_r(1),V_r(\phi)\right\}+\frac{r-1}{r}V_r(\phi), \quad (3.6)$$

$$V_{r-1}(1) \;=\; \frac{1}{r}\max\left\{\frac{r}{n}b_{r-1}(1)+qV_r(1),qV_r(\phi)\right\}+\frac{r-1}{r}V_r(1). \qquad (3.7)$$

These equations can be solved and then summarized as follows.

Theorem 3.5 The $(2,q)$-problem is distinguished into two cases depending on whether $q_2 \geq q$ or $q_2 < q$.

Case 1 : $q_2 \geq q$.

(i) *Form of the optimal policy.* The optimal policy is r_0-threshold rule.

(ii) *Formula for critical number.* r_0 is the smallest integer r such that

$$\frac{q}{1-q}+\frac{q_2-q}{n-1}\left[\frac{q(n-r)}{1-q}-\sum_{k=r}^{n-2}d_{r,k}\right]\geq d_{r,n-1}, \qquad (3.8)$$

where, for $r \leq k$,

$$d_{r,k}=\prod_{j=r}^{k}\left(1+\frac{q}{j}\right)-1.$$

(iii) *Probability of making an offer to the best.* V^* is given by

$$V^*=\frac{r_0-1}{qn}\left[\frac{q_2-q}{n-1}\sum_{k=r_0-1}^{n-2}d_{r_0-1,k}+d_{r_0-1,n-1}\right]. \qquad (3.9)$$

(iv) *Asymptotic formula for critical number.* Let $\tilde{r_0}=\lim_{n\to\infty} r_0/n$, then $\tilde{r_0}$ is the unique root x of the equation

$$2q(q_2-q)x^{1+q}-(1+q)(1-q+q_2)x^q+(1-q)(1+q_2)=0. \qquad (3.10)$$

(v) *Asymptotic value of V^*.* Let $\tilde{V}=\lim_{n\to\infty}V^*$. then

$$\tilde{V}=\frac{(\tilde{r_0})^p}{q(1+q)}[(1+q_2)-(1+q)(1-q+q_2)(\tilde{r_0})^q+q(q_2-q)(\tilde{r_0})^{1+q}]. \qquad (3.11)$$

Case 2: $q_2 < q$.

(i) *Form of the optimal policy.* The optimal policy can be described in terms of the two integers r_1 and $r_2 (1 \leq r_1 \leq r_2 \leq n)$ as follows: The optimal policy passes over the first $r_1 - 1$ applicants. On time interval $[r_1, r_2-1]$, the optimal policy makes an offer to candidates alternately, that is, it makes an offer to the first candidate; it then makes no offer to the second candidate; it then makes an offer to the third candidate, and so forth (when $r_1 = r_2$, interval $[r_1, r_2 - 1]$ shrinks). If no offer has been accepted by $r_2 - 1$, the optimal policy then makes an offer successively to each candidate that appears.

(ii) *Formulae for critical numbers.* r_2 is the smallest integer r such that

$$\frac{q_2}{1-q}+\frac{q-q_2}{n-1}\left[(n-r)+\sum_{k=r}^{n-2}d_{r,k}\right]\geq d_{r,n-1}. \qquad (3.12)$$

Define, for given r_2,

$$\begin{aligned} A_1 &= \frac{1}{2\sqrt{q}}\left[\sqrt{q}+(q-q_2)\left(\frac{n-r_2+1}{n-1}\right)+(1+\sqrt{q})A\right] \\ A_2 &= \frac{1}{2\sqrt{q}}\left[\sqrt{q}-(q-q_2)\left(\frac{n-r_2+1}{n-1}\right)-(1-\sqrt{q})A\right] \end{aligned}$$

where

$$A = d_{r_2-1,n-1} - \frac{q-q_2}{n-1}\left[(n-r_2+1)+\sum_{k=r_2-1}^{n-2} d_{r_2-1,k}\right].$$

Then if

$$\left(\frac{1-\sqrt{q}}{1+\sqrt{q}}\right)\left(\frac{A_1}{A_2}\right) \le 1,$$

r_1 is the smallest integer $r(\le r_2-1)$ such that

$$\prod_{j=r}^{r_2-2}\left(\frac{j-\sqrt{q}}{j+\sqrt{q}}\right) \ge \left(\frac{1-\sqrt{q}}{1+\sqrt{q}}\right)\left(\frac{A_1}{A_2}\right). \tag{3.13}$$

Otherwise $r_1 = r_2$.

(iii) *Probability of making an offer to the best.*

$$V^* = \frac{r_1-1}{\sqrt{q}n}\left[A_1\prod_{j=r_1-1}^{r_2-2}\left(1+\frac{\sqrt{q}}{j}\right) - A_2\prod_{j=r_1-1}^{r_2-2}\left(1-\frac{\sqrt{q}}{j}\right)\right]. \tag{3.14}$$

(iv) *Asymptotic formulae for critical numbers.* Let $\tilde{r_i} = \lim_{n\to\infty} r_i/n, i = 1,2$. Then $\tilde{r_2}$ is the unique root x of the equation

$$(1-q)(q-q_2)x^{1+q} - (1+q)(1-q+q_2)x^q + (1-q)(1+q_2) = 0, \tag{3.15}$$

and $\tilde{r_1}$ is given by

$$\tilde{r_1} = \left[\frac{(1-\sqrt{q})C_1}{(1+\sqrt{q})C_2}\right]^{\frac{1}{2\sqrt{q}}}, \tag{3.16}$$

where

$$\begin{aligned} C_1 &= \frac{(\tilde{r_2})^{\sqrt{q}}}{2q(1+q)}[(1-\sqrt{q})(q-q_2)q\tilde{r_2} + (\sqrt{q}+q)(1+q_2)(\tilde{r_2})^{-q} - \sqrt{q}(1+q)(1-q+q_2)] \\ C_2 &= \frac{(\tilde{r_2})^{-\sqrt{q}}}{2q(1+q)}[(1+\sqrt{q})(q-q_2)q\tilde{r_2} - (\sqrt{q}-q)(1+q_2)(\tilde{r_2})^{-q} + \sqrt{q}(1+q)(1-q+q_2)] \end{aligned}$$

(v) *Asymptotic value of* V^*. Let $\tilde{V} = \lim_{n\to\infty} V^*$, then

$$\tilde{V} = C_1(\tilde{r_1})^{1-\sqrt{q}} + C_2(\tilde{r_1})^{1+\sqrt{q}} \tag{3.17}$$

Proof. see Appendix B.2.

To have an intuitive feeling to the form of the optimal policies described in Theorem 3.5, it might be helpful to consider two extreme cases, i.e., $q_2 = 1$ and $q = 0$ as Case 1, and $q_2 = 0$ and $q = 1$ as Case 2. Imagine a situation where a candidate(referred to as A) rejected an offer and the next candidate(referred to as B) has just arrived. In the former case, B turns out to be the very best at this instant and consequently the optimal policy makes an offer to B, which implies that the optimal policy is a threshold rule. In the latter case, A turns out to be neither the best nor the second best at this instant and hence, if time is not matured, i.e., if the arrival time of B is not too late, B is judged to be non-best with large probability and possibly passed by. This is why the optimal policy is not necessarily a threshold rule in the latter case.

Tables 2 and 3 respectively give the numerical values of r_0 and V^* (in Case 1) and r_1, r_2 and V^* (in Case 2) for some values of n and (q_2, q). Table 4 contains the numerical values

Table 2

r_0 and V^* for some values of n and (q_2, q) in Case 1

(upper is r_0 and lower is V^*)

	(q_2, q)		
n	$(0.3, 0.1)$	$(0.5, 0.1)$	$(0.5, 0.3)$
10	4	4	4
	0.45573	0.48994	0.50809
30	11	10	9
	0.42910	0.46090	0.48289
50	17	17	15
	0.42425	0.45536	0.47776
80	27	26	24
	0.42155	0.45249	0.47478
100	34	32	30
	0.42065	0.45147	0.47377

Table 3

r_1, r_2 and V^* for some values of n and (q_2, q) in Case 2

(upper is r_1, middle is r_2 and lower is V^*)

	(q_2, q)		
n	$(0.1, 0.3)$	$(0.1, 0.5)$	$(0.3, 0.5)$
	5	5	4
10	5	5	4
	0.42201	0.42953	0.48841
	8	8	10
30	14	14	10
	0.40217	0.42740	0.46761
	13	14	13
50	23	24	16
	0.39895	0.42575	0.46482
	22	17	22
80	37	38	25
	0.40095	0.41439	0.46314
	31	26	26
100	46	47	32
	0.40429	0.42203	0.46193

Table 4

$\tilde{r_0}, \tilde{r_1}, \tilde{r_2}$ and $\tilde{V}$ for some pairs of (q_2, q)
Case 1 $(q_2 \geq q)$: upper is $\tilde{r_0}$ and lower is $\tilde{V}$.
Case 2 $(q_2 < q)$: upper is $\tilde{r_1}$,middle is $\tilde{r_2}$ and lower is $\tilde{V}$.

$q_2 \backslash q$	0.1	0.3	0.5	0.7	0.9
	0.34868	0.32134	0.28399	0.23159	0.14098
0.1		0.45279	0.46434	0.42154	0.26911
	0.38742	0.40171	0.42061	0.44676	0.49034
	0.33106	0.30455	0.26768	0.21331	0.11681
0.3			0.30912	0.26459	0.14236
	0.41706	0.43507	0.45900	0.49383	0.55716
	0.31629	0.28794	0.25000	0.19443	0.09803
0.5				0.20872	0.10547
	0.44754	0.46993	0.50000	0.54456	0.62802
	0.30389	0.27432	0.23512	0.17907	0.08572
0.7					0.08779
	0.47808	0.50564	0.54217	0.59691	0.70064
	0.29345	0.26308	0.22323	0.16724	0.07743
0.9					
	0.51036	0.54201	0.58517	0.65030	0.77426

of $\tilde{r_0}$ and $\tilde{V}$ (in Case 1) and $\tilde{r_1}, \tilde{r_2}$ and $\tilde{V}$ (in Case 2) simultaneously, letting q_2 and q run from 0.1 to 0.9 by 0.2. Some properties suggested from this table are in order.

(i) $\tilde{r_0}$ and $\tilde{r_1}$ are decreasing both in q_2 and q.

(ii) $\tilde{r_2}$ is decreasing in q_2, but not necessarily decreasing in q.

(iii) $\tilde{V}$ is increasing both in q_2 and q.

(iv) Whereas $\tilde{r_0}$ and $\tilde{r_1}$ are both no greater than e^{-1}, $\tilde{r_2}$ can be greater than e^{-1}.

3.3. 3-problem

Here we give some computational results for the 3-problem because solving this problem analytically seems difficult. The optimality equations of the $(3, q)$-problem are given as follows, each corresponding to the basic rejection histories (ϕ), (1), (2) and (1, 2),

$$V_{r-1}(\phi) = \frac{1}{r}\max\left\{\frac{r}{n} + V_r(1), V_r(\phi)\right\} + \frac{r-1}{r}V_r(\phi) \tag{3.18}$$

$$V_{r-1}(1) = \frac{1}{r}\max\left\{\frac{r}{n}b_{r-1}(1) + V_r(1,2), V_r(2)\right\} + \frac{r-1}{r}V_r(1) \tag{3.19}$$

$$V_{r-1}(2) = \frac{1}{r}\max\left\{\frac{r}{n}b_{r-1}(2) + qV_r(1), qV_r(\phi)\right\} + \frac{q}{r}V_r(\phi) + \frac{r-2}{r}V_r(2) \tag{3.20}$$

$$V_{r-1}(1,2) = \frac{1}{r}\max\left\{\frac{r}{n}b_{r-1}(1,2) + qV_r(1,2), qV_r(2)\right\} + \frac{q}{r}V_r(1) + \frac{r-2}{r}V_r(1,2). \tag{3.21}$$

Let

$$K_r \equiv V_r(\phi) - V_r(1), \qquad L_r \equiv V_r(2) - V_r(1,2),$$

and define

$$G(\phi) = \left\{r : K_r \leq \frac{r}{n}\right\}, \qquad G(1) = \left\{r : L_r \leq \frac{r}{n}b_{r-1}(1)\right\}$$

$$G(2) = \left\{r : K_r \leq \frac{r}{nq}b_{r-1}(2)\right\}, \qquad G(1,2) = \left\{r : L_r \leq \frac{r}{nq}b_{r-1}(1,2)\right\}.$$

Then, from (3.18)-(3.21), $G(\phi), G(1), G(2)$ and $G(1,2)$ represent the optimal stopping regions, each corresponding to the basic rejection histories (ϕ),(1),(2) and (1,2).

We examined 9^3=729 cases letting q_2, q_3 and q respectively run from 0.1 to 0.9 by 0.1 with n=100 fixed, and 5^2=25 cases letting q_2 and q respectively run from 0.1 to 0.9 by 0.2 with n=1000 and q=0.3 fixed(see Table 5 for the latter). Computational experiences show that, in each case, the optimal policy is time isotone, that is, there exist four integers $s(\phi), s(1), s(2)$ and $s(1,2)$ such that

$$G(\phi) = \{r : r \geq s(\phi)\}, \quad G(1) = \{r : r \geq s(1)\}$$
$$G(2) = \{r : r \geq s(2)\}, \quad G(1,2) = \{r : r \geq s(1,2)\}.$$

Table 5

The type of the optimal policy and the critical numbers for the 3-problem with n=10000 and q=0.3 fixed. For each type, only the effective numbers are given

$q_2\backslash q_3$	0.1	0.3	0.5	0.7	0.9
0.1	T_1 $s(\phi)$=332 $s(1)$=485 $s(2)$=368 $s(1,2)$=544	T_3 $s(\phi)$=322 $s(1)$=453	T_3 $s(\phi)$=312 $s(1)$=422	T_3 $s(\phi)$=302 $s(1)$=392	T_3 $s(\phi)$=293 $s(1)$=365
0.3	T_2 $s(\phi)$=317 $s(1)$=331 $s(2)$=343 $s(1,2)$=363	T_{4D} $s(\phi)$=305	T_{4A} $s(\phi)$=294	T_{4A} $s(\phi)$=284	T_{4A} $s(\phi)$=274
0.5	T_{4C} $s(\phi)$=300	T_{4D} $s(\phi)$=289	T_{4B} $s(\phi)$=278	T_{4B} $s(\phi)$=267	T_{4B} $s(\phi)$=257
0.7	T_{4C} $s(\phi)$=287	T_{4D} $s(\phi)$=275	T_{4B} $s(\phi)$=264	T_{4B} $s(\phi)$=254	T_{4B} $s(\phi)$=245
0.9	T_{4C} $s(\phi)$=275	T_{4D} $s(\phi)$=264	T_{4B} $s(\phi)$=253	T_{4B} $s(\phi)$=243	T_{4B} $s(\phi)$=235

Table 5 presents the numerical values of $s(\phi), s(1)$ and $s(1,2)$ and the types of the optimal policy for 5^2=25 cases when n=1000 and q=0.3 are fixed. The optimal policies are distinguished into four types T_1, T_2, T_3 and T_4(T_4 is further distinguished into T_{4A}, T_{4B}, T_{4C} and T_{4D}) as indicated below, depending on the magnitude of the values $s(\phi), s(1), s(2)$, and $s(1,2)$

$$\begin{aligned}
T_1 &: s(\phi) < s(2) < s(1) < s(1,2) \\
T_2 &: s(\phi) < s(1) < s(2) < s(1,2) \\
T_3 &: s(2) \leq s(\phi) < s(1,2) \leq s(1) \\
T_{4A} &: s(1,2) < s(2) < s(1) < s(\phi) \\
T_{4B} &: s(1,2) < s(1) < s(2) < s(\phi) \\
T_{4C} &: s(1) < s(1,2) < s(\phi) < s(2) \\
T_{4D} &: s(1) = s(1,2) \leq s(\phi) = s(2).
\end{aligned}$$

It is easy to see that, from Figure 1(b), if $s(\phi) \geq \max\{s(1), s(1,2)\}$, then the optimal policy becomes $s(\phi)$-threshod rule. Thus T_4 represents a threshold rule. However it is not easy to state T_1 verbally. For example, consider the behavior of T_1 on time interval $[s(\phi), s(2)-1]$.

Let $t_1, t_2, \cdots$ be the arrival times of the candidates on that interval. At time t_1, T_1 makes an offer to the first candidate because the basic rejection history is now (ϕ) and $t_1 \geq s(\phi)$. If this offer is rejected, then the basic rejection history changes from (ϕ) to (1) (see Figure 1(*b*)) and T_1 makes no offer to the second candidate due to $t_2 < s(1)$. The basic rejection history then changes from (1) to (2) and T_1 makes no offer to the third candidate due to $t_3 < s(2)$, and the basic rejection history again moves back to (ϕ) from (2). Thus T_1 restarts over again with basic rejection history (ϕ) from time t_3 onward, and so T_1 makes an offer to the fourth candidate and make no offer to the fifth and sixth candidates, and so forth. To describe the cyclic property of the optimal policy on each interval, it is convenient to introduce a cyclic rule $R_{m,n}, m, n \geq 0$, which makes an offer to the first m candidates successively and then makes no offer to the next n candidates, and then restarts over again so far as a candidate appears. Then, from the above, T_1 behaves like a $R_{1,2}$ on $[s(\phi), s(2)-1]$. The behavior of T_1 on other intervals can be also examined and summarized as follows :

$$T_1 = \begin{cases} R_{0,1} & \text{on } [1, s(\phi)-1] \\ R_{1,2} & \text{on } [s(\phi), s(2)-1] \\ R_{1,1} \text{ (or } R'_{1,1}) & \text{on } [s(2), s(1)-1] \\ R_{2,1} \text{ (or } R'_{2,1}) & \text{on } [s(1), s(1,2)-1] \\ R_{1,0} & \text{on } [s(1,2), n] \end{cases} \tag{3.22}$$

$R'_{1,1}$ and $R'_{2,1}$ in (3.22) represent a modified $R_{1,1}$ and $R_{2,1}$ respectively, *i.e.*, $R'_{1,1}$ makes no offer to the first candidate and then follows $R_{1,1}$, and $R'_{2,1}$ makes an offer to the first candidate and makes no offer to the second candidate and then follows $R_{2,1}$. Then it is easy to observe that if the process reaches $[s(2), s(1)-1]$ with basic rejection history (ϕ) or (2) (basic rejection history (1)), then T_1 behaves like a $R_{1,1}(R'_{1,1})$ on this interval. Similarly T_1 behaves like a $R_{2,1}$ or $R'_{2,1}$ according to whether the process reaches $[s(1), s(1,2)-1]$ with basic rejection history (ϕ) or (2), or basic rejection history (1). T_2 and T_3 are also described in terms of cyclic rules.

Table 5 shows that the optimal policy is a threshold rule as long as $q_2 \geq q_3 \geq q$. From these numerical results and the analytical results of the 0-,1- and 2- problems, we conclude this paper with a conjecture " If the sequence $\{q_j; 1 \leq j \leq n\}$ satisfies $q_1 \geq q_2 \geq \cdots \geq q_n$, then the optimal policy remains a threshold rule".

APPENDIX A

A.1 Proof of Lemma 2. 1

Let $p(j_1, j_2, \cdots, j_k; n \mid i_1, i_2, \cdots, i_k; r)$ be the joint probability that i_1-th best, i_2-th best,$\cdots$, and i_k-th best among the first r applicants are respectively j_1-th best, j_2-th best,$\cdots$, and j_k-th best among all n applicants, where $(i_1, i_2, \cdots, i_k) \subset N_r$ and $(j_1, j_2, \cdots, j_k) \subset N_n$. That is,

$$p(j_1, j_2, \cdots, j_k; n \mid i_1, i_2, \cdots, i_k; r) = P\{A(r, i_1; n) = j_1, A(r, i_2; n) = j_2, \cdots, A(r, i_k; n) = j_k\}.$$

Then, from the simple combinatorial argument

$$p(j_1, j_2, \cdots, j_k; n \mid i_1, i_2, \cdots, i_k; r) = \frac{\binom{j_1-1}{i_1-1}\binom{j_2-j_1-1}{i_2-i_1-1}\cdots\binom{j_k-j_{k-1}-1}{i_k-i_{k-1}-1}\binom{n-j_k}{r-i_k}}{\binom{n}{r}} \tag{A.1}$$

for $(j_1, j_2, \cdots, j_k) \in W_r(i_1, i_2, \cdots, i_k)$, where $W_r(i_1, i_2, \cdots, i_k)$ stands for the set of possible values $(j_1, j_2, \cdots, j_k)$, i.e., $W_r(i_1, i_2, \cdots, i_k) = \{(j_1, j_2, \cdots, j_k) : j_1 < j_2 < \cdots < j_k, i_s \leq j_s \leq n - r + i_s, 1 \leq s \leq k\}$. We sometimes write $W_r(i_1, i_2, \cdots, i_k; n)$ to clarify the dependence on n. Notice that, since $n!$ arrival orders are equally likely, arrival times of these k applicants are irrelevant.

From (A.1), (2.4) and (2.5) can be respectively written as

$$a_r(i_1, i_2, \cdots, i_k) = \sum_{j_1}\sum_{j_2}\cdots\sum_{j_k}\left(\prod_{t=1}^{k} q_{j_t}\right) p(j_1, j_2, \cdots, j_k; n \mid i_1, i_2, \cdots, i_k; r), \tag{A.2}$$

$$b_r(i_1, i_2, \cdots, i_k) = \sum_{j_1}\sum_{j_2}\cdots\sum_{j_k}\left(\prod_{t=1}^{k} q_{j_t+1}\right) p(j_1, j_2, \cdots, j_k; n-1 \mid i_1, i_2, \cdots, i_k; r), \tag{A.3}$$

where summations with respect to $(j_1, j_2, \cdots, i_k)$ are taken over $W_r(i_1, i_2, \cdots, i_k; n)$ for $a_r(i_1, i_2, \cdots, i_k)$ but over $W_r(i_1, i_2, \cdots, i_k; n-1)$ for $b_r(i_1, i_2, \cdots, i_k)$.

The following lemma presents some properties of $p(j_1, j_2, \cdots, j_k; n \mid i_1, i_2, \cdots, i_k; r)$ that will be used later.

Lemma A.1

(i) $$\frac{p(j_1, j_2, \cdots, j_k; n \mid (i_1, i_2 \cdots, i_k)^{\circ} i; r)}{p(j_1, j_2 \cdots, j_k; n \mid i_1, i_2 \cdots, i_k; r-1)}$$

$$= \begin{cases} \dfrac{r}{n-r+1}\left[\dfrac{j_1 - i_1}{i_1}\right] & 1 \leq i \leq i_1 \\ \dfrac{r}{n-r+1}\left[\dfrac{(j_t - i_t) - (j_{t-1} - i_{t-1})}{i_t - i_{t-1}}\right] & i_{t-1} < i \leq i_t \ (2 \leq t \leq k) \\ \dfrac{r}{n-r+1}\left[\dfrac{(n-r+1) - (j_k - i_k)}{r - i_k}\right] & i_k < i \leq r \end{cases}$$

(ii) $$p(j_1, j_2, \cdots, j_k; n \mid i_1, i_2, \cdots, i_k; r) = p(j_1, j_2, \cdots, j_{t-1}; j_t - 1 \mid i_1, i_2, \cdots, i_{t-1}; i_t - 1) p(j_t, j_{t+1}, \cdots, j_k; n \mid i_t, i_{t+1}, \cdots, i_k; r) \quad (2 \leq t \leq k)$$

(iii) $$p(j_1, j_2, \cdots, j_k; n \mid (i_1, i_2, \cdots, i_k)^{\circ} 1; r) = \frac{r}{n}\binom{j_1 - 1}{i_1} p(j_1 - 1, j_2 - 1, \cdots, j_k - 1; n-1 \mid i_1, i_2, \cdots, i_k; r-1)$$

(iv) $$p(j_1, j_2, \cdots, j_k; n \mid i_1, i_2, \cdots, i_k; r-1) = \frac{1}{r}\sum_{i=1}^{r} p(j_1, j_2, \cdots, j_k; n \mid (i_1, i_2, \cdots, i_k)^{\circ} i; r)$$

Proof. Straightforward from (A.1).

Proof of Lemma 2.1
Let $\tilde{p}(j_1, j_2, \cdots, j_k; n \mid i_1, i_2, \cdots, i_k; r-1)$ be the joint probability that i_1-th best, i_2-th best,$\cdots$, and i_k-th best among the first $r-1$ applicants are respectively j_1-th best, j_2-th best, $\cdots$, and j_k-th best among all n applicants, provided that all offers given to these k applicants have been rejected. Then, by the Bayes formula,

$$\tilde{p}(j_1, j_2, \cdots, j_k; n \mid i_1, i_2, \cdots, i_k; r-1) = \frac{(q_{j_1} q_{j_2} \cdots q_{j_k}) p(j_1, j_2, \cdots, j_k; n \mid i_1, i_2, \cdots, i_k; r-1)}{a_{r-1}(i_1, i_2, \cdots, i_k)} \tag{A.4}$$

We easily see that the probability distribution of X_r, given that i_1-th best, i_2-th best,$\cdots$, and i_k-th best among the first $r-1$ applicants are respectively j_1-th best, j_2-th best,$\cdots$, and j_k-th best among all n, is given by

$$P(X_r = i \mid j_1, j_2, \cdots, j_k) = \begin{cases} \dfrac{1}{i_1}\left[\dfrac{j_1 - i_1}{n-r+1}\right] & 1 \le i \le i_1 \\ \dfrac{1}{i_t - i_{t-1}}\left[\dfrac{(j_t - i_t) - (j_{t-1} - i_{t-1})}{n-r+1}\right] & i_{t-1} < i \le i_t \ (2 \le t \le k) \\ \dfrac{1}{r - i_k}\left[\dfrac{(n-r+1) - (j_k - i_k)}{n-r+1}\right] & i_k < i \le r. \end{cases} \tag{A.5}$$

Thus the result follows from (A.4), (A.5) and Lemma A.1(i), since $p_r(i; i_1, i_2, \cdots, i_k)$ is calculated through

$$p_r(i; i_1, i_2, \cdots, i_k) = \sum_{j_1}\sum_{j_2}\cdots\sum_{j_k} P(X_r = i \mid j_1, j_2, \cdots, j_k)\tilde{p}(j_1, j_2, \cdots, j_k; n \mid i_1, i_2, \cdots, i_k; r-1),$$

where summations with respect to $(j_1, j_2, \cdots, j_k)$ are taken over $W_{r-1}(i_1, i_2, \cdots, i_k)$.

A.2 Proof of Lemma 2. 2

The following lemma is concerned with another property of $a_r(i_1, \cdots, i_k)$ and the proof is straightforward from Lemma A.1(ii). We write $a_r(i_1, \cdots, i_k; n)$ to make explicit the dependence on n.

Lemma A. 2
For any $s (2 \le s \le k)$,

$$a_r(i_1, \cdots, i_k; n) = \sum_{j_s}\cdots\sum_{j_k} a_{i_s - 1}(i_1, \cdots, i_{s-1}; j_s - 1)\left(\prod_{t=s}^{k} q_{j_t}\right) p(j_s, \cdots, j_k; n \mid i_s, \cdots, i_k; r),$$

where summations with respect to $(j_s, \cdots, j_k)$ are taken over $W_r(i_s, \cdots, i_k)$.

Proof of Lemma 2.2.
We'll only derive $s_r(1; i_1, \cdots, i_k)$ for action a_1, since others can be obtained in a similar way. Suppose that we are in state $(1; i_1, \cdots, i_k) \in S_r$. Then the probability that the true ranks of the applicants constituting the rejection history are $j_1, \cdots, j_k$ is given by $\tilde{p}(j_1, \cdots, j_k; n \mid (i_1, \cdots, i_k)^\circ 1; r)$ (defined in (A.4)). On the other hand, if the true ranks of these k applicants are $j_1, \cdots, j_k$, action a_1 and the subsequent optimal continuation leads to a success with probability

$$p_1 p(1; j_1 - 1 \mid 1; i_1) + a_{i_1}(1; j_1 - 1) v_r((i_1, \cdots, i_k)^* 1).$$

The first term corresponds to acceptance of the offer and the second term corresponds to rejection. Thus we have

$$\begin{aligned} s_r(1; i_1, \cdots, i_k) &= \sum_{j_1} \cdots \sum_{j_k} [p_1 p(1; j_1 - 1 \mid 1; i_1) + a_{i_1}(1; j_1 - 1) v_r((i_1, \cdots, i_k)^* 1)] \\ &\quad \times \tilde{p}(j_1, \cdots, j_k; n \mid (i_1, \cdots, i_k)^\circ 1; r) \end{aligned} \tag{A.6}$$

where summations with respect to $(j_1, \cdots, j_k)$ are taken over $W_r((i_1, \cdots, i_k)^\circ 1)$.
From Lemma A.1(iii) and (A.3), the first term in the RHS of (A.6) can be reduced to

$$\begin{aligned} &\frac{p_1}{a_r((i_1, \cdots, i_k)^\circ 1)} \sum_{j_1} \cdots \sum_{j_k} \left(\prod_{t=1}^k q_{j_t} \right) p(j_1, \cdots, j_k; n \mid (i_1, \cdots, i_k)^\circ 1; r) \\ &= p_1 \left(\frac{r}{n} \right) \frac{1}{a_r((i_1, \cdots, i_k)^\circ 1)} \sum_{j_1} \cdots \sum_{j_k} \left(\prod_{t=1}^k q_{j_t} \right) p(j_1 - 1, \cdots, j_k - 1; n - 1 \mid i_1, \cdots, i_k; r - 1) \\ &= p_1 \left(\frac{r}{n} \right) \frac{b_{r-1}(i_1, \cdots, i_k)}{a_r((i_1, \cdots, i_k)^\circ 1)}. \end{aligned} \tag{A.7}$$

The second term can be written, from Lemma A.2, as

$$\begin{aligned} &\frac{v_r((i_1, \cdots, i_k)^* 1)}{a_r((i_1, \cdots, i_k)^\circ 1)} \sum_{j_1} \cdots \sum_{j_k} a_{i_1}(1; j_1 - 1) \left(\prod_{t=1}^k q_{j_t} \right) p(j_1, \cdots, j_k; n \mid (i_1, \cdots, i_k)^\circ 1; r) \\ &\qquad = v_r((i_1, \cdots, i_k)^* 1) \frac{a_r((i_1, \cdots, i_k)^* 1)}{a_r((i_1, \cdots, i_k)^\circ 1)}. \end{aligned} \tag{A.8}$$

Substituting (A.7) and (A.8) into (A.6) yields the desired result. $c_r(i; i_1, \cdots, i_k)$ is self evident.

A. 3 Proof of Lemma 2. 3

(i) is immediate from Lemma 2.1, since $\sum_{i=1}^r p_r(i; i_1, \cdots, i_k)$ must be unity.

(ii) is immediate from the definition of $a_r(i_1, \cdots, i_k)$.

(iii) We first consider the case where q_j is non-increasing in j and show (a) by induction on r. For $r = n$, (a) is evident from $a_n(i_1, \cdots, i_k) = \prod_{t=1}^k q_{i_t}$. Assume that (a) holds for r. Then the result is immediate from (2.8) since each term of the right hand side, i.e., $a_r((i_1, \cdots, i_k)^\circ i)$, is non-increasing in $i_t (1 \le t \le k)$ from the induction hypothesis. Because (a) implies that $a_r((i_1, \cdots, i_k)^\circ i) \le a_r((i_1, \cdots, i_k)^\circ r)$ for $1 \le i \le r$, it follows that

$$a_{r-1}(i_1, \cdots, i_k) = \frac{1}{r} \sum_{i=1}^r a_r((i_1, \cdots, i_k)^\circ i) \le \frac{1}{r} \sum_{i=1}^r a_r((i_1, \cdots, i_k)^\circ r) = a_r(i_1, \cdots, i_k),$$

which proves (b). When q_j is non-decreasing in j, all the inequalities involved in the above can be reversed.

A. 4 Proof of Lemma 2. 4

(i) We show (2.10) by induction on r. For $r = n - 1$, (2.10) is evident from the nature of $b_{n-1}(.)$(Lemma 2.3(ii)) through $V_{n-1}(i_1, \cdots, i_k) = b_{n-1}(i_1, \cdots, i_k)/n$. Assume now that (2.10) holds for some r. Then (2. 7) in fact holds from (2.6). Thus

$$\begin{aligned} &V_{r-1}((i_1, \cdots, i_k)^\circ i) - V_{r-1}((i_1, \cdots, i_k)^* i) \\ &= \frac{1}{r} \left[\max \left\{ \frac{r}{n} b_{r-1}((i_1, \cdots, i_k)^\circ i) + V_r(((i_1, \cdots, i_k)^\circ i)^* 1), V_r(((i_1, \cdots, i_k)^\circ i)^\circ 1) \right\} \right. \end{aligned}$$

$$
\begin{aligned}
&-\max\left\{\frac{r}{n}b_{r-1}((i_1,\cdots,i_k)^*i)+V_r(((i_1,\cdots,i_k)^*i)^*1),V_r(((i_1,\cdots,i_k)^*i)^\circ 1)\right\}\Big]\\
&+\frac{1}{r}\sum_{j=2}^{r}[V_r(((i_1,\cdots,i_k)^\circ i)^\circ j)-V_r(((i_1,\cdots,i_k)^*i)^\circ j)]\\
\geq\ &\frac{1}{r}\min\left[\frac{r}{n}\{b_{r-1}((i_1,\cdots,i_k)^\circ i)-b_{r-1}((i_1,\ cdots,i_k)^*i)\}\right.\\
&+\{V_r(((i_1,\cdots,i_k)^\circ i)^*1)-V_r(((i_1,\cdots,i_k)^*i)^*1)\},\{V_r(((i_1,\ cdots,i_k)^\circ i)^\circ 1)\\
&\left.-V_r(((i_1,\cdots,i_k)^*i)^\circ 1)\}\right]+\frac{1}{r}\sum_{j=2}^{r}[V_r(((i_1,\cdots,i_k)^\circ i)^\circ j)-V_r(((i_1,\cdots,i_k)^*i)^\circ j)]\\
=\ &\frac{1}{r}\min\left[\frac{r}{n}\{b_{r-1}((i_1,\cdots,i_k)^\circ i)-b_{r-1}((i_1,\cdots,i_k)^*i)\}\right.\\
&+\{V_r(((i_1,\cdots,i_k)^*1)^\circ(i+1))-V_r(((i_1,\cdots,i_k)^*1)^*(i+1))\},\{V_r(((i_1,\cdots,i_k)^\circ 1)^\circ(i+1))\\
&\left.-V_r(((i_1,\cdots,i_k)^\circ 1)^*(i+1))\}\right]\\
&+\frac{1}{r}\sum_{j=2}^{i}[V_r(((i_1,\cdots,i_k)^\circ j)^\circ(i+1))-V_r(((i_1,\cdots,i_k)^\circ j)^*(i+1))]\\
&+\frac{1}{r}\sum_{j=i+1}^{r}[V_r(((i_1,\cdots,i_k)^\circ(j-1))^\circ i)-V_r(((i_1,\cdots,i_k)^\circ(j-1))^*i)]\\
\geq\ &0,
\end{aligned}
$$

where the last equality follows from the easily verifiable fact that the rejection history $((i_1,\cdots,i_k)^\wedge i)^\circ j$ can be written as $((i_1,\cdots,i_k)^\circ j)^\wedge(i+1)$ or $((i_1,\cdots,i_k)^\circ(j-1))^\wedge i$ depending on whether $j\leq i$ or $j>i$ ($^\wedge$ denotes either $*$ or $^\circ$) and the last inequality follows from the induction hypothesis and Lemma 2.3(ii).

(ii) The proof is by induction on r, in the same manner as in Lemma 2.3(iii)(a).

APPENDIX B

B. 1 Proof of Lemma 3. 2

We show by induction on r. For $r=n-1$, the assertion is immediate from Lemma 3.1(ii) through $V_{n-1}(i_1,\cdots,i_k)=b_{n-1}(i_1,\cdots,i_k)/n$. We show the case $i_1\leq m-1$ and $i_l=m-1$, because other cases can be shown similarly. Assume now that the assertion holds for r. Then we have from (2.7), Lemma 3.1(ii) and the induction hypothesis

$$
\begin{aligned}
&V_{r-1}(i_1,\cdots,i_k)\\
=\ &\frac{1}{r}\max\left\{\frac{r}{n}q^{k-l}b_{r-1}(i_1,\cdots,i_l)+q^{k+1-l}V_r((i_1,\cdots,i_{l-1})^*1),q^{k+1-l}V_r((i_1,\cdots,i_{l-1})^\circ 1)\right\}\\
&+\frac{1}{r}\left\{\sum_{i=2}^{i_l}q^{k+1-l}V_r((i_1,\cdots,i_{l-1})^\circ i)+\sum_{i=i_l+1}^{r}q^{k-l}V_r((i_1,\cdots,i_l)^\circ i)\right\}\\
=\ &q^{k-l}V_{r-1}(i_1,\cdots,i_l),
\end{aligned}
$$

where the last equality again follows from (2.7) and the induction hypothesis.

B. 2 Proof of Theorem 3.5

Let $K_r\equiv V_r(\phi)-V_r(1)$ and define

$$G(\phi) = \left\{r : K_r \leq \frac{r}{n}\right\} \tag{B.1}$$

$$G(1) = \left\{r : K_r \leq \frac{r}{nq} b_{r-1}(1)\right\}. \tag{B.2}$$

Then, from (3.6) and (3.7), $G(\phi)$ and $G(1)$ represent the optimal stopping(offering) regions, each corresponding to the basic rejection histories (ϕ) and (1). Thus if $r \in G(\phi)(r \in G(1))$, the optimal policy makes an offer in state $(1;\phi)((1;1))$ at time r.

We begin with the following lemma.

Lemma B.1
$G(\phi)$ and $G(1)$ have the following properties.
Case 1 : $q_2 \geq q$

(i) $r \in G(\phi) \implies r \in G(1)$.

(ii) $r \notin G(1) \implies r-1 \notin G(1)$.

(iii) Let

$$s(1) = \min\left\{r : K_r \leq \frac{r}{nq} b_{r-1}(1)\right\}.$$

Then, for $r \geq s(1)$,

$$r \notin G(\phi) \implies r-1 \notin G(\phi).$$

Case 2 : $q_2 < q$

(i) $r \in G(1) \implies r \in G(\phi)$.

(ii) $r \notin G(\phi) \implies r-1 \notin G(\phi)$.

(iii) Let

$$s(\phi) = \min\left\{r : K_r \leq \frac{r}{n}\right\}.$$

Then, for $r \geq s(\phi)$,

$$r \notin G(1) \implies r-1 \notin G(1).$$

Proof.
Case 1.
(i) is immediate from (B.1), (B.2) and the fact that $b_{r-1}(1) \geq q$.
(ii) Assume that $r \notin G(1)$, then $r \notin G(\phi)$ from (i). Thus, from (3.6) and (3.7),

$$V_{r-1}(\phi) = V_r(\phi), \tag{B.3}$$

$$V_{r-1}(1) = \frac{q}{r} V_r(\phi) + \frac{r-1}{r} V_r(1). \tag{B.4}$$

Therefore, from (B.3) and (B.4),

$$K_{r-1} = V_{r-1}(\phi) - V_{r-1}(1) = K_r + \frac{1}{r}[V_r(1) - qV_r(\phi)] \geq K_r, \tag{B.5}$$

where the last inequality follows from the fact that $V_r(1) \geq qV_r(\phi)$, which comes from Lemma

2.4(ii) and Lemma 3.2. Considering that, from Lemma 2.3(iii)(b), $b_r(1)$ is non-decreasing in r, we have from (B.5) and the assumption that $r \notin G(1)$ that

$$K_{r-1} \geq K_r \geq \frac{r}{nq} b_{r-1}(1) \geq \frac{r-1}{nq} b_{r-2}(1),$$

which proves $r-1 \notin G(1)$.
(iii) Assume that $r \notin G(\phi)$ for some $r \geq s(1)$. Then, from (3.6),

$$V_{r-1}(\phi) = V_r(\phi), \qquad K_r > \frac{r}{n}. \quad (B.6)$$

On the other hand, from the definition of $s(1)$, we have $r \in G(1)$ which implies from (3.7)

$$V_{r-1}(1) = \frac{1}{n} b_{r-1}(1) + \left(1 - \frac{p}{r}\right) V_r(1). \quad (B.7)$$

Thus, from (B.6) and (B.7),

$$K_{r-1} = V_{r-1}(\phi) - V_{r-1}(1) = K_r - \frac{1}{n} b_{r-1}(1) + \frac{p}{r} V_r(1) > \frac{r-1}{n} \quad (\text{since} \quad b_{r-1}(1) \leq 1)$$

which implies $r-1 \notin G(\phi)$.

Case 2.
(i) is immediate from (B.1), (B.2) and the fact that $b_{r-1}(1) \leq q$.
(ii) Assume that $r \notin G(\phi)$, then $r \notin G(1)$ from (i). Thus, from (3.6) and (3.7),

$$V_{r-1}(\phi) = V_r(\phi), \qquad K_r > \frac{r}{n} \tag{B.8}$$

$$V_{r-1}(1) = \frac{q}{r} V_r(\phi) + \frac{r-1}{r} V_r(1). \tag{B.9}$$

Therefore, from (B.8) and (B.9),

$$K_{r-1} = V_{r-1}(\phi) - V_{r-1}(1) \geq \frac{r-1}{r} K_r > \frac{r-1}{n},$$

which proves $r-1 \notin G(\phi)$.
(iii) Assume that $r \notin G(1)$ for some $r \geq s(\phi)$. Then, from (3.7),

$$V_{r-1}(1) = \frac{q}{r} V_r(\phi) + \frac{r-1}{r} V_r(1) \tag{B.10}$$

$$K_r > \frac{r}{nq} b_{r-1}(1). \tag{B.11}$$

On the other hand, from the definition of $s(\phi)$, we have $r \in G(\phi)$ which implies from (3.6)

$$V_{r-1}(\phi) = \frac{1}{r}\left[\frac{r}{n} + V_r(1)\right] + \frac{r-1}{r} V_r(\phi). \tag{B.12}$$

Thus, from (B.12) and (B.10),

$$\begin{aligned} K_{r-1} &= V_{r-1}(\phi) - V_{r-1}(1) \\ &= \frac{1}{n} + \frac{r-1-q}{r} K_r + \frac{p}{r} V_r(1) \\ &\geq \frac{1}{n} + \frac{r-2}{r} K_r \\ &> \frac{1}{n}\left[1 + \frac{r-2}{q} b_{r-1}(1)\right] \qquad (\text{from} \quad (B.11)) \\ &= \frac{r-1}{nq} b_{r-2}(1), \qquad (\text{from} \quad (2.9)) \end{aligned}$$

which implies $r-1 \notin G(1)$ completing the proof.

Lemma B.1 can be summarized as follows : There exist two integers $s(\phi)$ and $s(1)$ such that

$$\begin{aligned} G(\phi) &= \{r : s(\phi) \le r \le n\} \\ G(1) &= \{r : s(1) \le r \le n\}, \end{aligned}$$

where $s(\phi)$ and $s(1)$ are respectively defined as

$$s(\phi) = \min\left\{r : K_r \le \frac{r}{n}\right\} \tag{B.13}$$

$$s(1) = \min\left\{r : K_r \le \frac{r}{nq} b_{r-1}(1)\right\} \tag{B.14}$$

and satisfy

$$s(\phi) \ge s(1) \text{ (Case 1)}, \qquad s(\phi) \le s(1) \text{ (Case 2)}. \tag{B.15}$$

Differential forms corresponding to (3.6) and (3.7) are given, from Lemma 3.3, by

$$\frac{d}{dx}V(x;\phi) = \frac{1}{x}V(x;\phi) - \frac{1}{x}\max\{x + V(x;1), V(x;\phi)\} \tag{B.16}$$

$$\frac{d}{dx}V(x;1) = \frac{1}{x}V(x;1) - \frac{1}{x}\max\{xb(x;1) + qV(x;1), qV(x;\phi)\}, \tag{B.17}$$

where

$$b(x;1) = q_2 x + q(1-x). \tag{B.18}$$

We are now ready to prove Theorem 3. 5.

Proof of Theorem 3.5

Case 1.

(i) Let $r_0 = s(\phi)$. Then, from (B.15) and Figure 1(a), the optimal policy is r_0-threshold rule.

(ii) For $r \ge r_0$, we have from (3.6) and (3.7) that

$$V_{r-1}(\phi) = \frac{1}{n} + \frac{1}{r}V_r(1) + \frac{r-1}{r}V_r(\phi), \tag{B.19}$$

$$V_{r-1}(1) = \frac{1}{n}b_{r-1}(1) + \frac{r-p}{r}V_r(1). \tag{B.20}$$

Since

$$b_r(1) = \frac{r}{n-1}q_2 + \left(1 - \frac{r}{n-1}\right)q,$$

starting with the boundary condition $V_n(1) = 0, V_r(1)$ can be solved from (B.20) recursively to yield

$$\begin{aligned} V_r(1) &= \frac{1}{n}\sum_{k=r}^{n-1} b_k(1) \prod_{j=r+1}^{k}\left(1 - \frac{p}{j}\right) \\ &= \frac{r}{n}\left[d_{r,n-1} + \frac{q_2 - q}{n-1}\left\{n - r + \sum_{k=r}^{n-2} d_{r,k}\right\}\right]. \end{aligned} \tag{B.21}$$

Substituting (B.21) into (B.19) and then solving (B.19), we have

$$\begin{aligned} V_r(\phi) &= \frac{r}{n}\left[\sum_{j=r}^{n-1}\frac{1}{j}+\sum_{j=r+1}^{n-1}\frac{n}{j(j-1)}V_j(1)\right] \\ &= \frac{r}{qn}\left[d_{r,n-1}+\frac{q_2-q}{n-1}\sum_{k=r}^{n-2}d_{r,k}\right]. \end{aligned} \tag{B.22}$$

Thus (3.8) follows by substituting (B.21) and (B.22) into (B.13).

(iii) (3.9) is obtained from (B.22) through $V^* = V_0(\phi) = \cdots = V_{r_0-1}(\phi)$.

(iv) For $x \geq \tilde{r_0}$, we have from (B.16) and (B.17) that

$$\frac{d}{dx}V(x;\phi) - \frac{1}{x}V(x;\phi) + \left[1+\frac{1}{x}V(x;1)\right] = 0 \tag{B.23}$$

$$\frac{d}{dx}V(x;1) - \frac{p}{x}V(x;1) + b(x;1) = 0. \tag{B.24}$$

These equations combined with (B.18) and $V(1;\phi) = V(1;1) = 0$ are solved as follows after a bit of calculation.

$$V(x;1) = \frac{x^p}{1+q}[(1+q_2)-(1+q)x^q-(q_2-q)x^{1+q}] \tag{B.25}$$

$$V(x;\phi) = \frac{x^p}{q(1+q)}[(1+q_2)-(1+q)(1-q+q_2)x^q+q(q_2-q)x^{1+q}]. \tag{B.26}$$

Thus (3.10) follows from (B.25) and (B.26) because $\tilde{r_0}$ can be defined from (B.13) as a unique root x of the equation $V(x;\phi) - V(x;1) = x$.

(v) (3.11) is immediate from (B.26) through $\tilde{V} = V(0^+;\phi) = V(\tilde{r_0};\phi)$.

Case 2.

(i) Let $r_1 = s(\phi)$ and $r_2 = s(1)$. Let $r(\geq r_1)$ be the time when the first offer is made but rejected. Then at this moment the basic rejection history changes from (ϕ) to (1)(see Figure 1(a)). Thus if the next candidate appears prior to r_2, the optimal policy makes no offer due to (B.15) and accordingly the basic rejection history changes from (1) to (ϕ), i.e., we restart over again with the basic rejection history (ϕ). Such change of the basic rejection history repeats itself as long as a candidate appears prior to r_2. From time r_2 onward, the optimal policy obviously makes an offer to each candidate successively. Thus the policy described in Theorem 3.5(i) turns out to be optimal.

(ii) For $r \geq r_2$, (B.19) and (B.20) also hold and hence $V_r(1)$ and $V_r(\phi)$ are given by (B.21) and (B.22) respectively. Thus (3.12) follows by substituting (B.21) and (B.22) into (B.14). For $r_1 \leq r < r_2$, (3.6) and (3.7) yield

$$\begin{aligned} V_{r-1}(\phi) &= \frac{1}{n}+\frac{1}{r}V_r(1)+\frac{r-1}{r}V_r(\phi) \\ V_{r-1}(1) &= \frac{q}{r}V_r(\phi)+\frac{r-1}{r}V_r(1), \end{aligned}$$

which can be written as

$$P_{r-1} = \frac{1}{r-1}Q_r + P_r \tag{B.27}$$

$$Q_{r-1} = \frac{q}{r-1}P_r + Q_r \tag{B.28}$$

through the transformation

$$V_r(\phi) = \frac{r}{n}P_r \quad \text{(B.29)}$$

$$V_r(1) = \frac{r}{n}[Q_r - 1]. \quad \text{(B.30)}$$

From (B.27) and (B.28), Q_r satisfies the 2nd order difference equation

$$[(r-1)^2 - q]Q_r - (r-1)(2r-3)Q_{r-1} + (r-2)(r-1)Q_{r-2} = 0. \quad \text{(B.31)}$$

Substituting the assumed form $Q_r = \prod_{j=r}^{m}(1 + a/j)$ into (B.31) yields $a = \pm\sqrt{q}$. This implies that the general solution to (B.31) can be expressed as

$$Q_r = A_1 \prod_{j=r}^{r_2-2}\left(1 + \frac{\sqrt{q}}{j}\right) + A_2 \prod_{j=r}^{r_2-2}\left(1 - \frac{\sqrt{q}}{j}\right), \quad \text{(B.32)}$$

where A_1 and A_2 are constants that must be determined.
Applying (B.32) to (B.28) yields

$$P_r = \frac{1}{\sqrt{q}}\left[A_1 \prod_{j=r}^{r_2-2}\left(1 + \frac{\sqrt{q}}{j}\right) - A_2 \prod_{j=r}^{r_2-2}\left(1 - \frac{\sqrt{q}}{j}\right)\right]. \quad \text{(B.33)}$$

Thus (3.13) is immediate from (B.32), (B.33), (B.29) and (B.30) since r_1 is defined as the smallest integer r such that $P_r \le Q_r$ through (B.13). A_1 and A_2 are determined by equating (B.29) and (B.30) with (B.22) and (B.21) respectively at $r = r_2 - 1$.

(iii) (3.14) is obtained from (B.29) and (B.33) through $V^* = V_0(\phi) = \cdots = V_{r_1-1}(\phi)$.

(iv) For $x \ge \tilde{r_2}$, (B.23) and (B.24) also hold and hence $V(x;1)$ and $V(x;\phi)$ are given by (B.25) and (B.26) respectively. Thus (3.15) follows from (B.25), (B.26) and (B.18) because $\tilde{r_2}$ can be defined from (B.14) as a unique root x of the equation

$$V(x;\phi) - V(x;1) = \frac{x}{q}b(x;1).$$

For $\tilde{r_1} \le x < \tilde{r_2}$, we have from (B.16) and (B.17)

$$\frac{d}{dx}V(x;\phi) - \frac{1}{x}V(x;\phi) + \left[1 + \frac{1}{x}V(x;1)\right] = 0$$

$$\frac{d}{dx}V(x;1) - \frac{1}{x}V(x;1) + \frac{q}{x}V(x;\phi) = 0,$$

which can be transformed into

$$\frac{d}{dx}P(x) + \frac{1}{x}Q(x) = 0 \quad \text{(B.34)}$$

$$\frac{d}{dx}Q(x) + \frac{q}{x}P(x) = 0 \quad \text{(B.35)}$$

through

$$V(x;\phi) = xP(x) \quad \text{(B.36)}$$

$$V(x;1) = x[Q(x) - 1]. \quad \text{(B.37)}$$

From (B.34) and (B.35), $Q(x)$ satisfies

$$x^2\frac{d^2}{dx^2}Q(x) + x\frac{d}{dx}Q(x) - qQ(x) = 0,$$

which is an Euler-type differential equation and is solved to yield

$$Q(x) \quad = \quad C_1 x^{-\sqrt{q}} + C_2 x^{\sqrt{q}}, \tag{B.38}$$

where C_1 and C_2 are constants to be determined.
Applying (B.38) to (B.35) yields

$$P(x) \quad = \quad \frac{1}{\sqrt{q}}[C_1 x^{-\sqrt{q}} - C_2 x^{\sqrt{q}}]. \tag{B.39}$$

Thus (3.16) is immediate from (B.38), (B.39), (B.36) and (B.37) since $\tilde{r_1}$ is defined as a unique root x of the equation

$$P(x) \quad = \quad Q(x) \tag{B.40}$$

through (B.13).

C_1 and C_2 are determined by equating (B.36) and (B.37) with (B.26) and (B.25) respectively at $x = \tilde{r_2}$.

(v) (3.17) is immediate from (B.36) and (B.40) through $\tilde{V} = V(\tilde{r_1}; \phi) = \tilde{r_1} P(\tilde{r_1}) = \tilde{r_1} Q(\tilde{r_1})$.

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