

## WEIGHT OF SHORTEST PATH ANALYSES FOR THE OPTIMAL LOCATION PROBLEM

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*Abstract* In this paper we consider the shortest path counting problem (*SPCP*): how many shortest paths contain each edge of the network  $N = (V, E)$  with the vertex set  $V$  and the edge set  $E$ ? There are  $n(n-1)$  shortest paths in the network with  $|V| = n$ , so among all these shortest paths the *SPCP* requires us to count the number of shortest paths passing each edge. Defining the weight of shortest paths for each edge in the network as the number of shortest paths contained, we can obtain theoretical results in the form of explicit expressions for special types of networks such as trees, grid type, circular type and so on. We apply these results to solve median and center location problems for special types of networks. Furthermore we consider the network connection problem asking how to connect two networks with a single edge or two edges from the viewpoint of minimizing the total weight of shortest paths in the newly combined network. Finally we summarize our theoretical results and show the correspondence between these theoretical results of the *SPCP* and their implication in the actual location problems.

### 1. Introduction

The shortest path counting problem (*SPCP*) originates from the following situation. Let us consider the road network in a certain city area. We call the road between two adjacent intersections or between an intersection and a neighboring terminal point as a road segment. It happens in the real road network that some road segments are crowded with automobiles, while others are not. Generally, road segments in the central part of the city road network are most crowded with automobiles, while those in the peripheral and surrounding parts are not. Then how can we estimate the “degree” of congestion of each road segment in the city road network? Crowdedness of each road segment may correspond to the “importance” of the road segment. Then how can we quantify the importance of each road segment in the real city road network? The *SPCP* builds up the correspondence between road segment and its importance in the road network. Theoretical results for the *SPCP* are obtained for special types of networks, *e.g.*, grid type and circular type (see Oyama-Taguchi [4, 5, 6, 7]). We apply the *SPCP* to the location problem by defining the weight of shortest paths induced at each vertex of the road network, then we apply these results for finding the median and the center corresponding to the optimal location of facilities and also determining the optimal connection of two networks. In this paper we propose an approach such that, given a road network, we can estimate the crowdedness of each road segment based upon the structure of the network. This leads to measurement of the importance of each road segment.

Given a network  $N = (V, E)$  with vertex set  $V = \{1, \dots, n\}$  consisting of  $n$  vertices and edge set  $E = \{(i, j) \mid i \in V, j \in V\}$ , let each edge  $(i, j) \in E$  in the network  $N = (V, E)$  have the “length” or the “distance”  $d_{ij}$ . The shortest path from vertex  $i \in V$  to vertex  $j \in V$  in the network  $N = (V, E)$  is defined as the path with the shortest length, *i.e.*, the smallest sum of lengths of all edges contained in the path from vertex  $i \in V$  to vertex

$j \in V$ . Then the shortest path counting problem (*SPCP*) follows: how many shortest paths contain each edge of the network  $N = (V, E)$ ? There are  $n(n - 1)$  shortest paths in the network with  $|V| = n$ , so among all these shortest paths the *SPCP* requires us to count the number of shortest paths passing each edge. Let  $w_e(e_{ij})$  denote the number of shortest paths containing the edge  $e_{ij} = (i, j) \in E$ , i.e., the number of shortest paths passing the edge  $e_{ij} = (i, j)$  among  $n(n - 1)$  shortest paths. Let the weight of each shortest path indicate the total number of all edges contained in the shortest path. Then the weight of shortest paths induced at vertex  $i \in V$  in the network  $N = (V, E)$  is defined to be the sum of weights of all shortest paths leaving vertex  $i \in V$  and reaching all other vertices  $j \in V, j \neq i$ , in the network. We denote the weight of shortest paths induced at vertex  $i \in V$  by  $w_v(i)$ . Furthermore we denote the total weight of all shortest paths for the network  $N = (V, E)$  by  $W_V(N)$  as follows.

$$W_V(N) = \sum_{i \in V} w_v(i) = \sum_{e_{ij} \in E} w_e(e_{ij}) \tag{1.1}$$

In the following sections we deal with the weight of shortest paths induced at each vertex for several special types of network.

### 2. Weight of Shortest Paths for the Tree Structured Network

First we consider the weight of shortest paths induced at each vertex in the tree structured network  $N = (V, E)$ . Let the length of each edge in the network  $N = (V, E)$  be  $d_{ij} = 1$  for each edge  $(i, j) \in E$ . The tree structured network has the property that there always exists a unique path between any two vertices of the network. Hence the weight of shortest paths induced at vertex  $i \in V$  in the tree structured network  $N = (V, E)$  is the sum of the number of edges contained in the path between vertex  $i \in V$  and all other vertices  $j \in V, j \neq i$ , in the network. Considering the set of all shortest paths leaving and entering a designated vertex for a special type of tree structured network, which we call a single path network  $L_n$  and a star network  $W_n$ , consisting of  $n$  vertices as illustrated in Figure 1 and Figure 2, respectively, we easily obtain the following theorem.

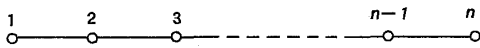


Figure 1. Single path  $L_n$

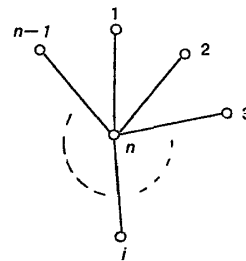


Figure 2. Star graph  $R_n$

**Theorem 2.1** *In the single path network  $L_n$  as illustrated in Figure 1 the weight of shortest paths induced at each vertex  $i \in V$  and the total weight of shortest paths for the network are given as follows.*

$$w_v(i) = \left(i - \frac{n + 1}{2}\right)^2 + \frac{n^2 - 1}{4} \tag{2.1}$$

$$W_V(L_n) = \frac{1}{3}n(n + 1)(n - 1) \tag{2.2}$$

*In the star network  $W_n$  as illustrated in Figure 2 the weight of shortest paths induced at each vertex  $i \in V$  is given as follows.*

$$w_v(i) = \begin{cases} 2n - 3 & 1 \leq i \leq n - 1 \\ n - 1 & i = n \end{cases} \quad (2.3)$$

$$W_V(W_n) = 2(n - 1)^2 \quad (2.4)$$

Results equivalent to (2.2) and (2.4) in the above theorem are shown in Oyama-Taguchi [4]. We omit the proof since it can be easily obtained. The above result shows that in the single path network and star network, weights of shortest paths induced are minimized when the vertex  $i \in V$  is a “central” point of the network.

**Theorem 2.2** *Total weight of all shortest paths for the tree structured network  $N = (V, E)$  with  $|V| = n$  is maximized when the network is a single path and minimized when it is a star network.*

**Proof** Given a tree structured network  $N = (V, E)$  with  $|V| = n$ , we choose the longest path  $P_1$  contained in the network. Without loss of generality, let the set of vertices in the longest path be  $\{1, 2, \dots, p\}$  and subtrees connected with the longest path  $P_1$  be denoted by  $T_1, T_2, \dots, T_q$  where the subtree  $T_i$  is connected with the vertex  $v_i$  in the path for  $1 \leq i \leq q$ . We assume  $v_1 \leq v_2 \leq \dots \leq v_q$  without losing generality. By assumption we know that the distance from vertex  $v_i$  to the farthest vertex in the subtree  $T_i$  is no larger than the distance from  $v_i$  to vertices 1 or  $p$ . Thus, by detaching the subtree  $T_q$  from the vertex  $v_q$  and reconnecting it with the end vertex  $p$  we can obtain the situation that all shortest paths from vertices in the subtree  $T_i$  are no smaller than the original ones. Then the newly built network denoted by  $N' = (V', E')$  having a larger longest path has larger total weight of shortest paths than the original network  $N = (V, E)$ . By repeating this process we finally obtain the single path network  $L_n$ , which has the largest total weight.

Regarding the minimization process, in order to obtain a new network having a smaller total weight of shortest paths, we make the length of the longest path smaller. We connect a part of the longest path from the vertex  $v_q + 1$  to the vertex  $p$  with another neighboring vertex  $v_q - 1$ . In the event that the distance from  $v_q$  to the farthest vertex in the subtree  $T_q$  is equal to the distance from  $v_q$  to  $p$  in the longest path, *i.e.*,  $p - v_q$ , we connect both subtree  $T_q$  and the single path from the vertex  $v_q + 1$  to  $p$ , having equal length, with the vertex  $v_q - 1$ . Thus the newly built network  $N'' = (V'', E'')$  having smaller longest path has smaller total weight of shortest paths than the original network  $N = (V, E)$ . By repeating this process we finally obtain the star network  $R_n$  which has the smallest total weight.  $\square$

Let us consider the implication of the above theorem. Let the vertex and the edge of the network represent city and road connecting two cities, respectively. Then the above theorem implies that the structure corresponding to a single path  $L_n$ , connecting  $n$  cities sequentially, maximizes the total weight of shortest paths while a star graph  $R_n$ , connecting  $n$  cities around one city, minimizes it when we connect these  $n$  cities with the minimum number of roads, *i.e.*,  $n - 1$ . From this we can conclude that connecting  $n$  cities with the star graph structure may be the most “efficient” from the viewpoint of traffic road congestion.

Given a general network  $N = (V, E)$  1-median of the network is defined to be the point such that the sum of lengths of all shortest paths between the point and all other vertices in the network is minimized. We try to express 1-median of the network using the weight of all shortest paths induced at. In the general tree structured network, the vertex at which the weight of shortest paths induced is to be minimized is obtained as in the following theorem.

**Theorem 2.3** *1-median of the tree structured network with arbitrary positive edge lengths*

minimizes the weight of all shortest paths induced at.

**Proof** Let the tree structured network  $N = (V, E)$  have the positive edge length  $d_{ij} > 0$  for each  $(i, j) \in E$ . Let the 1-median and the vertex minimizing the weight of shortest paths induced in the network  $N = (V, E)$  be denoted by  $s_1$  and  $s_t$ , respectively, *i.e.*, the unique path between vertices  $s_1$  and  $s_t$  consist of  $t - 1$  edges and the vertices between them are represented by  $s_1, s_2, \dots, s_t$ . Let the set of vertices contained in one of the subtrees obtained by deleting the edge  $(s_j, s_{j+1})$  and including the node  $s_j$  be denoted by  $N_j$ . Let the complement of  $N_j$  with respect to the vertex set  $V$  be  $\bar{N}_j$ . Then, obviously, we have the following relation

$$N_1 \subset N_2 \subset \dots \subset N_{t-1} \tag{2.5}$$

which implies that letting  $|N_j| = n_j$  for  $1 \leq j \leq t - 1$

$$n_1 \leq n_2 \leq \dots \leq n_{t-1}. \tag{2.6}$$

Now we calculate the total length of all shortest paths from the vertex  $s_j$  by  $l_v(s_j)$ . Denoting the length of the edge  $(s_j, s_{j+1})$  simply by  $d'_{jj+1}$  instead of  $d_{s_j s_{j+1}}$ , we obtain the following relation

$$l_v(s_j) - l_v(s_{j+1}) = d'_{jj+1}(|\bar{N}_j| - |N_j|) = d'_{jj+1}(n - 2n_j) \tag{2.7}$$

From the above relation we can express  $l_v(s_k)$  in terms of  $l_v(s_1)$  as follows.

$$\begin{aligned} l_v(s_k) &= l_v(s_1) - \sum_{j=1}^{k-1} (n - 2n_j) d'_{jj+1} = l_v(s_1) + \sum_{j=1}^{k-1} (2n_j - n) d'_{jj+1} \\ &\geq l_v(s_1) + (2n_1 - n) \sum_{j=1}^{k-1} d'_{jj+1} \end{aligned} \tag{2.8}$$

Since vertex  $s_1$  is the 1-median of the network  $N = (V, E)$ ,  $l_v(s_1)$  minimizes the total length of all shortest paths from the vertex. Thus  $l_v(s_j)$ 's are nondecreasing with respect to the suffix  $j$ . Similarly, defining the weight of all shortest paths induced at each vertex  $s_j$  by  $w_v(s_j)$ , we obtain the following relation.

$$w_v(s_k) = w_v(s_1) + \sum_{j=1}^{k-1} (2n_j - n) \geq w_v(s_1) + (k - 1)(2n_1 - n) \tag{2.9}$$

Hence we can conclude that  $w_v(s_j)$ 's are nondecreasing with respect to the suffix  $j$  while vertex  $s_t$  minimizes the weight  $w_v(s_t)$  of all shortest paths that are induced. Therefore vertices  $s_1$  and  $s_t$  have to coincide or be adjacent. Namely, 1-median and the vertex minimizing the total weight of all shortest paths have to coincide, *i.e.*, either  $t = 1$  or  $2n_1 = n$  and  $t = 2$ . In the former case there exists a unique 1-median vertex while in the latter case either vertex on the edge  $(s_1, s_2)$  can be a 1-median.  $\square$

Given a tree structured network  $N = (V, E)$  with  $|V| = n$ , to obtain a vertex minimizing the weight of shortest paths induced at is equivalent to finding a 1-median of the tree structured network, which requires  $O(n)$  complexity algorithm developed by Goldman [1] as the majority algorithm. As shown in Hakimi [2], 1-median of the tree structured network is on some vertex. This 1-median vertex can also be called a Weber point in the area of location analysis. In the general network  $N = (V, E)$   $p$ -median vertices are defined to be the set of  $p$  vertices such that the sum of lengths of all shortest paths between each vertex

and its nearest vertex in the set of  $p$  vertices is minimized. We consider the relation between  $p$ -median of the network and the weight of all shortest paths induced at.

Given a single path network  $L_n$  consisting of  $n$  vertices, we construct a new network  $H_n$  obtained by adding a new vertex  $v_0$  to the network  $L_n$  and connecting it with the set of  $p$  vertices  $i_1, i_2, \dots, i_p$  in the single path  $L_n$  defining the length of edge  $(v_0, i_j)$  to be a large number  $D$  for each  $j$ ,  $1 \leq j \leq p$ . Then  $p$ -median of the network  $L_n$  can be found by obtaining a set of  $p$  connecting vertices in the network  $H_n$  minimizing the total length of all shortest paths from  $v_0$ . Let the set of  $p$  connecting vertices be  $M_p = \{i_1, i_2, \dots, i_p\}$  and all edge lengths in  $L_n$  be constant. Then the total weight of all shortest paths between any two vertices in the single path network  $L_n$  is constant for a given  $n$ , which means that minimizing the total length of all shortest paths between the vertex  $v_0 \in H_n$  and all other vertices is equivalent to minimizing the total weight of all shortest paths induced at the vertex  $v_0$  in the whole network  $H_n$ . Thus  $p$ -median of the single path network  $L_n$  can be obtained as the set  $M_p$  minimizing the weight of shortest paths induced at the vertex  $v_0 \in H_n$  in the network.

Given two mutually disjoint tree structured networks  $N_1 = (V_1, E_1)$  and  $N_2 = (V_2, E_2)$  with  $|V_1| = n_1$  and  $|V_2| = n_2$ , respectively, we try to connect these two networks by a single edge, *i.e.*, a “bridge” so that the newly built tree network has the minimum total weight of shortest paths. When the connection attains the minimum total weight of all shortest paths, we call it an optimal connection of two mutually disjoint tree structured networks. We obtain the following theorem with respect to the optimal connection of two mutually disjoint tree structured networks.

**Theorem 2.4** *Given two tree structured networks  $N_1 = (V_1, E_1)$  and  $N_2 = (V_2, E_2)$  with  $|V_1| = n_1$ ,  $|V_2| = n_2$ , and  $V_1 \cap V_2 = \emptyset$ , respectively, optimal connection is obtained when we connect 1-median vertices of  $N_1$  and  $N_2$ , respectively.*

**Proof** The total weight of all shortest paths in the newly built network can be affected by these shortest paths connecting the vertices in these networks  $N_1 = (V_1, E_1)$  and  $N_2 = (V_2, E_2)$  and passing the bridge. Since the weight of shortest paths for the bridge is given by the constant term  $n_1 n_2$ , we need to minimize the weight of shortest paths connecting the vertex in the networks  $N_1 = (V_1, E_1)$  and  $N_2 = (V_2, E_2)$ , respectively, to and from the end vertex of the bridge. Hence, applying the result of Theorem 2.3 we obtain the theorem.  $\square$

The above theorem implies that we need to connect two “central” vertices when two tree structured networks  $N_1 = (V_1, E_1)$  and  $N_2 = (V_2, E_2)$  are single paths with  $|V_1| = n_1$  and  $|V_2| = n_2$ , respectively, in order to minimize the total weight of all shortest paths in the newly built tree network. When integers  $n_1$  and  $n_2$  are odd, we obtain the minimum total weight of shortest paths in the newly built tree network as follows.

$$\frac{1}{12}(4n_1 + 3n_2)(n_1^2 - 1) + \frac{1}{12}(3n_1 + 4n_2)(n_2^2 - 1) + 2n_1 n_2$$

Now we consider the problem of connecting two tree structured networks by two “bridges” so that we can minimize the total weight of all shortest paths in the newly built network. However this problem would be very difficult generally since the newly built network would not be tree structured any more if we connect them by two bridges. Incidentally, the new network has a single cycle. We give a result for a special case when two tree structured networks are identical single paths.

**Theorem 2.5** *Let two identical single path networks  $A_1$  and  $A_2$  have  $m$  edges with equal*

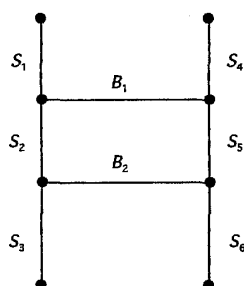


Figure 3. Connection of two identical single paths

edge length and be denoted by  $L_{m+1}$ . Let the first bridge  $B_1$  connect the grid point  $t_1 + 1$  in  $A_1$  with the symmetric point  $t_1 + 1$  in  $A_2$ , and the second bridge  $B_2$  connect the grid point  $t_1 + t_2 + 1$  in  $A_1$  with its symmetric point  $t_1 + t_2 + 1$  in  $A_2$ , where  $0 \leq t_1 \leq m$ ,  $0 \leq t_2 \leq m$  and  $0 \leq t_1 + t_2 \leq m$ , as illustrated in Figure 3. Suppose  $t_1^*$  and  $t_2^*$ , ignoring their integrality, minimize the total weight of all shortest paths for all edges in the network. Then  $t_1^*$  and  $t_2^*$  are given as follows.

$$t_1^* = \frac{m+1}{2} \left\{ 2 - \frac{1}{m+1} - \sqrt{2 + \frac{1}{3(m+1)^2}} \right\} \tag{2.10}$$

$$t_2^* = (m+1) \left\{ \sqrt{2 + \frac{1}{3(m+1)^2}} - 1 \right\} \tag{2.11}$$

**Proof** In Figure 3 we divide the set of vertices  $k \in L_{m+1}$  in the single path networks  $A_1$  and  $A_2$  into six subsets as follows.

$$\begin{aligned} S_1 &= \{k \in A_1 \mid 1 \leq k \leq t_1 + 1\} & S_2 &= \{k \in A_1 \mid t_1 + 2 \leq k \leq t_1 + t_2\} \\ S_3 &= \{k \in A_1 \mid t_1 + t_2 + 1 \leq k \leq m + 1\} & S_4 &= \{k \in A_2 \mid 1 \leq k \leq t_1 + 1\} \\ S_5 &= \{k \in A_2 \mid t_2 + 2 \leq k \leq t_1 + t_2\} & S_6 &= \{k \in A_2 \mid t_1 + t_2 + 1 \leq k \leq m + 1\} \end{aligned}$$

Regarding a shortest path containing the  $k$ -th edge segment  $E_k$  in the network  $L_{m+1}$ , we denote the starting and the arriving (terminal) vertex by  $p$  and  $q$ , respectively. Then the set of shortest paths containing the edge segment  $E_k$  can be classified into the following nine groups with respect to the starting and the terminal points and these shortest paths are counted as follows.

- (1)  $\{p \in S_1, q \in S_4\}, \{p \in S_4, q \in S_1\}$

$$w_e(E_k) = 2k(t_1 + 1) \quad 1 \leq k \leq t_1 \tag{2.12}$$

Adding (2.12) with respect to  $k$ , we obtain the following total.

$$T_1 = \sum_{k=1}^{t_1} w_e(E_k) = t_1(t_1 + 1)^2 \tag{2.13}$$

- (2)  $\{p \in S_1, q \in S_5\}, \{p \in S_5, q \in S_1\}$

$$w_e(E_k) = 2k(t_2 - 1) \quad 1 \leq k \leq t_1 \tag{2.14}$$

Adding (2.14) with respect to  $k$ , we obtain the following total:

$$T_2 = \sum_{k=1}^{t_1} w_e(E_k) = t_1(t_1 + 1)(t_2 - 1) \tag{2.15}$$

(3)  $\{p \in S_1, q \in S_6\}, \{p \in S_6, q \in S_1\}$

$$w_e(E_k) = \begin{cases} 2k(m+1-t_1-t_2) & 1 \leq k \leq t_1 \\ (t_1+1)(m+1-t_1-t_2) & t_1+1 \leq k \leq t_1+t_2 \end{cases} \quad (2.16)$$

Adding (2.16) with respect to  $k$ , we obtain the following total:

$$T_3 = \sum_{k=1}^{t_1+t_2} w_e(E_k) = (t_1+1)(t_1+t_2)(m+1-t_1-t_2) \quad (2.17)$$

(4)  $\{p \in S_3, q \in S_4\}, \{p \in S_4, q \in S_3\}$

$$w_e(E_k) = \begin{cases} (m+1-t_1-t_2)(t_1+1) & t_1+1 \leq k \leq t_1+t_2 \\ 2(m+1-k)(t_1+1) & t_1+t_2+1 \leq k \leq m \end{cases} \quad (2.18)$$

Adding the above with respect to  $k$  in the corresponding range, we obtain the following total:

$$T_4 = \sum_{k=t_1+1}^{t_1+t_2+1} w_e(E_k) = (t_1+1)(m+1-t_1-t_2)(m-t_1) \quad (2.19)$$

(5)  $\{p \in S_3, q \in S_5\}, \{p \in S_5, q \in S_3\}$

$$w_e(E_k) = 2(m+1-k)(t_2-1) \quad t_1+t_2+1 \leq k \leq m \quad (2.20)$$

Adding (2.20) with respect to  $k$ , we obtain the following total:

$$T_5 = \sum_{k=t_1+t_2+1}^m w_e(E_k) = (m+1-t_1-t_2)(m-t_1-t_2)(t_2-1) \quad (2.21)$$

(6)  $\{p \in S_3, q \in S_6\}, \{p \in S_6, q \in S_3\}$

$$w_e(E_k) = 2(m+1-k)(m+1-t_1-t_2) \quad t_1+t_2+1 \leq k \leq m \quad (2.22)$$

Adding (2.22) with respect to  $k$  in the corresponding range, we obtain the following total:

$$T_6 = \sum_{k=t_1+t_2+1}^m w_e(E_k) = (m+1-t_1-t_2)^2(m-t_1-t_2) \quad (2.23)$$

(7)  $\{p \in S_2, q \in S_4\}, \{p \in S_4, q \in S_2\}$

Similarly to the case (2) we obtain the following total.

$$T_7 = (t_1+1)t_2(t_2-1) \quad (2.24)$$

(8)  $\{p \in S_2, q \in S_6\}, \{p \in S_6, q \in S_2\}$

Similarly to the case (5) we obtain the following total.

$$T_8 = t_2(t_2-1)(m+1-t_1-t_2) \quad (2.25)$$

(9)  $\{p \in S_2, q \in S_5\}, \{p \in S_5, q \in S_2\}$

First we consider the case that the shortest path between two vertices can be uniquely determined. For the first case of passing through the bridge at  $k = t_1 + t_2 + 1$  we have

$$w_e(E_k) = (k-t_1-2)(k-t_1-1) \quad t_1+3 \leq k \leq t_1+t_2 \quad (2.26)$$

The next case of passing through the bridge at  $k = t_1 + 1$  gives the weight as follows.

$$w_e(E_k) = (t_1 + t_2 - k - 1)(t_1 + t_2 - k) \quad t_1 + 1 \leq t_1 + t_2 - 2 \quad (2.27)$$

Then we consider the case that there exists more than one shortest path between two vertices. In this case the weight can be counted from symmetry by

$$w_e(E_k) = t_2 \quad t_1 + 2 \leq k \leq t_1 + t_2 \quad (2.28)$$

Thus adding (2.26)-(2.28) with respect to  $k$  in the corresponding range, we obtain the following total:

$$T_9 = \frac{1}{3}t_2(t_2 - 1)(2t_2 - 1) \quad (2.29)$$

Therefore the total weight of all shortest paths for all edges denoted by  $Q$  can be obtained as follows:

$$\begin{aligned} Q = \sum_{i=1}^9 T_i &= t_1(t_1 + 1)(t_1 + t_2) + (t_1 + 1)(m + t_2)(m + 1 - t_1 - t_2) \\ &+ (m + 1 - t_1 - t_2)(m - t_1 - t_2)(m - t_1) + t_2(t_2 - 1)(m + 2 - t_2) \\ &+ \frac{1}{3}t_2(t_2 - 1)(2t_2 - 1) \end{aligned} \quad (2.30)$$

Suppose that  $t_1^*$  and  $t_2^*$  minimize the total weight denoted by  $Q$  in (2.30), then  $t_1^*$  and  $t_2^*$  should satisfy the following simultaneous equations with respect to variables  $t_1$  and  $t_2$  neglecting their integrality.

$$\frac{\partial Q}{\partial t_1} = 2(2t_1 + t_2 - m)(m + 1 - t_2) = 0 \quad (2.31)$$

$$\frac{\partial Q}{\partial t_2} = \frac{1}{2}\{t_2^2 + 2(m + 1)t_2 - m^2 - 2m - \frac{4}{3}\} = 0 \quad (2.32)$$

From the above equations we obtain the following.

$$(2t_1 + t_2 - m)(m + 1 - t_2) = 0 \quad (2.33)$$

$$t_2^2 + 2(m + 1)t_2 - m^2 - 2m - \frac{4}{3} = 0 \quad (2.34)$$

Solving the above equations simultaneously, we obtain the solution  $t_1^*$  and  $t_2^*$  given in (2.10) and (2.11), respectively.  $\square$

Obviously we know that in the single path network  $L_n$  with equal edge length 2-median vertices denoted by  $i_1^*$  and  $i_2^*$  can be obtained as follows.

$$i_1^* = \left\lfloor \frac{n}{4} \right\rfloor \quad i_2^* = \left\lfloor \frac{3n}{4} \right\rfloor$$

where  $\lfloor x \rfloor$  indicates the largest integer which is smaller than or equal to  $x$ . Comparing the above result with our result in Theorem 2.5, two optimal connecting vertices for minimizing the total weight of all shortest paths exist “inside” the 2-median vertices since  $t_2^*$  gives almost 40% (corresponding to  $\sqrt{2} - 1 = 0.41$ ) of the total length of the single path, while 2-median gives the corresponding length as almost 50%.



Now we consider the problem of connecting two tree structured networks by  $p$  “bridges” so that we can minimize the total weight of all shortest paths in the newly built network. We give a result for a special case in the following theorem.

**Theorem 2.6** *Let two tree structured networks  $A_1$  and  $A_2$  have  $n_1$  and  $n_2$  nodes, respectively, and all edges in  $A_1$  and  $A_2$  have equal length, e.g., 1. We connect these two networks by  $p$  bridges in such a way that all end points in the network  $A_1$  are in common, i.e., meet at one vertex. We also assume that the length of each bridge is much larger than the length of the longest paths in two tree structured networks  $A_1$  and  $A_2$ . Then the total weight of all shortest paths in the newly built network is minimized when we connect 1-median of the network  $A_1$  with  $p$ -median of the network  $A_2$ .*

**Proof** Shortest paths between two nodes in each network of  $A_1$  and  $A_2$  are unchanged since adding bridges doesn't affect them. We need to consider the shortest paths between two vertices in  $A_1$  and  $A_2$ . The total weight of shortest paths between two vertices in  $A_1$  and  $A_2$  can be minimized when the sum of lengths of all shortest paths is minimized. Thus the minimum of total weight of shortest paths is attained when the vertex in the network  $A_1$  is 1-median and these  $p$  connecting points in the network  $A_2$  coincide with the set of  $p$ -median of the network  $A_2$ .  $\square$

### 3. Weight of Shortest Paths for the Grid Type Network

The grid type network  $G(m, n)$  consists of  $m + 1$  horizontal lines and  $n + 1$  vertical lines with equal intervals whose intersections denoted by grid points form  $(m + 1)(n + 1)$  vertices and  $m$  vertical edges and  $n$  horizontal edges at each column and row directions as illustrated in Figure 4. In Oyama-Taguchi[4, 5, 6] we obtained explicit expressions for the weight of shortest paths for each edge in the special types of networks including the grid type network. Given a grid type network  $G(m, n)$  as shown in Figure 4 we know from Theorem 3.1 (refer to Oyama-Taguchi [4]) that the weight of shortest paths for edges  $V_{kl}$  and  $H_{kl}$  can be given as follows.

$$w_e(V_{kl}) = 2k(m + 1 - k)(n + 1) \quad 1 \leq k \leq m, \quad 1 \leq l \leq n + 1 \quad (3.1)$$

$$w_e(H_{kl}) = 2l(n + 1 - k)(m + 1) \quad 1 \leq k \leq m + 1, \quad 1 \leq l \leq n \quad (3.2)$$

Now regarding the total weight of all shortest paths for all edges in the grid type network, we obtain the following theorem.

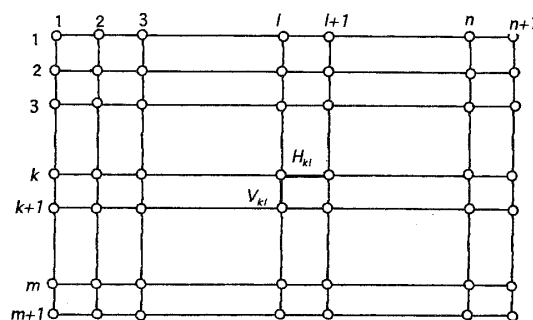


Figure 4. Grid type network  $G(m, n)$

**Theorem 3.1** *In the grid type network  $G(m, n)$  the sum of weights of vertical and hori-*

zontal edges are given as follows.

$$\sum_{k=1}^m \sum_{l=1}^{n+1} w_e(V_{kl}) = \frac{1}{3}m(m+1)(m+2)(n+1)^2 \tag{3.3}$$

$$\sum_{k=1}^{m+1} \sum_{l=1}^n w_e(H_{kl}) = \frac{1}{3}(m+1)^2n(n+2)(n+2) \tag{3.4}$$

Therefore the total weight of all shortest paths for all edges, denoted by  $T_w$ , can be written as follows.

$$\begin{aligned} T_w &= \sum_{k=1}^m \sum_{l=1}^{n+1} w_e(V_{kl}) + \sum_{k=1}^{m+1} \sum_{l=1}^n w_e(H_{kl}) \\ &= \frac{1}{3}(m+1)(n+1)\{m(m+2)(n+1) + (m+1)n(n+2)\} \end{aligned} \tag{3.5}$$

**Proof** First the total weight of shortest paths for vertical edges in the grid type network can be obtained as follows.

$$\begin{aligned} \sum_{k=1}^m \sum_{l=1}^{n+1} w_e(V_{kl}) &= (n+1)^2 \sum_{k=1}^m 2k(m-k-1) \\ &= -2(n+1)^2 \sum_{k=1}^m k^2 + 2(m+1)(n+1)^2 \sum_{k=1}^m k \\ &= \frac{1}{3}m(m+1)(m+2)(n+1)^2 \end{aligned}$$

Similarly we can obtain the total weight of all shortest paths for horizontal edges as in (3.4). Thus (3.5) can be easily obtained.  $\square$

From the above results we know that the total weight of all shortest paths for the edges  $V_{kl}$ 's has computational order  $\mathcal{O}(m^3n^2)$ , and it is considered as the product of the number of edges  $mn$  and the order  $\mathcal{O}(m^2n)$  with a maximum weight of  $w_e(V_{kl})$ . Similarly, the total weight of edges  $H_{kl}$ 's has  $\mathcal{O}(m^2n^3)$  and it is considered to be the product of the number of edges  $mn$  and the order  $\mathcal{O}(mn^2)$  with a maximum weight of  $w_e(H_{kl})$ .

Given a grid type network  $G(m, n)$  we consider the weight of all shortest paths induced at each vertex  $(k, l) \in V$  located at the intersection of the  $k$ -th horizontal line and the  $l$ -th vertical line in the network. We denote the weight of shortest paths induced at vertex  $v_{kl} = (k, l) \in V$  by  $w_v(v_{kl})$ . For each vertex of the grid type network  $G(m, n)$  the weight of shortest paths induced can be obtained as in the following theorem.

**Theorem 3.2** *Given a grid type network  $G(m, n)$  the weight of all shortest paths induced at vertex  $v_{kl} = (k, l) \in V$  can be expressed as follows.*

$$\begin{aligned} w_v(v_{kl}) &= (n+1)\left(k - \frac{m+2}{2}\right)^2 + (m+1)\left(l - \frac{n+2}{2}\right)^2 \\ &\quad + \frac{1}{4}m(m+2)(n+1) + \frac{1}{4}(m+1)n(n+2) \end{aligned} \tag{3.6}$$

**Proof** Let the vertical and horizontal edges connecting with the vertex  $v_{kl} = (k, l) \in V$  be  $V_{kl}$  and  $H_{kl}$ , respectively, as illustrated in Figure 4. Then, as we assumed in Oyama-Taguchi

[4], we apply the min-turn rule when choosing the shortest path: when there exist multiple shortest paths with the same total length, we choose the one with the minimum number of turns. Incidentally, this rule is unnecessary when we consider the total weight of shortest paths since they contribute equally to the total weight even though two shortest paths have a different number of turns.

Now we divide the set of vertices in the network  $G(m, n)$  into four subsets  $S_1, S_2, S_3$  and  $S_4$  as follows.

$$\begin{aligned} S_1 &= \{(i, j) \mid 1 \leq i \leq k-1, 1 \leq j \leq l\} \\ S_2 &= \{(i, j) \mid 1 \leq i \leq k, l+1 \leq j \leq n+1\} \\ S_3 &= \{(i, j) \mid k+1 \leq i \leq m+1, l+1 \leq j \leq n+1\} \\ S_4 &= \{(i, j) \mid k \leq i \leq m+1, 1 \leq j \leq l-1\} \end{aligned}$$

Then we can categorize the set of shortest paths between the designated vertex  $v_{kl}$  and its destination vertex  $v_{ij} = (i, j)$  into four groups as (i)  $(i, j) \in S_1$ , (ii)  $(i, j) \in S_2$ , (iii)  $(i, j) \in S_3$ , (iv)  $(i, j) \in S_4$ . For each group of these shortest paths letting the weights of the vertical and horizontal edges be denoted by  $w(V_{ij})$  and  $w(H_{ij})$ , respectively, these weights can be counted as follows.

(i)  $(i, j) \in S_1$

$$w(V_{il}) = il \quad 1 \leq i \leq k-1 \quad (3.7)$$

$$w(H_{ij}) = j \quad 1 \leq i \leq k-1, 1 \leq j \leq l-1 \quad (3.8)$$

(ii)  $(i, j) \in S_2$

$$w(V_{ij}) = i \quad 1 \leq i \leq k-1, l+1 \leq j \leq n+1 \quad (3.9)$$

$$w(H_{kj}) = k(n+1-j) \quad l \leq j \leq n \quad (3.10)$$

(iii)  $(i, j) \in S_3$

$$w(V_{il}) = (m+1-i)(n+2-l) \quad k \leq i \leq m \quad (3.11)$$

$$w(H_{ij}) = (n+1-j) \quad k+1 \leq i \leq m+1, l \leq j \leq n \quad (3.12)$$

(iv)  $(i, j) \in S_4$

$$w(V_{ij}) = (m+1-i) \quad k \leq i \leq m, 1 \leq j \leq l-1 \quad (3.13)$$

$$w(H_{kj}) = (m+2-k)j \quad 1 \leq j \leq l-1 \quad (3.14)$$

Hence denoting the total weight of these shortest paths for each group by  $w(S_i)$ ,  $i = 1, 2, 3, 4$ , respectively, we obtain these total weights by adding (3.7)-(3.14) as follows.

$$w(S_1) = \frac{1}{2}(k-1)kl + \frac{1}{2}(k-1)(l-1)l \quad (3.15)$$

$$w(S_2) = \frac{1}{2}k(k-1)(n+1-l) + \frac{1}{2}k(n+1-l)(n+2-l) \quad (3.16)$$

$$w(S_3) = \frac{1}{2}(m+1-k)(m+2-k)(n+2-l) + \frac{1}{2}(m+1-k)(n+1-l)(n+2-l) \quad (3.17)$$

$$w(S_4) = \frac{1}{2}(m+1-k)(m+2-k)(l-1) + \frac{1}{2}(m+2-k)l(l-1) \quad (3.18)$$

Adding (3.15)-(3.18), we obtain the weight of all shortest paths induced at vertex  $v_{kl} = (k, l) \in V$  denoted by  $w_v(v_{kl})$  as given in (3.6).  $\square$

Theorem 3.1 shows that the minimum weight of all shortest paths with respect to the edges  $V_{kl}$  and  $H_{kl}$  can be attained in the “central” part in both vertical and horizontal arrangements. We know the minimum weights of edges  $V_{kl}$  and  $H_{kl}$  have computational order  $\mathcal{O}(m^2n)$  and  $\mathcal{O}(mn^2)$ , respectively.

#### 4. Optimal Connection of Two Grid Type Networks

Suppose we connect two grid type networks  $A_1$  and  $A_2$  with exactly the same structure and the same size by adding a “bridge”  $B$  between two grid points in a mirror-like symmetric position as illustrated in Figure 5. We consider the problem asking where we should have a bridge in order to minimize the total weight of all shortest paths for all edges in the combined network in Figure 5. The following theorem solves this problem.

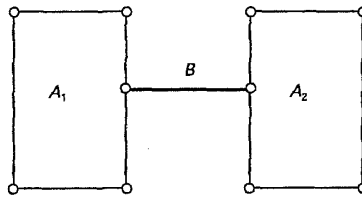


Figure 5. Connection of two identical grid type networks

**Theorem 4.1** *Let two grid type networks  $A_1$  and  $A_2$  be identically represented by  $G(m, n)$ , and these two networks  $A_1$  and  $A_2$  are connected by a bridge  $B$  as illustrated in Figure 5. Then the total weight of all shortest paths for all vertical and horizontal edges is minimized when the bridge connects two “central” points in both sides of these two identical grid type networks.*

**Proof** Suppose the bridge  $B$  connects the grid point  $(t + 1, n + 1)$  in  $A_1$  with  $(t + 1, 1)$  in  $A_2$ . We consider the total weight of all shortest paths between all pairs of vertices in the combined network. However, when two vertices are in a network, *e.g.*,  $A_1$  or  $A_2$ , we have already counted in Theorem 4.1, *i.e.*, they are constant for fixed  $m$  and  $n$ . Hence in order to minimize the total weight of all shortest paths for all edges we have only to consider the weight of shortest paths with respect to all pairs connecting two vertices, which are in  $A_1$  and  $A_2$ , respectively.

When choosing the shortest path, as we assumed in Oyama-Taguchi [4], we apply the min-turn rule again: when there exist multiple shortest paths with the same total length, we choose the one with the minimum number of turns. Incidentally, this rule is unnecessary when we consider the total weight for all edges since they contribute equally to the total weight. Using the result in Theorem 3.2 and letting  $k = t + 1$  and  $l = n + 1$  in the expression (3.6), we can obtain the weight of shortest paths induced at the vertex  $v_{kn+1}$  as follows.

$$w_v(v_{kn+1}) = (n + 1)\left(t - \frac{m}{2}\right)^2 a + \frac{1}{4}m(m + 2)(n + 1) + \frac{1}{2}(m + 1)n(n + 1) \quad (4.1)$$

Hence, from the above expression, we can conclude that the total weight of shortest paths is minimized when  $t = \lfloor \frac{m}{2} \rfloor$ , *i.e.*, when the bridge connects the “central” points in both sides of these two identical grid type networks  $A_1$  and  $A_2$ .  $\square$

Let us consider the case that we are allowed to have “two bridges”  $B_1$  and  $B_2$  connecting two grid type networks  $A_1$  and  $A_2$  as illustrated in Figure 6. Similar to the “one bridge”

case mentioned above, let two grid type networks  $A_1$  and  $A_2$  be given as  $G(m, n)$  as before. Again we consider the problem asking where we should have two bridges  $B_1$  and  $B_2$  in order that we minimize the total weight of all edges with respect to the shortest paths connecting any two grid points in the given network? The following theorem solves this problem for the case that each interval has equal length as in the previous theorem.

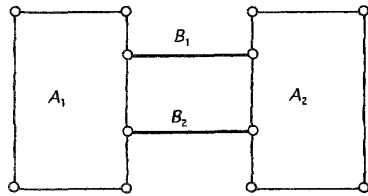


Figure 6. Connection of two identical grid type networks

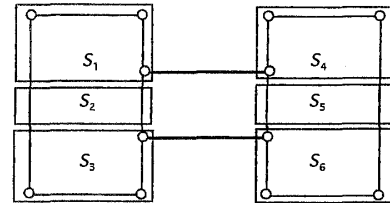


Figure 7. Connecting structure of two identical grid type networks

**Theorem 4.2** *Let two identical grid type networks  $A_1$  and  $A_2$  be given by  $G(m, n)$ . Let the first bridge  $B_1$  connect the point  $(t_1 + 1, n + 1)$  in  $A_1$  with the symmetric point  $(t_1 + 1, 1)$  in  $A_2$ , and the second bridge  $B_2$  connect the grid point  $(t_1 + t_2 + 1, n + 1)$  in  $A_1$  with its symmetric point  $(t_1 + t_2 + 1, 1)$  in  $A_2$ , where  $0 \leq t_1 \leq m$ ,  $0 \leq t_2 \leq m$  and  $0 \leq t_1 + t_2 \leq m$ , as shown in Figure 7. Suppose  $t_1^*$  and  $t_2^*$ , ignoring their integrality, minimize the total weight of all shortest paths for all edges in the network. Then  $t_1^*$  and  $t_2^*$  are given as follows.*

$$t_1^* = \frac{m + 1}{2} \left\{ 2 - \frac{1}{m + 1} - \sqrt{2 + \frac{1}{3(m + 1)^2}} \right\} \tag{4.2}$$

$$t_2^* = (m + 1) \left\{ \sqrt{2 + \frac{1}{3(m + 1)^2}} - 1 \right\} \tag{4.3}$$

Proof is omitted as it is similar to the case of Theorem 2.5, following the division as illustrated in Figure 7 and counting the weight of all shortest paths between all pairs of vertices existing for each pair of divided regions.

From the expression (4.3) in the above theorem we know that the total weight of all shortest paths can be minimized when  $t_2$  is almost equal to  $0.41(m + 1)$  for large  $m$ , *i.e.*, around 40% of the the total vertical length of the grid type network. Interestingly, note that optimal  $t_1^*$  and  $t_2^*$  are expressed exactly in the same form as (4.2) and (4.3) for the case of connecting two identical single path networks. Naturally, the actual values of  $t_1^*$  and  $t_2^*$  minimizing the total weight of all shortest paths for all edges in the network need to be integers. These optimal integer values are certainly expected to be close enough to the above values of  $t_1^*$  and  $t_2^*$ , respectively. Furthermore, note that  $t_1^*$  and  $t_2^*$  do not depend on the value of  $n$ , which is similar to the case of Theorem 2.5. However, this does not mean that the above result can hold for more general two grid type networks  $A_1$  and  $A_2$ , *e.g.*, with the sizes  $(m + 1) \times (n_1 + 1)$  and  $(m + 1) \times (n_2 + 1)$  if  $n_1 \neq n_2$ .

### 5. 1-median and 1-center for Grid Type and Circular Type Networks

We consider the 1-median and the 1-center for the grid type network first. Let the numbers of edge segments in vertical and horizontal directions be  $m$  and  $n$ , respectively, as illustrated in Figure 4. Also we denote the  $k$ -th horizontal and  $l$ -th vertical grid point by  $(k, l)$  as shown in previous sections. We assume that a client is at each grid point of the network, and has unit amount of demand for certain commodity. Then one facility location problem

requires us to establish one facility to supply each client with unit amount of demand so that the certain objective criteria should be optimized. Median and center problems are two major location problems. The former has the objective criteria minisum while the latter has minimax objective. Namely, 1-median problem tries to find an optimal site of the facility satisfying that the total distance from each client to the facility is minimized while 1-center problem requires that the farthest client from the facility needs to be minimized. Depending on whether the restriction that the potential site for the facility is on the nodes only or the whole network including the edge, we call the result a node median (center) problem or absolute median (center) problem, respectively. In this paper we are focused on the absolute median and center problems. In order to make our expression much simpler we assume, in what follows, that the integer  $n$  is even. We give the 1-median and the 1-center for the grid type network without proof since they can be easily obtained.

**Theorem 5.1** *Given the grid type network with the numbers of edge segments in vertical and horizontal directions  $m$  and  $n$ , respectively, 1-median  $(k_m, l_m)$  and 1-center  $(k_c, l_c)$  can be obtained as follows.*

$$k_m = \begin{cases} k & 1 + \frac{m-1}{2} \leq k \leq 1 + \frac{m+1}{2} \\ 1 + \frac{m}{2} \end{cases} \quad \begin{array}{l} m : \text{ odd} \\ m : \text{ even} \end{array} \quad (5.1)$$

$$k_c = 1 + \frac{m}{2} \quad (5.2)$$

$$l_m = l_c = 1 + \frac{n}{2} \quad (5.3)$$

Let us consider the 1-median and the 1-center for the circular type network. The circular type network  $K(m, n, \theta)$  with the polar angle  $\theta$  has the lengths  $r$  and  $s$  in the radial and circular directions, respectively, and is divided into  $m$  and  $n$  edge segments, respectively, in each direction, as illustrated in Figure 8. We obtained the explicit expression for the *SPCP* for the circular type network  $K(m, n, \theta)$  in Oyama-Taguchi [4, 5, 6]. Then we obtain the following theorems with respect to 1-median of the circular type network.

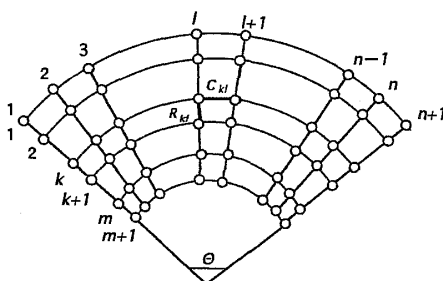


Figure 8. Circular type network  $K(m, n, \theta)$

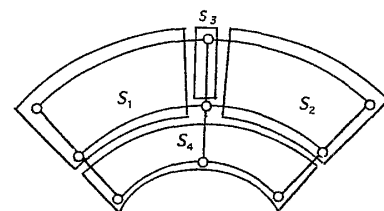


Figure 9. Structure of circular type network

**Theorem 5.2** *Given the circular type network  $K(m, n, \theta)$  with the polar angle  $\theta$  and the lengths of the radial and circular segments given by  $r$  and  $s$ , respectively, and their numbers of edge segments in radial and circular directions  $m$  and  $n$ :even, respectively. We assume that  $\theta \leq 4$ , the radius of the outer circle is 1, and the integer  $n$  is very large. Then the 1-median  $(k_m, l_m)$  for the circular type network can be approximately given as follows.*

$$k_m = \left(1 + \frac{m}{2}\right) \frac{1 - \frac{\theta}{8(m+2)}}{1 - \frac{\theta}{8}} \quad 0 < \theta \leq 4 - \frac{2}{m} \quad (5.4)$$

$$\begin{aligned}
 &= m + 1 & 4 - \frac{2}{m} < \theta < 4 \\
 l_m &= 1 + \frac{n}{2} & & (5.5)
 \end{aligned}$$

**Proof** From the node optimality theorem for the 1-median problem (see, *e.g.* Hakimi [2], Mirchandani-Francis [3]) we know that the 1-median can be obtained at a node of the given network when the number of grid points is odd. In the case that the number of grid points is even, we know that 1-median exists on the edge segment. We compute the weight of shortest paths induced at each grid point  $(k, l)$  of the circular type network  $K(m, n, \theta)$ . Note that the total length of all shortest paths induced at each grid point needs to be calculated as the sum of products of the weight of shortest paths on each edge and its corresponding length.

The circular coordinate of the 1-median  $l_m = 1 + \frac{n}{2}$  is obvious from symmetry, so we need to prove the radial coordinate  $k_m$  in (5.4). We consider the total length of all shortest paths induced at each grid point  $(k, l)$ , hence calculating the total length of all shortest paths originating from the point  $(k, l)$  with the destination of another point  $(i, j)$ . Now we divide the set of destination vertices  $(i, j)$  in the network  $K(m, n, \theta)$  into four subsets  $S_1, S_2, S_3$  and  $S_4$  as follows, as illustrated in Figure 9.

$$\begin{aligned}
 S_1 &= \{(i, j) \mid 1 \leq i \leq k, 1 \leq j \leq l - 1\} \\
 S_2 &= \{(i, j) \mid 1 \leq i \leq k, l + 1 \leq j \leq n + 1\} \\
 S_3 &= \{(i, j) \mid 1 \leq i \leq k - 1\} \\
 S_4 &= \{(i, j) \mid k + 1 \leq i \leq m + 1, 1 \leq j \leq n + 1\}
 \end{aligned}$$

Then we can categorize the set of edges according to the routes of shortest paths between the designated vertex  $(k, l)$  and its destination vertex  $(i, j)$  into three groups as (i)  $(i, j) \in S_1 \cup S_2 \cup S_3$ , (ii)  $(i, j) \in S_1 \cup S_2$ , (iii)  $(i, j) \in S_4$ . For each group of these shortest paths letting the weights of the radial and circular edges be denoted by  $w(R_{ij})$  and  $w(C_{ij})$ , respectively, these weights can be counted as follows.

(i)  $(i, j) \in S_1 \cup S_2 \cup S_3$

$$w(R_{ij}) = i\Delta_r \quad 1 \leq i \leq k - 1, 1 \leq j \leq n + 1 \tag{5.6}$$

where

$$\Delta_r = \frac{r}{m}.$$

(ii)  $(i, j) \in S_1 \cup S_2$

$$w(C_{kj}) = jk\Delta_{S_k} \quad 1 \leq j \leq l - 1 \tag{5.7}$$

$$w(C_{kj}) = (n + 1 - j)k\Delta_{S_k} \quad l \leq j \leq n \tag{5.8}$$

where

$$\Delta_{S_k} = \left\{1 - \frac{(k - 1)r}{m}\right\} \frac{\theta}{n}.$$

(iii)  $(i, j) \in S_4$

$$w(R_{il}) = (m + 1 - i)(n + 1)\Delta_r \quad k \leq i \leq m \tag{5.9}$$

$$w(C_{ij}) = \begin{cases} j\Delta_{S_i} & k + 1 \leq i \leq m + 1, 1 \leq j \leq l - 1 \\ (n + 1 - j)\Delta_{S_i} & k + 1 \leq i \leq m + 1, l \leq j \leq n. \end{cases} \tag{5.10}$$

Adding the above expressions (5.6)-(5.10) with respect to indices  $k$  and  $l$  in the corresponding range, we obtain the total length  $L$  as follows.

$$\begin{aligned}
 L &= \sum_{i=1}^{k-1} i(n+1)\Delta_r + \sum_{j=1}^{l-1} jk\Delta_{S_k} + \sum_{j=l}^n (n+1-j)k\Delta_{S_k} \\
 &+ \sum_{i=k}^m (m+1-i)(n+1)\Delta_r + \sum_{i=k+1}^{m+1} \sum_{j=1}^{l-1} j\Delta_{S_i} + \sum_{i=k+1}^{m+1} \sum_{j=l}^n (n+1-j)\Delta_{S_i} \\
 &= \frac{1}{2}k(k-1)(n+1)\Delta_r + \frac{1}{2}l(l-1)k\Delta_{S_k} + \frac{1}{2}(n+1-l)(n+2-l)k\Delta_{S_k} \\
 &+ \frac{1}{2}(m+1-k)(m+2-k)(n+1)\Delta_r + \frac{1}{2}l(l-1) \sum_{i=k+1}^{m+1} \Delta_{S_i} \\
 &+ \frac{1}{2}(n+1-l)(n+2-l) \sum_{i=k+1}^{m+1} \Delta_{S_i}
 \end{aligned}
 \tag{5.11}$$

1-median of the circular type network exists on the central axis  $l = 1 + \frac{n}{2} = l_m$  from the symmetry, and also using the following relation

$$k\Delta_{S_k} + \sum_{i=k+1}^{m+1} \Delta_{S_i} = \left\{ -\frac{r}{2m}k^2 + \frac{r}{2m}k + \frac{1}{2}(m+1)(2-r) \right\} \frac{\theta}{n}
 \tag{5.12}$$

we can write the total weight expression (5.11) as follows.

$$\begin{aligned}
 L &= \left\{ n+1 - \frac{1}{8}(n+2)\theta \right\} \Delta_r k^2 - \left\{ (m+2)(n+1) - \frac{1}{8}(n+2)\theta \right\} \Delta_r k \\
 &+ \frac{1}{2}(m+1)(m+2)(n+1)\Delta_r + \frac{1}{8}(2-r)(m+1)(n+2)\theta
 \end{aligned}
 \tag{5.13}$$

From the above expression we know that the total weight of shortest paths induced at the point  $(k, l)$  can be minimized at  $(k_m, 1 + \frac{n}{2})$  such that

$$\begin{aligned}
 k_m &= \frac{(m+2)[n+1 - \frac{(n+2)\theta}{8(m+2)}]}{2[n+1 - \frac{(n+2)\theta}{8}]} = \frac{m+2}{2} \frac{1 - \frac{(n+2)\theta}{8(m+2)(n+1)}}{1 - \frac{(n+2)\theta}{8(n+1)}} \\
 &\Rightarrow \frac{m+2}{2} \frac{1 - \frac{\theta}{8(m+2)}}{1 - \frac{\theta}{8}} \quad \text{as } n \text{ goes to } +\infty.
 \end{aligned}
 \tag{5.14}$$

In the above expression  $k_m < m+1$  implies  $\theta < 4 - \frac{2}{m}$ , while  $k_m \geq m+1$ , i.e.,  $k_m = m+1$  implies  $4 - \frac{2}{m} \leq \theta \leq 4$ . Therefore, letting  $n$  be large in (5.14), we know that  $k_m$  can be expressed as in (5.4).  $\square$

The above theorem implies that the grid type network, which corresponds to the ultimate case  $\theta = 0$ , has 1-median at the grid point  $(1 + \frac{m}{2}, 1 + \frac{n}{2})$  and furthermore it moves closer to the middle point at the “bottom” line up to the ultimate point  $(m+1, 1 + \frac{n}{2})$  as  $\theta$  approaches 4. Regarding the case  $\theta \geq 4$ , we obtain the following theorem.

**Theorem 5.3** *Given the circular type network  $K(m, n, \theta)$  with the polar angle  $\theta$  and the lengths of the radial and circular segments given by  $r$  and  $s$ , respectively, and their numbers of edge segments in radial and circular directions  $m$  and  $n$ :even, respectively, suppose that*



$\theta \geq 4$ , the radius of the outer circle is 1 and integer  $n$  be very large. Then the 1-median  $(k_m, l_m)$  for the circular type network can be obtained at  $k_m = m + 1$  and  $l_m = 1 + \frac{n}{2}$ .

**Proof** Circular coordinate  $l_m = 1 + \frac{1}{2}$  is obvious from symmetry, so we prove that  $k_m = 1 + \frac{m}{2}$ .

We consider the weight of shortest paths induced at each grid point  $(k, l_m)$ , hence calculating the total weight of all shortest paths originating from the point  $(k, l_m)$  with destination at another point  $(i, j)$ . Now we divide the set of destination vertices  $(i, j)$  in the network  $K(m, n, \theta)$  into six subsets  $S_1, S_2, S_3, S_4, S_5$  and  $S_6$  as follows.

$$\begin{aligned} S_1 &= \{(i, j) \mid 1 \leq i \leq k, l_m - p_0 \leq j \leq l_m - 1\} \\ S_2 &= \{(i, j) \mid 1 \leq i \leq k, l_m + 1 \leq j \leq l_m + p_0\} \\ S_3 &= \{(i, l_m) \mid 1 \leq i \leq k - 1\} \\ S_4 &= \{(i, j) \mid k + 1 \leq i \leq m + 1, l_m - p_0 \leq j \leq l_m + p_0\} \\ S_5 &= \{(i, j) \mid 1 \leq i \leq m + 1, 1 \leq j \leq l_m - p_0 - 1\} \\ S_6 &= \{(i, j) \mid 1 \leq i \leq m + 1, l_m + p_0 + 1 \leq j \leq n + 1\} \end{aligned}$$

where  $p_0$  is defined to be the integer satisfying

$$\frac{p_0\theta}{n} \leq 2 \quad \text{and} \quad \frac{(p_0 + 1)\theta}{n} > 2$$

i.e.,

$$p_0 = \left\lfloor \frac{2n}{\theta} \right\rfloor \quad (5.15)$$

where  $[x]$  is the greatest integer which is not greater than  $x$ .

Then we can categorize the set of edges according to the routes of shortest paths between the designated vertex  $(k, l)$  and its destination vertex  $(i, j)$  into three groups as (i)  $(i, j) \in S_1 \cup S_2 \cup S_3$ , (ii)  $(i, j) \in S_4$ , (iii)  $(i, j) \in S_5 \cup S_6$ . For each group of these shortest paths letting the weights of the radial and circular edges be denoted by  $w(R_{ij})$  and  $w(C_{ij})$ , respectively, these weights can be counted as follows.

(i)  $(i, j) \in S_1 \cup S_2 \cup S_3$

$$w(R_{ij}) = i\Delta_r \quad 1 \leq i \leq k - 1, \quad l_m - p_0 \leq j \leq l_m + p_0 \quad (5.16)$$

$$w(C_{kj}) = \begin{cases} (j - l_m + p_0 + 1)k\Delta_{S_k} & l_m - p_0 \leq j \leq l_m - 1 \\ (l_m + p_0 - j)k\Delta_{S_k} & l_m \leq j \leq l_m + p_0 - 1 \end{cases} \quad (5.17)$$

(ii)  $(i, j) \in S_4$

$$w(R_{ij}) = (m + 1 - i)(2p_0 + 1)\Delta_r \quad k \leq i \leq m \quad (5.18)$$

$$w(C_{ij}) = \begin{cases} (j + 1 - l_m + p_0)\Delta_{S_i} & k + 1 \leq i \leq m + 1, l_m - p_0 \leq j \leq l_m - 1 \\ (l_m + p_0 - j)\Delta_{S_i} & k + 1 \leq i \leq m + 1, l_m \leq j \leq l_m + p_0 - 1 \end{cases} \quad (5.19)$$

(iii)  $(i, j) \in S_5 \cup S_6$

$$w(R_{ij}) = i\Delta_r \quad 1 \leq i \leq m, 1 \leq j \leq l_m - p_0 - 1, l_m + p_0 + 1 \leq j \leq n + 1 \quad (5.20)$$

$$w(R_{il_m}) = (m + 1)(n - 2p_0)\Delta_r \quad k \leq i \leq m \quad (5.21)$$

$$w(C_{m+1j}) = \begin{cases} (m + 1)j\Delta_{s_{m+1}} & 1 \leq j \leq l_m - p_0 - 1 \\ (m + 1)(n + 1 - j)\Delta_{s_{m+1}} & l_m + p_0 \leq j \leq n \end{cases} \quad (5.22)$$

$$w(C_{m+1j}) = \begin{cases} (m + 1)(l_m - p_0 - 1)\Delta_{s_{m+1}} & l_m - p_0 \leq j \leq l_m - 1 \\ (m + 1)(n + 1 - l_m - p_0)\Delta_{s_{m+1}} & l_m \leq j \leq l_m + p_0 - 1 \end{cases} \quad (5.23)$$

Adding the above expressions (5.16)-(5.23) with respect to all the indices  $i$  and  $j$  in the corresponding range, we obtain the total length  $L$  as follows.

$$\begin{aligned}
 L &= \sum_{i=1}^{k-1} \sum_{j=l_m-p_0}^{l_m+p_0} i\Delta_r + \sum_{j=l_m-p_0}^{l_m-1} (j-l_m+p_0+1)k\Delta_{s_k} \\
 &+ \sum_{j=l_m}^{l_m+p_0-1} (l_m+p_0-j)k\Delta_{s_k} + \sum_{i=k}^m (m+1-i)(2p_0+1)\Delta_r \\
 &+ \sum_{i=k+1}^{m+1} \sum_{j=l_m-p_0}^{l_m-1} (j-l_m+p_0+1)\Delta_{s_i} + \sum_{i=k+1}^{m+1} \sum_{j=l_m}^{l_m+p_0-1} (l_m+p_0-j)\Delta_{s_i} \\
 &+ \sum_{i=1}^m \sum_{j=1}^{l_m-p_0+1} i\Delta_r + \sum_{i=1}^m \sum_{j=l_m+p_0+1}^{n+1} i\Delta_r + \sum_{j=1}^{l_m-p_0-1} (m+1)j\Delta_{s_{m+1}} \\
 &+(m+1)\left\{ \sum_{j=l_m+p_0}^n (n+1-j) + \sum_{j=l_m-p_0}^{l_m-1} (l_m-p_0-1) \right\} \Delta_{s_{m+1}} \\
 &+ \sum_{i=k}^m (m+1)(n-2p_0)\Delta_r + \sum_{j=l_m}^{l_m+p_0-1} (m+1)(n+1-l_m-p_0)\Delta_{s_{m+1}} \\
 &= \frac{1}{2}k(k-1)(2p_0+1)\Delta_r + p_0(p_0+1)k\Delta_{s_k} \\
 &+ \frac{1}{2}(m+1-k)(m+2-k)(2p_0+1)\Delta_r + p_0(p_0+1) \sum_{i=k+1}^{m+1} \Delta_{s_i} \\
 &+ \frac{1}{2}m(m+1)(n-2p_0)\Delta_r + (m+1)(m-k+1)(n-2p_0)\Delta_r \\
 &+ \frac{1}{2}(m+1)(l_m-p_0)(l_m-p_0-1)\Delta_{s_{m+1}} \\
 &+ \frac{1}{2}(m+1)(n+2-l_m-p_0)(n-l_m-p_0+1)\Delta_{s_{m+1}} \\
 &+(m+1)(l_m-p_0-1)p_0\Delta_{s_{m+1}} + (m+1)(n+1-l_m-p_0)p_0\Delta_{s_{m+1}}
 \end{aligned}
 \tag{5.24}$$

In the above expression we know that the coefficients of variables  $k^2$  and  $k$  are given, respectively, as follows.

$$k^2 : \Delta_r \left\{ 2p_0 + 1 - \frac{1}{2}p_0(p_0 + 1) \frac{\theta}{n} \right\} \tag{5.25}$$

$$k : \Delta_r \left\{ \frac{1}{2}p_0(p_0 + 1) \frac{\theta}{n} - (m + 1)(n - 2p_0) - (m + 2)(2p_0 + 1) \right\} \tag{5.26}$$

Letting  $n$  be large in (5.25) and (5.26) and using the fact that  $\frac{p_0\theta}{n} \rightarrow 2$  as  $n$  becomes large, we know that  $k_m$  can be expressed as

$$\begin{aligned}
 k_m &= \frac{(m+1)(n-2p_0) + (m+2)(2p_0+1) - \frac{1}{2}p_0(p_0+1)\frac{\theta}{n}}{2\left\{(2p_0+1) - \frac{1}{2}p_0(p_0+1)\frac{\theta}{n}\right\}} \\
 &= \frac{m\left(1 + \frac{1}{n}\right) + 1 + \frac{2p_0+2}{n} - \frac{1}{2}\frac{p_0+1}{n}\frac{p_0\theta}{n}}{2\left(\frac{2p_0+1}{n} - \frac{1}{2}\frac{p_0+1}{n}\frac{p_0\theta}{n}\right)} \\
 &\Rightarrow \frac{1}{2} + (m+1)\frac{\theta}{4}.
 \end{aligned}
 \tag{5.27}$$

From the assumption  $\theta > 4$  the above expression (5.23) implies that  $k_m > m + 1$ , i.e., the total weight of all shortest paths induced at the point  $(k, l)$  in the circular type network can be minimized at the central point  $(m + 1, 1 + \frac{n}{2})$ .  $\square$

We consider the 1-center  $(k_c, l_c)$  for the grid type network  $G(m, n)$ , with the number of edge segments in the vertical and horizontal directions being  $m$  and  $n$ :even, respectively, as illustrated in Figure 4. In this grid type network  $G(m, n)$  the 1-center  $(k_c, l_c)$  can be given as follows.

$$k_c = 1 + \frac{m}{2}, \quad l_c = 1 + \frac{n}{2} \tag{5.28}$$

In the above expression  $k_c$  for odd  $m$  indicates the central point on the edge segment  $R_{kl}$ .

Next we consider the 1-center  $(k_c, l_c)$  for the circular type network  $K(m, n, \theta)$ , with the vertical and horizontal lengths  $r$  and  $s$  respectively, and the number of edge segments in the vertical and horizontal directions being  $m$  and  $n$ :even respectively, as illustrated in Figure 8. The following theorem gives the 1-center  $(k_c, l_c)$  for the circular type network  $K(m, n, \theta)$  with  $\theta \leq 2$

**Theorem 5.4** *Given the circular type network  $K(m, n, \theta)$  with the polar angle  $\theta$  and the radial and circular lengths  $r$  and  $s$ , respectively, and numbers of edge segments in radial and circular directions  $m$  and  $n$ :even, respectively, we assume that the radius of the outer circle is 1 and that integer  $n$  is very large. Then the 1-center  $(k_c, l_c)$  for the circular type network  $K(m, n, \theta)$  can be given approximately as follows.*

$$\begin{aligned} k_c &= 1 + \frac{m(1 - \frac{\theta}{2})}{2 - \frac{\theta}{2}} && \text{if } 0 \leq \theta < 2 \\ &= k && 1 \leq k \leq m + 1 \quad \text{if } \theta = 2 \\ &= m + 1 && \text{if } 2 < \theta \\ l_c &= 1 + \frac{n}{2} \end{aligned} \tag{5.29}$$

$$\tag{5.30}$$

**Proof** Since we know that  $l_c = 1 + \frac{n}{2}$  from symmetry we calculate the distances (i) from the point  $(k, l_c)$  through  $(m + 1, l_c)$  to  $(m + 1, 1)$  and also (ii) from the point  $(k, l_c)$  through  $(k, 1)$  to  $(1, 1)$ . Then we try to find the index  $k$  such that the 1-center can exist on the edge segment  $R_{kl}$  by comparing these distances. We denote the distance (i) between  $(k, l_c)$  and  $(m + 1, 1)$  by  $d_l$ , and (ii) between  $(k, l_c)$  and  $(1, 1)$  by  $d_u$ , respectively. Then these distances  $d_l$  and  $d_u$  can be given as follows.

$$d_l = (m - k + 1)\Delta_r + \sum_{j=1}^{l_c-1} \Delta_{s_{m+1}} \tag{5.31}$$

$$\begin{aligned} &= (m - k + 1)\Delta_r + (l_c - 1)(1 - r)\frac{\theta}{n} \\ &= (m - k + 1)\Delta_r + \frac{1}{2}(1 - r)\theta \\ d_u &= (k - 1)\Delta_r + (l_c - 1)\Delta_{s_k} \end{aligned} \tag{5.32}$$

$$\begin{aligned} &= (k - 1)\Delta_r + (l_c - 1)(1 - \frac{k-1}{m}r)\frac{\theta}{n} \\ &= (k - 1)\Delta_r + \frac{1}{2}\{1 - \frac{(k-1)r}{m}\}\theta \end{aligned}$$

From the above expressions the difference  $d_l - d_u$  can be written as follows.

$$d_l - d_u = \{m - 2k + 2 + \frac{\theta}{2}(k - m - 1)\}\Delta_r \tag{5.33}$$

$$= \left\{ \left( \frac{\theta}{2} - 2 \right) (k - 1) + m \left( 1 - \frac{\theta}{2} \right) \right\} \Delta_r$$

Suppose  $0 < \theta < 2$ , then the index  $k_c$  equalizing  $d_l$  and  $d_u$ , i.e.,  $d_l - d_u = 0$  can be obtained as

$$k_c = 1 + \frac{m \left( 1 - \frac{\theta}{2} \right)}{2 - \frac{\theta}{2}}.$$

This implies that  $k_c = 1 + \frac{m}{2}$  as  $\theta$  approaches 0. Suppose  $\theta = 2$ , then  $d_l - d_u = (1 - k)\Delta_r$  and  $d_u = 1$  for any  $k$ . This implies that  $k_c = k$  for all  $1 \leq k \leq m$ .

Suppose  $\theta > 4$ , then the longest route between  $(k_c, l_c)$  and  $(1, 1)$  can be obtained as  $(k, l_c)$  through  $(m + 1, l_c)$ ,  $(m + 1, 1)$  and  $(1, 1)$ . Thus the longest distance can be minimized when  $k_c = m + 1$ .  $\square$

The above theorem implies that when  $\theta = 0$ , corresponding to the grid type network  $G(m, n)$ , the 1-center  $(k_c, l_c)$  can be attained at the central point  $(1 + \frac{m}{2}, 1 + \frac{n}{2})$ ; while it comes closer to the point  $(1, 1 + \frac{n}{2})$  when  $\theta$  approaches 2 in the circular type network  $K(m, n, \theta)$ . When  $\theta = 2$ , the 1-center  $(k_c, l_c)$  can be any points on the central line from  $(1, l_c)$  to  $(m + 1, l_c)$ ; then, for  $\theta > 2$ , the 1-center  $(k_c, l_c)$  remains at the point  $(m + 1, 1 + \frac{n}{2})$ . Thus we can conclude that the 1-center of the circular type network  $K(m, n, \theta)$  with the polar angle  $\theta$  moves from the central point  $(1 + \frac{m}{2}, 1 + \frac{n}{2})$  for the grid type network  $G(m, n)$  corresponding to  $\theta = 0$  to  $(1, 1 + \frac{n}{2})$  as  $\theta$  comes closer to 2. Then at  $\theta = 2$  the 1-center of the circular type network extends to the whole line segment from  $(1, 1 + \frac{n}{2})$  to  $(m + 1, 1 + \frac{n}{2})$ , finally for all  $\theta > 2$  it remains at  $(m + 1, 1 + \frac{n}{2})$ .

## 6. Application and Summary

The *SPCP* provides us with interesting and important insights into the real world by applying the problem to the road network in several cities. We have applied the *SPCP* to real road networks in the cities of Tokyo and Kofu in Japan. We found that edges with larger weight of shortest paths almost coincided with those road segments with heavier traffic congestion (Oyama-Taguchi [6]).

Looking at the *SPCP* from the point of view of traffic congestion, Theorem 2.2 implies that, given a set of  $n$  cities, connecting them as a star network, i.e., one city as a center and other cities as its satellites, is the most efficient, while connecting them sequentially as a single path is the least efficient.

In this paper we also investigated the weight of shortest paths induced at each vertex of the network. When the network has the special structure, e.g., tree structured or grid type network, we can obtain explicit expressions for the weight of shortest paths. Connecting two networks by a single bridge and minimizing the total weight of shortest paths we found that we should connect two 1-median vertices of these networks. Namely connecting two tree structured networks with a single edge is the most efficient from the viewpoint of the weight of shortest paths when we connect two 1-median vertices. However, when we connect them by two bridges rather than one, minimizing the total weight of shortest paths, they should be between two vertices "inside" 2-medians. When we connect two single paths with different lengths, we don't know how we can connect two networks in order to minimize the total weight of shortest paths in the combined network.

We are presently working on revising our model to more general forms such as capacitated edges and non-uniform "traffic demand", which means not all pairs of vertices has equal traffic demand since some origin-destination combinations should occur more frequently than others. Also we are very much interested in the application of *SPCP* to some other

related areas such as network reliability problem and analyzing network connectivity. In order to measure the strength of the connectedness of the network we are hopeful that results of the *SPCP* can be applied.

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## References

- [1] A.J. Goldman: Optimal center location in simple networks. *Transportation Science*, **5**(1971) 212–221.
- [2] S.L. Hakimi: Optimal location of switching centers and the absolute centers and medians of a graph. *Operations Research*, **12** (1964) 450–459.
- [3] P.B. Mirchandani and R.L. Francis: *Discrete Location Theory* (John Wiley & Sons, 1990), 555p.
- [4] T. Oyama and A. Taguchi: On some results of the shortest path counting problem. *Abstracts of the OR Society Spring Meeting* (Kitakyushu, 1991), 102–103.
- [5] T. Oyama and A. Taguchi: Further results on the shortest path counting problem. *Abstracts of the OR Society Fall Meeting* (Osaka, 1991), 166–167.
- [6] T. Oyama and A. Taguchi: Shortest path counting problem and evaluating the congestion of road segments. *PAT News, The Institute of Statistical Research*, **1** (1992) 13–19 (in Japanese).
- [7] T. Oyama and A. Taguchi: Application of the shortest path counting problem to evaluate the importance of the city road segments in Japan. *Perspectives of Advanced Technology Society*, In Y. Matsuda and M. Fushimi (eds.): *Urban Life and Traffic*, **3** (Maruzen Planet Co., Tokyo, Japan, 1996), 3–20.
- [8] A. Taguchi and T. Oyama: Evaluating the importance of the city road segments based on the network structure—application on the city road network. *Communications of Operations Research*, **38-9** (1993), 465–470 (in Japanese).

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