Journal of the Operations Research
Society of Japan
Vol. 43, No. 1, March 2000

CHANCE CONSTRAINED BOTTLENECK SPANNING TREE PROBLEM WITH FUZZY RANDOM EDGE COSTS

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(Received February 2, 1999; Revised September 22, 1999)

Abstract This paper considers a generalized fuzzy random version of bottleneck spanning tree problem in which edge costs are fuzzy random variables. The problem is to find an optimal spanning tree under chance constraint with respect to possibility measure of bottleneck (maximum cost) edge of spanning tree. The problem is first transformed into a deterministic equivalent problem. Then its subproblem is introduced and a close relation between these problems is clarified. Finally, fully utilizing this relation, we propose a polynomial order algorithm that finds an optimal spanning tree under two special functions.

1. Introduction

There are two different kinds of decision-making method by mathematical programming in uncertain environment. One is based on a stochastic programming and the other a fuzzy programming. Many researches so far about fuzzy programming have been summarized in [4]. These two programmings have been compared in [5, 17, 20]. This paper treats a decision-making problem under the situation where fuzzy factor and random factor may be included at the same time. Kwakernaak [13] introduced fuzzy random variable to represent the element containing both of fuzzy and random factors simultaneously, and its applications have been investigated in, for example [3, 23]. Luhandjula [14] developed a linear programming model for the situation with both fuzziness and randomness. His model uses the concept of fuzzy event, which was introduced by Zadeh [21], to treat the vagueness of random events. On the other hand, this paper investigates the ambiguity of random variable. In this paper, we introduce a fuzzy random variable into the spanning tree problem, which is one of discrete network optimization problems. The spanning tree problem together with its variations has a wide range of applications especially to computer network communication, and is an important problem to investigate. Ishii et al. [6, 7] have proposed stochastic minimum spanning tree problem with random edge costs, while Itoh et al. [9] have proposed a fuzzy version.

This paper proposes a generalized version of spanning tree problem, i.e., fuzzy random bottleneck spanning tree problem, which is to find an optimal spanning tree under a chance constraint with respect to possibility measure of bottleneck edge of spanning tree. In other words, the problem is a fuzzy random version of [8].

Section 2 gives definition of fuzzy random variables. Section 3 formulates a fuzzy stochastic bottleneck spanning tree problem and shows that it is transformed into a deterministic equivalent problem P by using results of stochastic programming. Section 4 introduces maximum spanning tree problem P^h with parameter h as a subproblem of P. The close relation between P and P^h is derived and an optimal solution of P can be found from a certain subproblem P^h. Further utilizing this relation, Section 5 proposes an algorithm that finds an optimal spanning tree under two special functions of g in a polynomial time. Finally, Section 6 concludes this paper and discusses further research problems.

2. Fuzzy Random Variable

In real system, we often face with the situation where two different types of uncertainty, fuzziness and randomness, appear simultaneously. Randomness involves uncertainties in the outcome of experiment, on the other hand, fuzziness involves uncertainties in the meaning of data. Experiment involving vague data such as linguistic data is regarded as a phenomenon containing randomness and fuzziness. A typical example can be seen in knowledge-based system, in which the combined knowledge of a group of experts. Randomness occurs when each expert is selected at random. The knowledge given by each expert is vague.

The fuzzy random variables is a concept that can be applied to such a situation and was first introduced by Kwakernaak [13]. Kruse and Meyer [12] slightly modified the definition of Kwakernaak.

Puri and Ralescu [16] defined a slightly different definition and established the mathematical basis of fuzzy random variables. The concept introduced by Puri and Ralescu is a generalized theory of random set and is more general due to the measurability condition which is commonly employed to formalize it. Under the condition that fuzzy random variable is unimodal, Kwakernaak - Kruse and Meyer's and Puri and Ralescu's definition coincide (see [22]).

N. Watanabe [19] gave a simple but universal definition for fuzzy random variables, which is useful in applications. In this paper, we choose it as definition of fuzzy random variables.

Definition 1 *Let (Ω, B_Ω, P) be a probability space and (Λ, B_Λ) a measurable space, where Ω is a set, Λ is a class of fuzzy set, B_Ω and B_Λ are $\sigma-$algebras, and P is a probability measure. A fuzzy random variable Y is a measurable mapping of Ω into Λ. This means that $\{\omega | Y(\omega) \in A\} \in B_\Omega$ for arbitrary $A \in B_\Lambda$.*

The following theorem is sufficient conditions for Definition 1.

Theorem 1 *Let y be a measurable mapping of a probability space (Ω, B_Ω, P) into a measurable space (Γ, B_Γ) and Y a mapping of Ω into Λ. If there exists a bijection $h : \Lambda \to \Gamma$, then there exists a measurable space (Λ, B_Λ), and a mapping Y of (Ω, B_Ω, P) into (Λ, B_Λ) is a fuzzy random variable.*

The above theorem implies the next corollary immediately.

Corollary 1 *Let Y be a mapping of Ω into Λ. Suppose that for $\forall\omega \in \Omega$, the membership function $\mu_{X(\omega)}$ of a fuzzy set $Y(\omega)$ can be represented as $\mu_{Y(\omega)}(u) = f(u; y(\omega))$ for some function $f(u;\theta)$, where θ is a parameter vector such that $\theta_1 \neq \theta_2$ implies $f(u;\theta_1) \neq f(u;\theta_2)$. Then Y is a fuzzy random variable.*

If the membership function of a fuzzy set Y is determined by the location parameter y and if y is a random variable, then Y is a fuzzy random variable from corollary. In this paper, we treat a simple fuzzy random variable, where the mean of fuzzy number is replaced by a random variable. It is satisfied with the above corollary. The conditions in corollary is fairly restrictive, but useful in application.

3. Problem Formulation

Let $G = (N, E)$ denote undirected graph consisting of vertex set $N = \{v_1, v_2, \cdots, v_n\}$ and edge set $E = \{e_1, e_2, \cdots, e_m\} \subset N \times N$. Moreover cost c_j is attached to edge e_j. A spanning tree $T = (N, S)$ of G is a partial graph satisfying the following conditions.

1. T has the same vertex set as G.
2. $|S| = n-1$ where $|S|$ denotes the cardinality of set S.
3. T is connected.

T can be denoted with $0-1$ variables $x_1, x_2, \ldots, x_m$ as follows.

$$T \;:\; x_i = 1, \quad e_i \in S.$$
$$x_i = 0, \quad e_i \notin S.$$

Conversely, if $\{e_i | x_i = 1\}$ becomes a spanning tree of G with vertex set N, then $X = (x_1, x_2, \cdots, x_m)$ is also called spanning tree hereafter in this paper.

The minimal spanning tree problem is to seek a spanning tree X minimizing $\sum_{j=1}^{m} c_j x_j$ and can be solved by using some greedy-type algorithm [11, 15, 18, 2]. In this paper, we consider a bottleneck spanning tree problem, which has been first introduced by Ishii et al.[8]. When each cost of edge is a real value, the problem is formulated as follows.

$$P_0 : \quad \begin{array}{ll} \text{minimize} & \max\{c_i | x_i = 1\} \\ \text{subject to} & \boldsymbol{x} \in X. \end{array}$$

P_0 can be also solved by using the algorithm for a minimal spanning tree problem. Ishii et al. has investigated the problem where each edge cost is a random variable.

In actual system, decision-maker often face with a situation where there exist both fuzziness and randomness. Let us consider the construction of a communication network that connects some cities directly or indirectly. If each communication quantity per unit time between one city and another city is constant, the problem of minimizing maximal capacity necessary for handling these quantities becomes a bottleneck spanning tree problem. In reality, however, there is a situation where these quantities vary randomly with time and some expert can estimate these quantities approximately. In such a case, these quantities can be considered as fuzzy random variables and we consider a spanning tree problem where c_i is a fuzzy random variable characterized by the following membership function;

$$\mu_{C_i(\omega)}(y) = L\left(\frac{y - d_i(\omega)}{\beta_i}\right).$$

Each $d_i(\omega)$ is assumed to be distributed according to the normal distribution $N(\mu_i, \sigma_i^2)$ with mean μ_i and variance σ_i^2, and $d_i(\omega)$ and $d_j(\omega)$ $(i \neq j)$ are mutually independent, and β_i is a spread and is a positive real value. $L(\cdot)$ is a reference function and is the following linear function;

$$L(t) = \max(0,\ 1 - |t/t_0|),$$

where t_0 is a positive real value.

Since each edge cost contains both of fuzziness and randomness, we cannot use the optimal criterion and the solution method for a usual minimal spanning tree. As a matter of course, the less each edge cost involved in minimal spanning tree is, the better it is. Accordingly, we give the fuzzy goal "Each edge cost involved in a minimal spanning tree is about less than f_1", and represent it with a fuzzy set characterized by the following membership function μ_G;

$$\mu_G(y) \;=\; \begin{cases} 1 & (y \leq f_1) \\ \dfrac{y - f_1}{f_0 - f_1} & (f_1 < y < f_0) \\ 0 & (y \geq f_0), \end{cases}$$

where $f_1 < f_0$. We give the possibility measure with respect to the fuzzy goal as follows.

$$\mathcal{P}_{C_i(\omega)}(G) = \sup_y \min\{\mu_{C_i(\omega)}(y),\ \mu_G(y)\}.$$

$\mathcal{P}_{C_i(\omega)}(G)$ represents a degree of possibility that each edge cost involved in a minimal spanning tree is about less than f_1 under the possibility distribution of the edge cost is given, and varies randomly due to the randomness of $\mu_{C_i(\omega)}$. Then we propose the following problem P_1 as a decision-making method of P_0, which is a chance constrained programming problem and is based on the possibilistic programming by Inuiguchi et al. [4].

$$\begin{array}{lll} P_1: & \textit{maximize} & h + g(\alpha) \\ & \textit{subject to} & Pr\left(\min\{\mathcal{P}_{C_i}(y) \mid e_i \in S\} \geq h\right) \geq \alpha, \\ & & 0 \leq h \leq 1,\ \dfrac{1}{2} < \alpha < 1, \\ & & \boldsymbol{x} \in X, \end{array} \tag{3.1}$$

where $g(\alpha)$ is a differentiable and nondecreasing function of α. In the previous paper [10], we have investigated the problem maximizing only possibility measure, h. P_1 is to maximize not only possibility measure but probability measure and so is a more generalized problem. Next, we transform P_1 into the deterministic equivalent problem.

$\mathcal{P}_{C_i(\omega)}(y) \geq h$ implies

$$\begin{array}{ll} & \sup_y \min\{\mu_{C_i(\omega)}(y),\ \mu_G(y)\} \geq h \\ \Leftrightarrow & \exists\, y : \mu_{C_i(\omega)}(y) \geq h,\ \mu_G(y) \geq h \\ \Leftrightarrow & \exists\, y : L\left(\dfrac{y - d_i(\omega)}{\beta_i}\right) \geq h,\ \mu_G(y) \geq h \\ \Leftrightarrow & \exists\, y : y \geq d_i(\omega) - L^*(h)\beta_i,\ y \leq \mu_G^*(h) \\ \Leftrightarrow & d_i(\omega) - L^*(h)\beta_i \leq \mu_G^*(h), \end{array}$$

where $\mu_G^*(\cdot)$ and $L^*(\cdot)$ are the following pseudo inverse functions.

$$L^*(h) = t_0(1-h), \tag{3.2}$$

$$\mu_G^*(h) = h(f_1 - f_0) + f_0. \tag{3.3}$$

Therefore Eq(3.1) is transformed as follows.

$$\begin{array}{ll} & Pr\left(\min\{\mathcal{P}_{C_i(\omega)}(y) \mid e_i \in S\} \geq h\right) \geq \alpha \\ \Leftrightarrow & Pr\left(\bigcap_{e_i \in S} \left\{\mathcal{P}_{C_i(\omega)}(y) \geq h\right\}\right) \geq \alpha \\ \Leftrightarrow & \prod_{e_i \in S} Pr\left(\mathcal{P}_{C_i(\omega)}(y) \geq h\right) \geq \alpha \\ \Leftrightarrow & \prod_{e_i \in S} Pr(d_i(\omega) \leq L^*(h)\beta_i + \mu_G^*(h)) \geq \alpha. \end{array}$$

Since $Pr\left(d_i(\omega) \leq L^*(h)\beta_i + \mu_G^*(h)\right) = Pr\left[(d_i(\omega) - \mu_i)/\sigma_i \leq (L^*(h)\beta_i + \mu_G^*(h) - \mu_i)/\sigma_i\right]$ and $(d_i(\omega) - \mu_i)/\sigma_i$ is a mutually independent random variable distributed according to a standard normal distribution $N(0,1)$,

$$\prod_{e_i \in S} Pr(d_i(\omega) \leq L^*(h)\beta_i + \mu_G^*(h)) \geq \alpha$$

$$\Leftrightarrow \prod_{e_i \in S} F\left(\frac{L^*(h)\beta_i + \mu_G^*(h) - \mu_i}{\sigma_i}\right) \geq \alpha$$

$$\Leftrightarrow \sum_{e_i \in S} \log F\left(\frac{L^*(h)\beta_i + \mu_G^*(h) - \mu_i}{\sigma_i}\right) \geq \log \alpha$$

$$\Leftrightarrow \sum_{i=1}^{m} \log F\left(\frac{L^*(h)\beta_i + \mu_G^*(h) - \mu_i}{\sigma_i}\right) x_i \geq \log \alpha,$$

where F denotes the distribution function of $N(0,1)$. Thereby P_0 is equivalent to the following problem, which is a deterministic one;

$$\begin{aligned} P_1' : \quad & \textit{maximize} \quad h + g(\alpha) \\ & \textit{subject to} \quad \sum_{i=1}^{m} \log F\left(\frac{L^*(h)\beta_i + \mu_G^*(h) - \mu_i}{\sigma_i}\right) x_i \geq \log \alpha, \quad \frac{1}{2} < \alpha < 1, \\ & \qquad 0 \leq h \leq 1, \ \boldsymbol{x} \in X. \end{aligned}$$

Putting (3.2) and (3.3) into (3.1), P_1' is transformed into the following problem;

$$\begin{aligned} P : \quad & \textit{maximize} \quad h + g(\alpha) \\ & \textit{subject to} \quad \sum_{i=1}^{m} \log F\left(\frac{h(f_1 - f_0 - \beta_i t_0) + \beta_i t_0 + f_0 - \mu_i}{\sigma_i}\right) x_i \geq \log \alpha, \quad \frac{1}{2} \leq \alpha < 1, \\ & \qquad 0 \leq h \leq 1, \ \boldsymbol{x} \in X. \end{aligned}$$

Hereafter we discuss the solution method of P.

4. Subproblem P^h and Its Relation to P

In order to solve P, we introduce the following subproblem; with parameter h.

$$\begin{aligned} P^h : \quad & \textit{maximize} \quad \sum_{i=1}^{m} \log F(c_i(h)) x_i \\ & \textit{subject to} \quad \boldsymbol{x} \in X, \end{aligned}$$

where $c_i(h) = (h(f_1 - f_0 - \beta_i t_0) + \beta_i t_0 + f_0 - \mu_i)/\sigma_i$, which is a strictly decreasing function of h since $f_0 > f_1$ and t_0, β_i, $\sigma_i > 0$. P^h is an ordinary maximal spanning tree problem with edge cost $\log F(c_i(h))$ and can be efficiently solved. Let X^h denote an optimal solution of P^h and Z_h the optimal value of P.

Lemma 1 *Z_h is a strictly decreasing function of h.*

Proof. For $h_1 < h_2$, from the optimality of X^{h_2},

$$Z_{h_2} = \sum_{i=1}^{m} \log F(c_i(h_2)) x_i^{h_2} \geq \sum_{i=1}^{m} \log F(c_i(h_2)) x_i^{h_1} > \sum_{i=1}^{m} \log F(c_i(h_1)) x_i^{h_1} = Z_{h_1}.$$

The last inequality holds because $F(c_i(h))$ is a strictly decreasing function of h. □

Let (X^*, h^*, α^*) denote the optimal solution of P. Then the following theorem holds.

Theorem 2

1. $Z_h > \log \alpha^* \Longleftrightarrow h^* > h$.
2. $Z_h = \log \alpha^* \Longleftrightarrow h^* = h$.

3. $Z_h < \log \alpha^* \iff h^* < h$.

Proof. Clearly Z_h is a continuous function of h.

(1)$\Longrightarrow$

If $Z_h = \sum_{i=1}^m \log F(c_i(h))x_i^h > \log \alpha^*$, then since $\log F(\cdot)$ is strictly increasing and continuous,

$$\sum_{i=1}^m \log F(c_i(h))x_i^h > \sum_{i=1}^m \log F(c_i(h_1))x_i^h \geq \log \alpha^*.$$

holds for h_1 which is sufficiently close to but greater than h. The above relation shows (X^h, h_1, α) is feasible for P, that is, $h < h_1 \leq h^*$.

(1)$\Longleftarrow$

From the monotonicity of $F(c_i(h))$ and feasibility of (X^*, h^*, α^*), for $h < h^*$,

$$\log \alpha^* \leq \sum_{i=1}^m \log F(c_i(h^*))x_i^* < \sum_{i=1}^m \log F(c_i(h))x_i^* \leq Z_h.$$

(3)$\Longrightarrow$

First, note that

$$\log \alpha^* > Z_h = \sum_{i=1}^m \log F(c_i(h))x_i^h \geq \sum_{i=1}^m \log F(c_i(h))x_i^*.$$

The above relation and monotonicity of $F(c_i(h))$ show $h^* < h$.

(3)$\Longleftarrow$

$Z_h > \log \alpha^*$ contradicts the optimality of h^* from $h > h^*$ since it implies the feasibility of (X^h, h, α^*).

(2) Proof is automatically done after (1) and (3) are shown.

□

By theorem 2, the feasible solutions $(X^h,\ h,\ \alpha)$ satisfying $\sum_{i=1}^m \log F\,(c_i(h))\,x_i^h = \log \alpha$ include an optimal solution (X^*, h^*, α^*). Now, let $t = \log \alpha$, that is, then $t = \sum_{i=1}^m \log F(c_i(h))x_i^h$ holds.

Property 1 *t is a strictly decreasing and continuous function of h.*

Proof. Note that t is a total edge cost involved in a maximal spanning tree when h is fixed. Since c_i is strictly decreasing and continuous and F and logarithm function are strictly increasing and continuous, $\log F(c_i(h))$ is a strictly decreasing and continuous function of h. When the order of edge cost is given, a set of a maximal spanning tree can be determined uniquely. There exists an unique point h_{ij} where $\log F(c_i(h_{ij})) = \log F(c_j(h_{ij}))$, i.e., a crossing point where the order of two cost changes. In other words, $\log F(c_i(h_{ij})) \geq \log F(c_j(h_{ij}))$ in the interval $(-\infty, h_{ij}]$ and $\log F(c_i(h_{ij})) \leq \log F(c_j(h_{ij}))$ in the interval $[h_{ij}, \infty)$. Therefore there is a possibility that the edge set consisting of a spanning tree change at the point h_{ij};

$$h_{ij} = \{\sigma_i(\beta_j t_0 + f_0 - \mu_j) - \sigma_j(\beta_i t_0 + f_0 - \mu_i)\}/\{\sigma_j(f_1 - f_0 - \beta_i t_0) - \sigma_i(f_1 - f_0 - \beta_j t_0)\}.$$

The h_{ij} that have a value between $(0, 1)$ are sorted in the nonincreasing order as follows.

$$0 = h_0 < h_1 < \cdots < h_s < h_{s+1} = 1,$$

where s is the number of different value of h_{ij}. In the subinterval (h_i, h_{i+1}), the order of each edge cost is uniquely determined. Therefore the set of a maximal spanning tree is also uniquely determined. Since only one pair of edges are exchanged in the endpoint of each interval, t is continuous at the endpoints of the interval. Since t is apparently a strictly decreasing function, t is a strictly decreasing and continuous function. □

If neither h_{ij} is included in $(0,1)$, then the order of edge cost does not change in $[0,1]$ and hence the set of a maximal spanning tree is uniquely determined.

Let f be a probability density function of a standard normal distribution $N(0,1)$. Then the following property holds.

Property 2 *t is a concave and strictly decreasing function of h in each subinterval.*

Proof. t is clearly differentiable in each subinterval (h_j, h_{j+1}), $j = 1,2,\ldots,s$. The first and the second derivative function is calculated as follows.

$$\frac{dt}{dh} = \sum_{i=1}^{m} \frac{f(c_i(h))}{F(c_i(h))} \cdot \frac{f_1 - f_0 - \beta_i t_0}{\sigma_i} x_i^h = \sum_{i=1}^{m} \frac{\exp[-\frac{1}{2}\{c_i(h)\}^2]}{F(c_i(h))} \cdot \frac{f_1 - f_0 - \beta_i t_0}{\sqrt{2\pi}\sigma_i} x_i^h < 0$$

$$\frac{d^2 t}{dh^2} = \sum_{i=1}^{m} \frac{-f(c_i(h))\{c_i(h)F(c_i(h)) + f(c_i(h))\}}{F(c_i(h))^2} \times \left(\frac{f_1 - f_0 - \beta_i t_0}{\sigma_i}\right)^2 x_i^h < 0$$

since $c_i(h) > 0$ from the following conditions

$$\prod_{e_i \in S} F(c_i(h)) \geq \alpha > 1/2$$
$$f_1 - f_0 - \beta_i t_0 < 0.$$

□

Now we substitute $\alpha = e^t$ into $g(\alpha)$ and let $v(t) = g(e^t)$. Further, let

$$u(h) = h + g(\alpha) = h + v(t) = h + v\left(\sum_{i=1}^{m} \log F(c_i(h))\, x_i^h\right),$$

which is a objective function of the original problem P. Thus we seek h^* maximizing $u(h)$ and then $(X^*, h^*, \tilde{\alpha})$ becomes an optimal solution of P where $\tilde{\alpha}$ corresponds to h^*, i.e., $g(\tilde{\alpha}) = u(h^*) - h^*$. By the chain rule,

$$\frac{du}{dh} = 1 + \frac{dv}{dt}\frac{dt}{dh},$$
$$\frac{d^2u}{dh^2} = \frac{dv}{dt}\frac{d^2t}{dh^2} + \frac{d^2v}{dt^2}\left(\frac{dt}{dh}\right)^2.$$

Combining the above results, if $dv/dt < 0$ and $d^2v/dt^2 > 0$, then u is a convex function of h in each subinterval and then the endpoints of subintervals $[h_j,\ h_{j+1}]$, $j = 0,\ldots,s$ include the optimal value of h, h^*. While if $dv/dt > 0$ and $d^2v/dt < 0$, then u is a concave function in each subinterval and then the endpoints of subintervals or the point h such that $(dv/dt)(dt/dh) = -1$ include the optimal value of q, h^*. In the next section, we shall consider two special types of $q(\alpha)$ satisfying the concavity of $u(h)$.

5. Some Typical Cases of $g(\alpha)$ and Solution Procedure

In this section, we investigate two special cases of $g(\alpha)$, that is, $g(\alpha) = \lambda \log \alpha$ and $g(\alpha) = -\lambda/\alpha$, where λ is a positive constant value. These cases are especially given as examples, which can be solved easily.

5.1. Two special cases of $g(\alpha)$

Case (a) $g(\alpha) = \lambda \log \alpha$

In this case, $u(h) = h + \lambda \log \alpha = q + \lambda t$. For each subinterval $[h_{k-1},\ h_k]$, $k = 1, \ldots, s+1$, $u(h)$ is described as follows.

$$u(h) = u_k(h) = h + \lambda \sum_{i=1}^{m} \log F\left(c_i(h)\right) x_i^h, \quad (q \in [h_{k-1},\ h_k],\ k = 1,\ \ldots,\ s+1).$$

By differentiating $u(h)$ with h in each subinterval (h_{k-1}, h_k)

$$\frac{du}{dh} = 1 + \lambda \frac{dt}{dh},\ \frac{d^2u}{dh^2} = \lambda \frac{d^2t}{dh^2} < 0$$

since $d^2t/dh^2 \leq 0$ and $\lambda > 0$. That is, $u(h)$ is a concave function of h in each subinterval. Therefore, the possible candidate points maximizing $u(q)$ are $h_1, \ldots, h_s$ or the points such that $dt/dh = -1/\lambda$.

Case (b) $g(\alpha) = -\lambda/\alpha$

In this case,

$$u(h) = h - \frac{\lambda}{\alpha} = h - \lambda e^{-t}.$$

For each subinterval $[h_{k-1},\ h_k]$, $k = 1, \ldots, s+1$, $u(h)$ is described as follows:

$$u(h) = u_k(h) = h - \lambda \prod_{e_j \in S^h} \frac{1}{F(c_j(h))}, \quad (h \in [h_{k-1},\ h_k],\ k = 1, \ldots, s+1),$$

where S^h is the edge set of the maximum spanning tree corresponding to h, that is, $\prod_{e_j \in S^h} 1/F(c_j(h))$ is the product of $F(c_j(h))$ for $x_j^h = 1$, i.e., the jth element of $X^h = 1$. By differentiating $u(h)$ with h in each subinterval (h_{k-1}, h_k),

$$\begin{aligned} \frac{du}{dh} &= 1 + \lambda e^{-t} \frac{dt}{dh}, \\ \frac{d^2u}{dh^2} &= -\lambda e^{-t} \left(\frac{dt}{dh}\right)^2 + \lambda e^{-t} \frac{d^2t}{dh^2} < 0 \end{aligned}$$

since $d^2t/dh^2 < 0$ and $\lambda > 0$. In this case, $u(h)$ is also a concave function of h in each subinterval. Therefore, the possible candidate points maximizing $u(h)$ are $h_1, \ldots, h_s$ or the points such that $dt/dh = -(1/\lambda)e^t$.

In the above cases, the following property holds.

Property 3 *In each subinterval (h_j, h_{j+1}), there exists at most one point satisfying $du/dh =$ 0.*

Proof. In each subinterval, du/dh is a strictly decreasing function from $d^2u/dh^2 < 0$ and is continuous. If $du/dh|_{h_{k-1}+0} > 0$ and $du/dh|_{h_k-0} < 0$, then there is only one point h satisfying $du/dh = 0$ from the mean value theorem. Otherwise, there is not such a point that satisfies the condition $du/dh = 0$. □

5.2. Solution procedure

As mentioned earlier t is not be differentiable at $h_1, \ldots, h_s$. Thereby we obtain the following solution procedure:

1. Find all the crossing points $h_1, \ldots, h_s$. Then calculate the left and the right derivative of du/dh at each h_r, $r = 1, \ldots, s$ that is, $L_r = du/dh|_{h_r+0}$, $R_r = du/dh|_{h_r-0}$.
2. Find the subintervals $[h_r,\ h_{r+1}]$ such that $R_r > 0$ and $L_{r+1} < 0$, and find the points h_r^λ satisfying $du/dh = 0$ in these subintervals.
3. Let $h_\lambda = \{h_r^\lambda | dt/dh = -1/\lambda\}$. Compare $u(h_r^\lambda)$, $h_r^\lambda \in h_\lambda$ and $u(h_k)$, $k = 1, \ldots, s$ and choose the maximizer h^* of $u(\cdot)$ among $h \in h_\lambda \cup \{h_1, \ldots, h_s\}$. Then $(X^{h^*}, h^*, t(h^*))$ is an optimal solution of P.

Let $T_{MST}(n, m)$ denote the time to compute minimal spanning tree, where n is the number of vertices and m that of edges. Then the following theorem is obtained.

Theorem 3 *The above procedure finds an optimal solution of P in at most $O(m^2 T_{MST}(n, m))$ computational time if h_r^λ can be found in at most $T_{MST}(n, m)$ computational time.*

Proof. The validity is clear from the above discussion.
(Complexity) The calculation of $h_1, \ldots, h_s$ takes at most $O(m^2 \log m)$ computational time because there exist at most $O(m^2)$ crossing points of $c_i(h) = c_j(h)$, $i < j \leq m$, and sorting them takes $O(m^2 \log m)$ and the time complexity of the spanning tree algorithm is $T_{MST}(n, m)$. $|h_\lambda|$ is at most $O(m^2)$ and L_r, R_r can be calculated in $O(m)$. The total time complexity is $\max\{O(m^2 \log m),\ O(m^2) \cdot T_{MST}(n, m)\} = O(m^2 T_{MST}(n, m))$ since the time to compute a minimum spanning tree takes more than $O(\log m)$. □

6. Conclusion

We have considered a bottleneck spanning tree problem with fuzzy random edge cost and formulate the problem based on maximizing a possibility measure under a certain chance constraint. We do not treat the model based on maximizing a necessity measure, but it can be solved by the similar method proposed in this paper. Since we have proposed solution procedures for only two special cases. it is necessary to consider more general types of $g(\alpha)$. Furthermore we are trying to extend the idea in this paper to some other fuzzy random combinatorial optimization problems.

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