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A TWO-PERSON ZERO-SUM GAME WITH FRACTIONAL LOSS FUNCTION

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Abstract In this paper, we investigate a two-person zero-sum game with fractional loss function, which we call the two-person zero-sum fractional game. We are interested in a game value and a saddle point of the two-person zero-sum game and observe that they exist under various conditions. But, in many cases, it seems to be difficult to search directly for a game value and a saddle point of the fractional game. Thus, we study the existence and the properties of a game value and a saddle point for the game with the loss function including a parameter, which we call the two-person zero-sum parametric game. We show that, under various conditions, a game value and a saddle point of the parametric game with a special parameter are those of the fractional game.

1. Introduction

Recently, many concepts and terms about game theory have been introduced and have been investigated by many authors. Both individual stability and collective stability have been studied in practical game problems. In view of individual stability in two-person game, concepts of a game value and a saddle point in the two-person zero-sum game were introduced. Then, we are interested in a saddle point of the two-person zero-sum game, and the existence of a saddle point has been actively investigated under various conditions. We often observe two-person zero-sum games with fractional loss function in many economic problems. However, as far as we know, we think that there are few papers which treat such games.

In this paper, we study the existence and some properties of a saddle point for the two-person zero-sum game with fractional loss function, which we call the two-person zero-sum fractional game. However, in many cases, it seems to be difficult to search directly for a saddle point of the fractional game. Because, even if two functions are convex or concave, the loss function constructing by fractional expression of them is not necessarily convex or concave. Thus, we consider a two-person zero-sum game with loss function including a parameter, which we call the two-person zero-sum parametric game. We study the existence and properties of a saddle point for the parametric game. For the analysis of the parametric game, the mini-max theorem in convex analysis plays an important role. Then, we show that under various conditions, a saddle point of the parametric game with a special parameter is that of the fractional game. Moreover, using a game value of the game with a special parameter with respect to $\varepsilon > 0$, we show that there exists an ε-saddle point of the fractional game.

This paper is organized in the following way. In Section 2, we formulate a two-person zero-sum parametric game and characterize a game value and a saddle point of the parametric game. Section 3 is the main part of this paper. Associated with the results of Section 2, we discuss relations between the two-person zero-sum parametric game and the two-person zero-sum fractional game. Moreover, we show that there exists an ε-saddle point of the fractional game.

2. A Two-Person Zero-Sum Parametric Game

We begin by describing a two-person zero-sum parametric game (GP_θ) by the following strategic (or normal) form:

$$(X, Y, f, g, \theta, F_\theta), \tag{2.1}$$

where

1. X is a subset of a Banach space E, which is called the strategy set of player 1,
2. Y is a subset of a Banach space E, which is called the strategy set of player 2,
3. $f : X \times Y \to R$ and $g : X \times Y \to R_+$, where $R_+ = (0, \infty)$,
4. θ is a real number, which is called a parameter of the game,
5. $F_\theta = f - \theta g : X \times Y \to R$, that is, for all $(x, y) \in X \times Y$, $F_\theta(x, y) = f(x, y) - \theta g(x, y)$, is a loss function of player 1 and $-F_\theta(x, y)$ is a loss function of player 2.

In general, $\overline{F}_\theta = \inf_{x\in X} \sup_{y\in Y} F_\theta(x, y)$ is called the minimal worst loss of player 1 and $\underline{F}_\theta = \sup_{y\in Y} \inf_{x\in X} F_\theta(x, y)$ is called the maximal worst gain of player 2. Further, we call the **duality gap** the interval $[\underline{F}_\theta, \overline{F}_\theta]$.

Definition 2.1 *The two-person zero-sum game (GP_θ) has a* **game value** *(in short, a value), if*

$$\overline{F}_\theta = \underline{F}_\theta = F_\theta^*$$

and this common value F_θ^* is called the **game value** *(in short, the value) of the game (GP_θ).*

Definition 2.2 *A strategy $y^* \in Y$ is said to be a* **max-inf** *of the game (GP_θ) if*

$$\inf_{x\in X} F_\theta(x, y^*) = \inf_{x\in X} \sup_{y\in Y} F_\theta(x, y)$$

and a strategy $x^* \in X$ is said to be a **mini-sup** *of the game (GP_θ) if*

$$\sup_{y\in Y} F_\theta(x^*, y) = \sup_{y\in Y} \inf_{x\in X} F_\theta(x, y).$$

Definition 2.3 *A pair of strategies, $(x^*, y^*) \in X \times Y$, is said to be a* **saddle point** *of the game (GP_θ) if*

$$\inf_{x\in X} F_\theta(x, y^*) = F_\theta(x^*, y^*) = \sup_{y\in Y} F_\theta(x^*, y).$$

Then, the following fundamental result is known from J.-P. Aubin [1], Chapter 6.

Proposition 2.1 *A pair of strategies, (x^*, y^*), is a saddle point of the game (GP_θ) if and only if x^* and y^* are a mini-sup and a max-inf of the game (GP_θ), respectively.*

Lemma 2.1 *Let X and Y be subsets of a Banach space E and let X be a compact convex set. Suppose that a function $\varphi : X \times Y \to R$ satisfies the following conditions:*

(1) *$\varphi(x, y)$ is lower semi-continuous and convex with respect to x for all $y \in Y$;*
(2) *$\varphi(x, y)$ is concave with respect to y for all $x \in X$.*

Then, there exists $x^* \in X$, which satisfies

$$\sup_{y\in Y} \min_{x\in X} \varphi(x, y) = \sup_{y\in Y} \varphi(x^*, y) = \min_{x\in X} \sup_{y\in Y} \varphi(x, y), \tag{2.2}$$

that is, x^* is a mini-sup of the game.

The proof of this lemma is shown by Ky Fan's system theorem. See J.-P. Aubin [1, 2], and K. Fan [5] in detail.

Lemma 2.2 *Let X and Y be subsets of a Banach space E and let Y be a compact convex set. Suppose that a function $\varphi : X \times Y \to R$ satisfies the following conditions:*

(1) *$\varphi(x, y)$ is upper semi-continuous and concave with respect to y for all $x \in X$;*

(2) *$\varphi(x, y)$ is convex with respect to x for all $y \in Y$.*

Then, there exists $y^* \in Y$, which satisfies

$$\max_{y \in Y} \inf_{x \in X} \varphi(x, y) = \inf_{x \in X} \varphi(x, y^*) = \inf_{x \in X} \max_{y \in Y} \varphi(x, y), \tag{2.3}$$

that is, y^* is a max-inf of the game.

Corollary 2.1 *Let X and Y be compact convex subsets in a Banach space E. Suppose that a function $\varphi : X \times Y \to R$ satisfies the following conditions:*

(1) *$\varphi(x, y)$ is lower semi-continuous and convex with respect to x for all $y \in Y$;*

(2) *$\varphi(x, y)$ is upper semi-continuous and concave with respect to y for all $x \in X$.*

Then, there exists a saddle point $(x^*, y^*) \in X \times Y$ of the game.

Since there exist a *mini-sup* and a *max-inf* of the game under the conditions, the proof is given by Proposition 2.1.

We have the following minimax theorems and its corollaries for the game (GP_θ).

Theorem 2.1 *Suppose that Y is a compact convex set and that functions f and g satisfy the following conditions:*

(1) *$f(x, y)$ is convex with respect to x for all $y \in Y$;*

(2) *$f(x, y)$ is upper semi-continuous and concave with respect to y for all $x \in X$;*

(3) *$g(x, y)$ is concave with respect to x for all $y \in Y$;*

(4) *$g(x, y)$ is lower semi-continuous and convex with respect to y for all $x \in X$.*

Then, for all $\theta \geq 0$, there exists $y^* \in Y$ such that

$$\overline{F}_\theta = \underline{F}_\theta = \inf_{x \in X} F_\theta(x, y^*), \tag{2.4}$$

that is, y^* is a max-inf of the game (GP_θ).

Proof. Since θ is non-negative, from (2) and (4) in the theorem, it follows that the function $F_\theta(x, y)$ is upper semi-continuous and concave with respect to y for all $x \in X$. Similarly, from (1) and (3) in the theorem, we get that the function $F_\theta(x, y)$ is convex with respect to x for all $y \in Y$. Thus, Lemma 2.2 shows that $\overline{F}_\theta = \underline{F}_\theta$ and that there exists $y^* \in Y$ which is a *max-inf* of the game (GP_θ).

Corollary 2.2 *Suppose that X and Y are compact convex sets and that functions f and g satisfy the following conditions:*

(1) *$f(x, y)$ is lower semi-continuous and convex with respect to x for all $y \in Y$;*

(2) *$f(x, y)$ is upper semi-continuous and concave with respect to y for all $x \in X$;*

(3) *$g(x, y)$ is upper semi-continuous and concave with respect to x for all $y \in Y$;*

(4) *$g(x, y)$ is lower semi-continuous and convex with respect to y for all $x \in X$.*

Then, for all $\theta \geq 0$, there exists a saddle point $(x^*, y^*) \in X \times Y$ of the game (GP_θ).

From Theorem 2.1, there exist a *mini-sup* x^* and a *max-inf* y^* of the game (GP_θ). Thus, the proof of the corollary is easily given.

3. A Two-Person Zero-Sum Fractional Game

We define a two-person zero-sum game with fractional loss function (GP) as follows:

$$(X, Y, f, g, G), \tag{3.1}$$

where X and Y are subsets of a Banach space E, which are called the strategy sets of player 1 and player 2, respectively. Then, using functions $f : X \times Y \to R$ and $g : X \times Y \to R_+$, a loss function G of player 1 is given by $G = f/g$, that is, for all $(x, y) \in X \times Y$

$$G(x,y) = \frac{f(x,y)}{g(x,y)} \tag{3.2}$$

and $-G(x, y)$ is a loss function of player 2. Further, two parameters $\overline{\theta}$ and $\underline{\theta}$ are given by

$$\overline{\theta} := \inf_{x\in X} \sup_{y\in Y} G(x,y) \quad \text{and} \quad \underline{\theta} := \sup_{y\in Y} \inf_{x\in X} G(x,y) \tag{3.3}$$

In general, it holds that

$$\underline{\theta} \le \overline{\theta}$$

and the interval $[\underline{\theta}, \overline{\theta}]$ is called "duality gap" of the game (GP).

Definition 3.1 *The game (GP) has a* **value** *if*

$$\underline{\theta} = \overline{\theta} = \theta^*.$$

Further, $y^* \in Y$ is said to be a **max-inf** *of the game (GP) if*

$$\inf_{x\in X} G(x, y^*) = \inf_{x\in X} \sup_{y\in Y} G(x,y) = \overline{\theta}. \tag{3.4}$$

Similarly, $x^* \in X$ is said to be a **mini-sup** *of the game (GP) if*

$$\sup_{y\in Y} G(x^*, y) = \sup_{y\in Y} \inf_{x\in X} G(x,y) = \underline{\theta}. \tag{3.5}$$

From now, we will study some relations between the game (GP_θ) and the game (GP). Using the parameter $\overline{\theta}$ given by (3.3), we consider the game $(GP_{\overline{\theta}})$ with the loss function such as

$$F_{\overline{\theta}} = f - \overline{\theta} g : \quad X \times Y \to R, \tag{3.6}$$

that is, for each $(x, y) \in X \times Y$, $F_{\overline{\theta}}(x, y) = f(x, y) - \overline{\theta} g(x, y)$.

Then, in order to show the main theorems, we observe properties of $\overline{F}_\theta$ and $\underline{F}_\theta$ given by $\overline{F}_\theta = \inf_{x\in X} \sup_{y\in Y} F_\theta(x, y)$ and $\underline{F}_\theta = \sup_{y\in Y} \inf_{x\in X} F_\theta(x, y)$. We shall mention the following lemmas.

Lemma 3.1 *$\overline{F}_\theta$ has the following properties.*

(a) *$\overline{F}_\theta$ is non-increasing with respect to θ.*
(b) *If $\overline{F}_\theta < 0$, it holds that $\theta \ge \overline{\theta}$.*
(c) *If $\overline{F}_\theta > 0$, it holds that $\theta \le \overline{\theta}$.*
(d) *If $\theta > \overline{\theta}$, it holds that $\overline{F}_\theta \le 0$.*
(e) *If $\theta < \overline{\theta}$, it holds that $\overline{F}_\theta \ge 0$.*

Furthermore, under the conditions that Y is compact and that $f(x, y)$ and $g(x, y)$ are continuous with respect to y for all $x \in X$, it holds that

(f) $\overline{F}_\theta < 0$ *holds if and only if* $\theta > \overline{\theta}$ *holds,*
(g) $\overline{F}_{\overline{\theta}} \geq 0$ *holds.*

Proof. (a) If $\theta_1 < \theta_2$, then we get $\theta_1 g < \theta_2 g$ on $X \times Y$, because g is positive on $X \times Y$. Thus, it follows that $f(x,y) - \theta_1 g(x,y) > f(x,y) - \theta_2 g(x,y)$ for all $(x,y) \in X \times Y$, that is,

$$F_{\theta_1}(x,y) > F_{\theta_2}(x,y). \tag{3.7}$$

Therefore, we get that

$$\overline{F}_{\theta_1} = \inf_{x\in X} \sup_{y\in Y} F_{\theta_1}(x,y) \geq \inf_{x\in X} \sup_{y\in Y} F_{\theta_2}(x,y) = \overline{F}_{\theta_2}.$$

Hence, this shows that $\overline{F}_\theta$ is non-increasing with respect to θ.
(b) Since $\overline{F}_\theta < 0$, from the definition of $\overline{F}_\theta$, there exists $\overline{x} \in X$ such that

$$\sup_{y\in Y} F_\theta(\overline{x},y) < 0,$$

that is, for all $y \in Y$,

$$F_\theta(\overline{x},y) = f(\overline{x},y) - \theta g(\overline{x},y) < 0. \tag{3.8}$$

This shows that for all $y \in Y$, $G(\overline{x},y) = \dfrac{f(\overline{x},y)}{g(\overline{x},y)} < \theta$, that is

$$\sup_{y\in Y} G(\overline{x},y) \leq \theta. \tag{3.9}$$

From the definition of $\overline{\theta}$ and (3.9), it follows that $\overline{\theta} \leq \theta$.
(c) Since $\overline{F}_\theta > 0$, that is, for all $x \in X$, $\sup_{y\in Y} F_\theta(x,y) > 0$, there exists $y_x \in Y$, which depends on x, such that

$$F_\theta(x,y_x) = f(x,y_x) - \theta g(x,y_x) > 0. \tag{3.10}$$

From (3.10), it follows that $G(x,y_x) = \dfrac{f(x,y_x)}{g(x,y_x)} > \theta$, that is, for all $x \in X$,

$$\sup_{y\in Y} G(x,y) \geq G(x,y_x) > \theta,$$

which shows that $\overline{\theta} \geq \theta$.
(d) Since $\theta > \overline{\theta}$, from the definition of $\overline{\theta}$, there exists $\overline{x} \in X$ such that

$$\theta > \sup_{y\in Y} G(\overline{x},y),$$

that is, for all $y \in Y, \theta > G(\overline{x},y)$. This shows that for all $y \in Y$, $F_\theta(\overline{x},y) < 0$. Hence, we get that

$$0 \geq \sup_{y\in Y} F_\theta(\overline{x},y) \geq \inf_{x\in X} \sup_{y\in Y} F_\theta(x,y) = \overline{F}_\theta.$$

(e) Since $\overline{\theta} > \theta$, from the definition of $\overline{\theta}$, it follows that for all $x \in X$,

$$\sup_{y\in Y} G(x,y) > \theta.$$

Thus, there exists $y_x \in Y$, which depends on x, such that $G(x, y_x) > \theta$, that is, $F_\theta(x, y_x) > 0$. Then, we get that for all $x \in X$,

$$\sup_{y\in Y} F_\theta(x, y) \geq F_\theta(x, y_x) > 0.$$

Hence, we get that

$$\overline{F}_\theta = \inf_{x\in X} \sup_{y\in Y} F_\theta(x, y) \geq 0.$$

(f) Since $\overline{F}_\theta < 0$, there exists $\overline{x} \in X$ such that $\sup_{y\in Y} F_\theta(\overline{x}, y) < 0$, that is, for all $y \in Y$, $G(\overline{x}, y) < \theta$. Noting that Y is compact and $G(\overline{x}, y)$ is continuous on Y, we get that

$$\theta > \sup_{y\in Y} G(\overline{x}, y) \geq \inf_{x\in X} \sup_{y\in Y} G(x, y) = \overline{\theta}.$$

Thus, it holds that $\theta > \overline{\theta}$.

Conversely, since $\theta > \overline{\theta}$, from the proof of (b), it follows that there exists $\overline{x} \in X$ such that for all $y \in Y$, $F_\theta(\overline{x}, y) < 0$. Noting that Y is compact and $F_\theta(\overline{x}, y)$ is continuous on Y, we get that

$$0 > \sup_{y\in Y} F_\theta(\overline{x}, y) \geq \inf_{x\in X} \sup_{y\in Y} F_\theta(x, y) = \overline{F}_\theta.$$

Thus, the proof of (f) is completed.

(g) Suppose that $\overline{F}_{\overline{\theta}} < 0$ holds, there exists $\overline{x} \in X$ such that $\sup_{y\in Y} F_{\overline{\theta}}(\overline{x}, y) < 0$. Noting that Y is compact and $F_{\overline{\theta}}(\overline{x}, y)$ is continuous on Y, we get that for all $y \in Y$, $F_{\overline{\theta}}(\overline{x}, y) < 0$, that is, for all $y \in Y$, $\overline{\theta} > G(\overline{x}, y)$. This shows that

$$\overline{\theta} > \sup_{y\in Y} G(\overline{x}, y) \geq \inf_{x\in X} \sup_{y\in Y} G(x, y) = \overline{\theta}.$$

We arrive at a contradiction. Hence, it is proved that $\overline{F}_{\overline{\theta}} \geq 0$ holds.

Lemma 3.2 *$\underline{F}_\theta$ has the following properties.*

(a) *$\underline{F}_\theta$ is non-increasing with respect to θ.*
(b) *If $\underline{F}_\theta < 0$, it holds that $\theta \geq \underline{\theta}$.*
(c) *If $\underline{F}_\theta > 0$, it holds that $\theta \leq \underline{\theta}$.*
(d) *If $\theta > \underline{\theta}$, it holds that $\underline{F}_\theta \leq 0$.*
(e) *If $\theta < \underline{\theta}$, it holds that $\underline{F}_\theta \geq 0$.*

Furthermore, under the conditions that X is compact and that $f(x, y)$ and $g(x, y)$ are continuous with respect to x for all $y \in Y$, it holds that

(f) *$\underline{F}_\theta > 0$ holds if and only if $\theta < \underline{\theta}$ holds,*
(g) *$\underline{F}_{\underline{\theta}} \leq 0$ holds.*

Proof. Using $\underline{F}_\theta$ and $\underline{\theta}$ instead of $\overline{F}_\theta$ and $\overline{\theta}$ respectively, we can prove this lemma by similar arguments to the previous one.

We have the following relations between the game (GP_θ) and (GP).

Theorem 3.1 *Suppose that $y^* \in Y$ is a max-inf of the game (GP). Then, it holds that*

(i) *the game (GP) has a value θ^*,*
(ii) *If $\overline{F}_{\theta^*} \leq 0$, y^* is a max-inf of the game (GP_{θ^*}).*

Proof. (i) From the definition of $\overline{\theta}$ and $\underline{\theta}$, in general it holds that $\overline{\theta} \geq \underline{\theta}$.

On the other hand, since $y^* \in Y$ is a *max-inf* of the game (GP), it follows that

$$\overline{\theta} = \inf_{x \in X} G(x, y^*) \leq \sup_{y \in Y} \inf_{x \in X} G(x, y) = \underline{\theta}.$$

Thus, $\overline{\theta} = \underline{\theta}$ holds, that is, the game (GP) has a value θ^*.
(ii) Since y^* is a *max-inf* of the game (GP), it holds that for all $x \in X$,

$$\theta^* = \inf_{x \in X} G(x, y^*) \leq G(x, y^*),$$

that is, for all $x \in X$,

$$0 \leq F_{\theta^*}(x, y^*) \leq \sup_{y \in Y} F_{\theta^*}(x, y).$$

Thus, we arrive at the following inequality,

$$0 \leq \inf_{x \in X} F_{\theta^*}(x, y^*) \leq \inf_{x \in X} \sup_{y \in Y} F_{\theta^*}(x, y) = \overline{F}_{\theta^*} \leq 0.$$

This shows that

$$\inf_{x \in X} F_{\theta^*}(x, y^*) = \inf_{x \in X} \sup_{y \in Y} F_{\theta^*}(x, y).$$

That is, y^* is a *max-inf* of the game (GP_{θ^*}).

Corollary 3.1 *Suppose that $(x^*, y^*) \in X \times Y$ is a saddle point of the game (GP). Then, it holds that*

(i) $F_{\theta^*}(x^*, y^*) = 0$,
(ii) (x^*, y^*) *is a saddle point of the game* (GP_{θ^*}).

Theorem 3.2 *Suppose that the game (GP) has a value θ^* and that $\overline{F}_{\theta^*} \geq 0$ holds. Then, if $y^* \in Y$ is a max-inf of the game (GP_{θ^*}), y^* is a max-inf of the game (GP).*

Proof. Since $\overline{F}_{\theta^*} \geq 0$ and y^* is a *max-inf* of the game (GP_{θ^*}), it follows that

$$0 \leq \overline{F}_{\theta^*} = \inf_{x \in X} \sup_{y \in Y} F_{\theta^*}(x, y) = \inf_{x \in X} F_{\theta^*}(x, y^*) \leq F_{\theta^*}(x, y^*), \quad \text{for all } x \in X,$$

which implies that for all $x \in X$

$$\theta^* \leq G(x, y^*) \leq \sup_{y \in Y} G(x, y). \tag{3.11}$$

Therefore, we get that

$$\theta^* \leq \inf_{x \in X} G(x, y^*) \leq \inf_{x \in X} \sup_{y \in Y} G(x, y) = \theta^*.$$

This shows that y^* is *max-inf* of the game (GP).

Corollary 3.2 *Suppose that the game (GP) has a value θ^* and that a saddle point (x^*, y^*) of the game (GP_{θ^*}) satisfies $F_{\theta^*}(x^*, y^*) = 0$. Then, (x^*, y^*) is a saddle point of the game (GP).*

Theorem 3.3 *Suppose that Y is a compact convex set and that functions f and g satisfy the following conditions:*

1. *$f(x, y)$ is convex with respect to x for all $y \in Y$;*
2. *$f(x, y)$ is continuous and concave with respect to y for all $x \in X$;*

3. *$g(x,y)$ is concave with respect to x for all $y \in Y$;*
4. *$g(x,y)$ is continuous and convex with respect to y for all $x \in X$.*

Then, if $\overline{\theta} \geq 0$, it holds that

(i) *the game (GP) has a value θ^*,*

(ii) *There exists $y^* \in Y$ such that y^* is a max-inf of the game (GP).*

Proof. (i) Since $\overline{\theta} \geq 0$, it follows from Lemma 3.1 (g) and Theorem 2.1 that

$$\underline{F}_{\overline{\theta}} = \overline{F}_{\overline{\theta}} \geq 0 \tag{3.12}$$

and that there exists $y^* \in Y$, which is a *max-inf* of the game $(GP_{\overline{\theta}})$, that is,

$$\underline{F}_{\overline{\theta}} = \sup_{y\in Y} \inf_{x\in X} F_{\overline{\theta}}(x,y) = \inf_{x\in X} F_{\overline{\theta}}(x,y^*). \tag{3.13}$$

From (3.12) and (3.13), it follows that

$$0 \leq \sup_{y\in Y} \inf_{x\in X} F_{\overline{\theta}}(x,y) = \inf_{x\in X} F_{\overline{\theta}}(x,y^*) \leq F_{\overline{\theta}}(x,y^*), \quad \text{for all } x \in X. \tag{3.14}$$

From (3.14), it holds that for all $x \in X, \overline{\theta} \leq G(x,y^*)$. Consequently, we get

$$\overline{\theta} \leq \inf_{x\in X} G(x,y^*) \leq \sup_{y\in Y} \inf_{x\in X} G(x,y) = \underline{\theta}.$$

This shows that $\overline{\theta} = \underline{\theta}$ holds.

(ii) Since $\overline{F}_{\theta^*} \geq 0$ and $y^* \in Y$ is a *max-inf* of the game (GP_{θ^*}), from Theorem 3.2, it follows that y^* is a *max-inf* of the game (GP). Thus, the proof of the theorem is completed.

Theorem 3.4 *Suppose that Y is a compact convex set and that functions f and g satisfy the following conditions:*

1. *$f(x,y)$ is convex with respect to x for all $y \in Y$;*
2. *$f(x,y)$ is continuous and concave with respect to y for all $x \in X$;*
3. *$g(x,y)$ is concave with respect to x for all $y \in Y$;*
4. *$g(x,y)$ is continuous and convex with respect to y for all $x \in X$.*

Then, if $\overline{\theta} \geq 0$, for any $\varepsilon > 0$, there exists a point $(x^*,y^*) \in X \times Y$ such that

$$\theta^* \leq G(x^*,y^*) < \theta^* + \varepsilon,$$

where θ^* is the value of the game (GP).

Proof. From Theorem 3.3, we have $\overline{\theta} = \underline{\theta} = \theta^*$. Since $\theta^* \geq 0$, it follows from Lemma 3.1 (g) and Theorem 2.1, that there exists $y^* \in Y$ such that

$$\underline{F}_{\theta^*} = \overline{F}_{\theta^*} = \inf_{x\in X} F_{\theta^*}(x,y^*) \geq 0. \tag{3.15}$$

Then, from (3.15), we get that for all $x \in X$, $F_{\theta^*}(x,y^*) \geq 0$, that is, for all $x \in X$

$$\theta^* \leq G(x,y^*). \tag{3.16}$$

On the other hand, since $\overline{\theta} < \theta^* + \varepsilon$, it follows from Lemma 3.1 (f) that

$$\overline{F}_{\theta^*+\varepsilon}(x,y) = \inf_{x\in X} \sup_{y\in Y} F_{\theta^*+\varepsilon}(x,y) < 0,$$

which shows that there exists $x^* \in X$ such that $\sup_{y\in Y} F_{\theta^*+\varepsilon}(x^*,y) < 0$. Hence, we get that for all $y \in Y$, $F_{\theta^*+\varepsilon}(x^*,y) < 0$, that is, for all $y \in Y$,

$$G(x^*,y) < \theta^* + \varepsilon.$$

Thus, for the particular $y^* \in Y$ satisfying (3.16), it holds that

$$G(x^*,y^*) < \theta^* + \varepsilon. \tag{3.17}$$

Combining (3.16) with (3.17), we show that

$$\theta^* \leq G(x^*,y^*) < \theta^* + \varepsilon.$$

Thus, the proof of the theorem is completed.

Theorem 3.5 *Suppose that X and Y are compact convex sets, and that functions f and g satisfy the following conditions:*

1. *$f(x,y)$ is continuous and convex with respect to x for all $y \in Y$;*
2. *$f(x,y)$ is continuous and concave with respect to y for all $x \in X$;*
3. *$g(x,y)$ is continuous and concave with respect to x for all $y \in Y$;*
4. *$g(x,y)$ is continuous and convex with respect to y for all $x \in X$.*

Then, if $\overline{\theta} \geq 0$, there exists a saddle-point $(x^*,y^*) \in X \times Y$ such that for all x and y,

$$G(x^*,y) \leq \theta^* \leq G(x,y^*).$$

Using Theorem 3.3 and Corollary 2.1, we can easily prove the theorem.

References

[1] J. -P. Aubin: *Mathematical Methods of Game and Economic Theory*, Revised Edition (North-Holland, Amsterdam, 1982).

[2] J.-P. Aubin: *Optima and Equilibria* (Springer-Verlag, New York, 1993).

[3] J. P. Crouzeix, J. A. Ferland and S. Schaible: Duality in generalized linear fractional programming. *Mathematical Programming*, **27** (1983) 342–354.

[4] J. P. Crouzeix, J. A. Ferland and S. Schaible: An algorithm for generalized fractional programs. *Journal of Optimization Theory and Applications*, **47**-1 (1985) 35–49.

[5] K. Fan: Minimax theorems. *Proc. Nat. Acad. Sci. USA*, **39** (1953) 42–47.

[6] R. Jagannathan and S. Schaible: Duality in generalized fractional programming via Farkas's lemma. *Journal of Optimization Theory and Applications*, **41** (1983) 417–424.

[7] Y. Kimura, Y. Sawasaki and K. Tanaka: A perturbation on two-person zero-sum games. to appear in *Annals of Dynamic Game.*

[8] Y. Kimura, Y. Sawasaki and K. Tanaka: A non-cooperative equilibrium of n-person game with fractional loss functions. *RIMS Kokyuroku* (Kyoto Univ., Kyoto, Japan), **1043** (1998) 215–220.

[9] R.T. Rockafellar: *Convex Analysis* (Princeton University Press, 1970).

[10] K. Tanaka and K. Yokoyama: On ε-equilibrium point in a noncooperative n-person game. *Journal of Mathematical Analysis and Applications*, **160** (1991) 413–423.

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