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SMOOTHING METHODS FOR COMPLEMENTARITY PROBLEMS AND THEIR APPLICATIONS: A SURVEY

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Abstract We present an introduction to a class of smoothing methods for complementarity problems and their applications. We first discuss the features that characterize the smoothing methods for complementarity problems. We then outline the algorithms and convergence analysis. We finally give a brief view of smoothing methods for variational inequalities, semi-infinite programs, constrained optimization problems and mathematical programming with equilibrium constraints.

1. Introduction

Smoothing methods have been developed for solving many important optimization problems including complementarity problems [3, 5, 8, 10, 13, 16, 35, 37, 39, 49, 50, 56, 59], variational inequalities [2, 17, 19, 32, 51], optimal control problems [41], semi-infinite programs [57], mathematical programs with equilibrium constraints [29, 36], constrained optimization problems [1] and semidefinite complementarity problems [21]. A feature of these problems is that these problems or their constraints can be reformulated as piecewise differentiable equations. Using this feature, smoothing methods bring these problems close to continuously differentiable equations or continuously differentiable programming problems for which there are rich theory and abundant algorithms.

The main feature of smoothing methods is to approximate the nonsmooth (nondifferentiable) problems by a sequence of parameterized smooth (continuously differentiable) problems, and to trace the smooth path which leads to solutions.

The complementarity problem provides the prime candidate for illustrating the methodology of smoothing methods. For this reason we focus on it. Smoothing methods for complementarity problems are closely related to interior point methods. Both methods are based on homotopy continuation techniques [19, 42]. However, in contrast to interior point methods, iterates of smoothing methods do not have to stay in the feasible set, and the initial point can be chosen arbitrarily.

Complementarity problems can be reformulated to nonsmooth equations in several ways, see [22, 47, 50, 52]. Smoothing methods for complementarity problems may be considered as Newton-type methods for solving a special class of nonsmooth equations. Using smooth approximation functions in Newton-type methods for nonsmooth equations has been studied for more than thirty years [18, 34, 45, 60, 64]. In the last decade, many smooth approximation functions and Newton-type methods using smoothing functions for complementarity problems have been developed [8, 16, 26, 32, 37, 53]. In this paper, we intend to illustrate some basic approaches and results of smoothing Newton methods for complementarity problems and their applications.

In section 2, we study how to reformulate complementarity problems to nonsmooth equations and how to construct a sequence of smoothing functions. We also discuss the properties of smooth paths formed by the solutions of the smooth equations. In section 3, we describe an outline of smoothing methods for complementarity problems and discuss the convergence results. In section 4, we briefly discuss applications of smoothing methods to variational inequalities, semi-infinite programs, constrained optimization problems and mathematical programming with equilibrium constraints.

2. Smooth Approximations

Let $F : R^n \to R^n$ be continuously differentiable. The complementarity problem, denoted by CP(F), is to find a vector $z \in R^{2n}$ such that

$$z = \begin{pmatrix} x \\ y \end{pmatrix} \geq 0, \quad y = F(x), \quad \text{and} \quad x^T y = 0.$$

The CP(F) is called the linear complementarity problem (LCP(M, q)) if F is an affine mapping of the form

$$F(x) = Mx + q,$$

where $M \in R^{n\times n}$ and $q \in R^n$. Otherwise, the CP(F) is called the nonlinear complementarity problem (NCP(F)).

Many algorithms developed for CP(F) are based on reformulating the CP(F) as a system of equations [22, 46, 52] or an optimization problem using suitable merit functions [28, 38, 61]. Smoothing Newton methods are based on reformulating the CP(F) as a system of nonsmooth equations by using a NCP function or the Robinson normal map [52]. We call $\phi_0 : R^2 \to R$ an NCP-function if ϕ_0 satisfies

$$\phi_0(a, b) = 0 \iff ab = 0,\ a \geq 0,\ b \geq 0.$$

Obviously, z solves CP(F) if and only if z solves

$$H_0(z) = \begin{pmatrix} F(x) - y \\ \phi_0(x_1, y_1) \\ \vdots \\ \phi_0(x_n, y_n) \end{pmatrix} = 0, \tag{1}$$

or

$$G_0(x) = \begin{pmatrix} \phi_0(x_1, F_1(x)) \\ \vdots \\ \phi_0(x_n, F_n(x)) \end{pmatrix} = 0. \tag{1'}$$

In this paper, we restrict our attention to (1), but similar results are applicable for (1)'.

Many NCP-functions have been discovered in the past two decades. The well used NCP-functions in smoothing methods are the "min" function

$$\phi_0(a, b) = \min(a, b)$$

and the Fischer-Burmeister function

$$\phi_0(a, b) = \frac{1}{2}(a + b - \sqrt{a^2 + b^2}). \tag{2}$$

Notice that the original Fischer-Burmeister function is $\phi_{FB}(a, b) = \sqrt{a^2 + b^2} - a - b$. In this paper, we use the version (2), in order to discuss smoothing methods in a unified framework.

The "min" function is a piecewise smooth function whose nondifferentiable points form the line : $\{(a,b)^T \in R^2 \mid a=b\}$. The Fischer-Burmeister function is differentiable everywhere except at the point $(a,b)=(0,0)$. The "min" function provides finite convergence property for Newton-type methods to solve LCP(M,q), but the Fischer-Burmeister function does not [19, 27]. On the other hand, the Fischer-Burmeister function has a continuously differentiable natural merit function, which provides global convergence property for Newton-Armijo methods, but the "min" function does not [7, 39].

Nevertheless, Tseng [58] showed that the two NCP functions have the same growth rate by the following inequalities

$$\frac{1}{\sqrt{2}+2}|\min(a,b)| \le \frac{1}{2}|a+b-\sqrt{a^2+b^2}| \le \frac{\sqrt{2}+2}{2}|\min(a,b)|, \quad \text{for all } (a,b)\in R^2. \tag{3}$$

Now we consider how to construct smoothing functions of the "min" function and the Fischer-Burmeister function.

The "min" function can be written as

$$\min(a,b) = a - \max(0, a-b).$$

Several smoothing functions to the "min" function have been given by Smale [53], Chen and Harker [8], Chen and Qi [16], Kanzow [37], Zang [62], Teo, Rehbock and Jennings [57]. Recently, Chen and Mangasarian [13] introduced a family of smoothing functions, which unified the smoothing functions studied in [8, 16, 37, 53, 57, 62]. The Chen-Mangasarian family is built as follows. Let $\rho : R \to [0,\infty)$ be a piecewise continuous density function satisfying

$$\int_{-\infty}^{\infty}\rho(s)ds = 1 \quad \text{and} \quad \kappa_1 := \int_{-\infty}^{\infty}|s|\rho(s)ds < \infty.$$

Then a smoothing function of the "min" function is defined by

$$\phi(a,b,\epsilon) = a - \int_{-\infty}^{\infty}\max(0, a-b-\epsilon s)\rho(s)ds. \tag{4}$$

Specific cases of interest in this approach are

$$\rho(s) = \frac{e^{-s}}{(1+e^{-s})^2} \quad \Longrightarrow \quad \phi(a,b,\epsilon) = b - \epsilon\ln(1+e^{(b-a)/\epsilon})$$

$$\rho(s) = \frac{2}{(s^2+4)^{\frac{3}{2}}} \quad \Longrightarrow \quad \phi(a,b,\epsilon) = \frac{1}{2}\left(a+b-\sqrt{(a-b)^2+4\epsilon^2}\right)$$

$$\rho(s) = \begin{cases} 1 & \text{if } |s|\le \frac{1}{2} \\ 0 & \text{otherwise} \end{cases} \quad \Longrightarrow \quad \phi(a,b,\epsilon) = \begin{cases} a - \frac{1}{2\epsilon}(a-b+\frac{\epsilon}{2})^2 & \text{if } |a-b|\le\frac{\epsilon}{2} \\ \min(a,b) & \text{otherwise} \end{cases}$$

$$\rho(s) = \frac{1}{2}e^{-|s|} \quad \Longrightarrow \quad \phi(a,b,\epsilon) = \begin{cases} a - \frac{\epsilon}{2}e^{(a-b)/\epsilon} & \text{if } a\le b \\ b - \frac{\epsilon}{2}e^{(b-a)/\epsilon} & \text{if } a \ge b, \end{cases}$$

where the first ϕ is called the Neural Networks smoothing function, the second ϕ is called the CHKS (Chen-Harker-Kanzow-Smale) smoothing function, the third ϕ is called the uniform smoothing function and the fourth ϕ is called the Picard smoothing function.

A version of FB smoothing functions for the Fischer-Burmeister function is

$$\phi(a,b,\epsilon) = \frac{1}{2}(a+b-\sqrt{a^2+b^2+2\epsilon^2}), \quad \text{for } \epsilon > 0. \tag{5}$$

The Kanzow smooth function [37] is

$$\hat{\phi}(a,b,\epsilon) = a + b - \sqrt{a^2 + b^2 + 2\epsilon}, \quad \text{for } \epsilon > 0. \tag{6}$$

The term 2ϵ in (6) was replaced by ϵ or ϵ^2 in several papers [29, 36].

Although the FB smoothing function defined in (5) does not belong to the Chen-Mangasarian family, it shares the following common properties with the family.

Proposition 1 *The smoothing functions defined by (4) and (5) satisfy the following properties.*

(1) For any fixed $\epsilon > 0$, $\phi(a,b,\epsilon)$ is continuously differentiable for all $(a,b)^T \in R^2$, and the partial derivative satisfies

$$(0,\, 0) \le (\frac{\partial \phi(a,b,\epsilon)}{\partial a}, \frac{\partial \phi(a,b,\epsilon)}{\partial b}) \le (1,\, 1).$$

(2) For any fixed $(a,b)^T \in R^2$, $\phi(a,b,\epsilon)$ is continuously differentiable, monotonically decreasing and concave with respect to $\epsilon > 0$. In particular, for $\epsilon_1 \ge \epsilon_2 \ge 0$

$$0 \le \phi(a,b,\epsilon_2) - \phi(a,b,\epsilon_1) \le \kappa(\epsilon_1 - \epsilon_2),$$

where $\kappa = \max(\kappa_1, 1/\sqrt{2})$. Furthermore, $\phi(a,b,0) = \phi_0(a,b)$, and $\phi(a,b,\epsilon) \to -\infty$ as $\epsilon \to \infty$.

(3) For any fixed $(a,b)^T \in R^2$, the limit $\lim_{\epsilon \downarrow 0}(\frac{\partial \phi(a,b,\epsilon)}{\partial a}, \frac{\partial \phi(a,b,\epsilon)}{\partial b})$ exists. We define

$$\phi^o(a,b) := (\phi^o_a(a,b), \phi^o_b(a,b)) := \lim_{\epsilon \downarrow 0}(\frac{\partial \phi(a,b,\epsilon)}{\partial a}, \frac{\partial \phi(a,b,\epsilon)}{\partial b}).$$

Moreover for any $(a,b)^T, h^T \in R^2$

$$\lim_{h \to 0} \frac{\phi_0((a,b)+h) - \phi_0(a,b) - \phi^o((a,b)+h)h^T}{\|h\|} = 0. \tag{7}$$

(4) If the density function ρ satisfies $\rho(s) = \rho(-s)$ and supp(ρ)=$\{s : \rho(s) > 0\} = R$, and its second moment is bounded (see a weaker condition in [19]), then

$$\phi(a,b,\epsilon) = 0 \quad \Longrightarrow \quad a > 0,\, b > 0,\, ab \le \mu\epsilon^2,$$

where μ is a positive constant. Specially, the CHKS smoothing function and the FB smoothing function satisfy

$$\phi(a,b,\epsilon) = 0 \quad \Longleftrightarrow \quad a > 0,\, b > 0,\, ab = \epsilon^2.$$

Proof: The proof for the smooth approximation function defined by (4) can be found in [5] and its references. Now we show this proposition for ϕ defined by (5).

Result 1. Straightforward calculation shows

$$\frac{\partial \phi(a,b,\epsilon)}{\partial a} = \frac{1}{2}(1 - \frac{a}{\sqrt{a^2+b^2+2\epsilon^2}}) \quad \text{and} \quad \frac{\partial \phi(a,b,\epsilon)}{\partial b} = \frac{1}{2}(1 - \frac{b}{\sqrt{a^2+b^2+2\epsilon^2}}). \tag{8}$$

It is not difficult to see that the derivatives are in $(0,1)$.

Result 2. Calculating the derivatives gives

$$\frac{\partial\phi(a,b,\epsilon)}{\partial\epsilon}=-\frac{\epsilon}{\sqrt{a^2+b^2+2\epsilon^2}}\in(-\frac{1}{\sqrt{2}},0)$$

and

$$\frac{\partial^2\phi(a,b,\epsilon)}{\partial^2\epsilon}=-\frac{1}{\sqrt{a^2+b^2+2\epsilon^2}}+\frac{2\epsilon^2}{(a^2+b^2+2\epsilon^2)^{3/2}}\leq 0.$$

Hence ϕ is strictly decreasing and concave with respect to $\epsilon>0$. The last part of result 2 follows immediately.

Result 3. Letting ϵ go to zero in (8), we can see

$$\lim_{\epsilon\downarrow 0}(\frac{\partial\phi(a,b,\epsilon)}{\partial a},\ \frac{\partial\phi(a,b,\epsilon)}{\partial b})=\begin{cases}\frac{1}{2}(1-\frac{a}{\sqrt{a^2+b^2}},\ 1-\frac{b}{\sqrt{a^2+b^2}}), & \text{if } (a,b)\neq(0,0)\\ \frac{1}{2}(1,1), & \text{otherwise.}\end{cases}$$

Since ϕ_0 is continuously differentiable everywhere except $(a,b)=0$, we only need to show (7) in the case where $(a,b)=0$. By a simple calculation, we have $\phi^o(h)=\phi_0'(h)$ and

$$\phi_0(h)-\phi_0(0,0)-\phi^o(h)h^T=0$$

for all $h\neq 0$.

Result 4 follows from Lemma 2.2 in [37]. ▮

Based on Proposition 1, we can simply construct a smoothing function of H_0 by replacing ϕ_0 by ϕ, that is,

$$H(z,\epsilon)=\begin{pmatrix}F(x)-y\\ \phi(x_1,y_1,\epsilon)\\ \vdots\\ \phi(x_n,y_n,\epsilon)\end{pmatrix}.$$

Result 1 of Proposition 1 shows that for any fixed $\epsilon>0$, $H(z,\epsilon)$ is continuously differentiable for all $z\in R^{2n}$. The Jacobian of $H(\cdot,\epsilon)$ is given by

$$H_z(z,\epsilon)=\begin{pmatrix}F'(x) & -I\\ \mathrm{diag}(\frac{\partial\phi(x_i,y_i,\epsilon)}{\partial x_i}) & \mathrm{diag}(\frac{\partial\phi(x_i,y_i,\epsilon)}{\partial y_i})\end{pmatrix}.$$

Result 2 of Proposition 1 implies that the error of $H(z,\epsilon)$ to $H_0(z)$ is bounded by the smoothing parameter ϵ, namely,

$$||H(z,\epsilon)-H_0(z)||_\infty\leq\kappa\epsilon,\quad \text{for all } z\in R^{2n}.$$

Result 3 of Proposition 1 provides the limiting behavior of the Jacobian $H'(z,\epsilon)$ as the smoothing parameter ϵ approaches zero. In particular, we can define

$$H^o(z):=\lim_{\epsilon\downarrow 0}H_z(z,\epsilon)=\begin{pmatrix}F'(x) & -I\\ \mathrm{diag}(\phi_a^o(x_i,y_i)) & \mathrm{diag}(\phi_b^o(x_i,y_i))\end{pmatrix}.$$

Moreover, we have

$$\lim_{h\to 0}\frac{H_0(z+h)-H_0(z)-H^o(z+h)h}{||h||}=0.$$

This result is useful for designing locally fast convergent algorithms. Notice that $H^o : R^{2n} \to R^{2n}$ is a single valued function, which can be employed to design a superlinearly convergent Newton-type method

$$z^{k+1} = z^k - H^o(z^k)^{-1}H(z^k).$$

See [15].

From result 4 of Proposition 1, we can see that the smooth path

$$S_0 = \{z \in R^{2n} \mid H(z, \epsilon) = 0, \epsilon > 0\}$$

is in the interior of the feasible set

$$S = \{z \in R^{2n} \mid z \geq 0, y = F(x)\},$$

and the complementarity gap goes to zero quadratically in ϵ. Hence, the smooth path, if it exists, converges to the solution set of the CP(F) as $\epsilon \to 0$. For the existence and limiting properties of the smooth path, see papers [10, 19].

It is notable that the central path

$$C = \{z \in R^{2n} \mid z > 0, y = F(x),\ x_i y_i = \epsilon^2, i = 1, 2, \dots n\}$$

is a special smooth path as it can be derived from the CHKS smoothing function and the FB smoothing function. A common basic idea of the interior point methods is tracing the central path [42]. It will be interesting to investigate if we can trace a smooth path other than the central path to solve some open problems in interior point methods.

3. Smoothing Methods for CP(F)

Many smoothing methods have been developed in the past few years. We give an algorithm for illustrating a class of smoothing methods. This algorithm is based on smoothing methods presented in [17, 19, 39, 49], but it uses the reformulation of nonsmooth equations in [3, 5, 37, 58].

For simplicity, we assume that the density function in (4) satisfies

$$\{s \mid \rho(s) > 0\} = R.$$

In addition, all norms are Euclidean norms.

Let

$$\Theta(z) = \frac{1}{2}\|H_0(z)\|^2 \quad \text{and} \quad \theta(z, \epsilon) = \frac{1}{2}\|H(z, \epsilon)\|^2.$$

Algorithm 1 *Given $\rho, \alpha, \eta \in (0, 1)$, and a starting point $z^0 \in R^{2n}$. Choose a scalar $\sigma \in (0, 1 - \alpha)$. Let $\nu = \frac{\alpha}{2\sqrt{n}\kappa}$. Let $\beta_0 = \|H_0(z^0)\|$ and $\epsilon_0 = \nu\beta_0$.*

For $k \geq 0$:

1. Find a solution $\hat{d}^k$ of the system of linear equations

$$H_0(z^k) + H^o(z^k)d = 0.$$

If $\|H_0(z^k + \hat{d}^k)\| \leq \eta\beta_k$, let $z^{k+1} = z^k + \hat{d}^k$, and perform Step 3. Otherwise perform Step 2.

2. Find a solution d^k of the system of linear equations

$$H_0(z^k) + H_z(z^k, \epsilon_k)d = 0.$$

Let m_k be the smallest nonnegative integer m such that

$$\theta(z^k + \rho^m d^k, \epsilon_k) - \theta(z^k, \epsilon_k) \leq -\sigma\rho^m\Theta(z^k). \tag{9}$$

Set $t_k = \rho^{m_k}$ and $z^{k+1} = z^k + t_k d^k$.

3. 3.1 If $\|H_0(z^{k+1})\| = 0$, terminate.

3.2 If

$$0 < \|H_0(z^{k+1})\| \leq \max\{\eta\beta_k, \alpha^{-1}\|H_0(z^{k+1}) - H(z^{k+1}, \epsilon_k)\|\},$$

let

$$\beta_{k+1} = \|H_0(z^{k+1})\| \quad \text{and} \quad \epsilon_{k+1} = \min\{\nu\beta_{k+1}, \frac{\epsilon_k}{2}\}.$$

3.3 Otherwise, let $\beta_{k+1} = \beta_k$ and $\epsilon_{k+1} = \epsilon_k$.

Set k:=k+1.

To discuss convergence of Algorithm 1, we restate the following definition.

Definition 1 *A matrix $M \in R^{n\times n}$ is said to be a*

(i) P_0 matrix if all principal minors of M are nonnegative;

(ii) P matrix if all principal minors of M are positive;

(iii) R_0 matrix if $LCP(M, 0)$ has $z^* = (0, 0) \in R^{2n}$ as its unique solution.

A function F is said to be a

(a) P_0 function if

$$\max_{i:x_i^1\neq x_i^2} (F_i(x^1) - F_i(x^2))(x_i^1 - x_i^2) \geq 0 \quad \text{for all} \quad x^1, x^2 \in R^n, x^1 \neq x^2,$$

(b) uniform P function, if for some $\gamma > 0$,

$$\max_{1\leq i\leq n} (F_i(x^1) - F_i(x^2))(x_i^1 - x_i^2) \geq \gamma\|x^1 - x^2\|^2 \quad \text{for all} \quad x^1, x^2 \in R^n,$$

(c) monotone function if

$$(F(x^1) - F(x^2))^T(x^1 - x^2) \geq 0 \quad \text{for all } x^1, x^2 \in R^n.$$

It is not difficult to see that every P matrix is both a P_0 matrix and R_0 matrix. Furthermore, the class of P_0 functions includes monotone functions and uniform P functions. In the linear case $F(x) = Mx + q$, F is a uniform P function if and only if M is a P matrix.

In order for Algorithm 1 to be well defined, one has to guarantee $H_z(z, \epsilon)$ is nonsingular for all $z \in R^{2n}$ and $\epsilon > 0$. By Theorem 4.3 in [32] and result 1 of Proposition 1, we can give a necessary and sufficient condition for the nonsingularity.

Lemma 1 *$H_z(z, \epsilon)$ is nonsingular for all $z \in R^{2n}$ and $\epsilon > 0$ if and only if F is a P_0 function.*

Following Lemma 3.1 in [17], we can show that the line search step (9) is well defined, i.e., there exists a finite nonnegative integer m_k such that (9) holds. Hence we have

Lemma 2 *Algorithm 1 is well defined if F is a P_0 function.*

To show global convergence of Algorithm 1, we need the following assumption.

A1. The level sets

$$D(\Gamma) = \{z \in R^{2n} \mid \|H_0(z)\| \leq \Gamma\}$$

are bounded for all positive numbers Γ.

Theorem 1 (Global Convergence) *Suppose that F is a P_0 function and assumption A1 holds. Then for any starting point $z^0 \in R^{2n}$, Algorithm 1 is well defined and the generated sequence $\{z^k\}$ remains in $D((1+\alpha)\|H_0(z^0)\|)$ and satisfies*

$$\lim_{k\to\infty} \|H_0(z^k)\| = 0. \tag{10}$$

Theorem 2 (Superlinear, quadratic, finite convergence) *Suppose that assumptions of Theorem 1 hold. Assume that for an accumulation point z^* of $\{z^k\}$, there are an open ball $\hat{B} := \hat{B}(z^*, \bar{r}) = \{z \mid \|z - z^*\| < \bar{r}\}$ and a positive number Υ such that for any $z \in \hat{B}$, $H^o(z)$ is nonsingular and $\|H^o(z)^{-1}\| \leq \Upsilon$. Then z^* is a (unique) solution of (1) and $\{z^k\}$ converges to z^* superlinearly in the sense*

$$\|z^{k+1} - z^*\| \leq o(\|z^k - z^*\|).$$

Moreover, if F has a locally Lipschitz continuous derivative around x^*, the convergence rate is quadratic, i.e.,

$$\|z^{k+1} - z^*\| \leq c\|z^k - z^*\|^2,$$

for a positive constant c. In addition, if F is an affine function and H_0 is defined by the "min" function, then the convergence is finite.

We can prove Theorem 1 and Theorem 2 by using the similar technique in the proof of Theorem 3.1 and Theorem 3.2 in [19]. The uniqueness of z^* is from Proposition 7 in [5].

If F is a uniform P function, then all assumptions of Theorem 2 hold.

Assumption A1 ensures that the sequence generated by Algorithm 1 remains in a bounded set, and so its accumulation points exist and are solutions of the CP(F).

Lemma 3 *The following assumptions are equivalent to Assumption A1.*

A1' *The level sets*

$$\hat{D}(\Gamma) = \{x \in R^n \mid \|\min(x, F(x))\| \leq \Gamma\}$$

are bounded for all positive numbers Γ.

A1" *The level sets*

$$\tilde{D}(\Gamma) = \{x \in R^n \mid \left(\sum_{i=1}^{n}\left[x_i + F_i(x) - \sqrt{x_i^2 + F_i^2(x)}\right]^2\right)^{\frac{1}{2}} \leq \Gamma\}$$

are bounded for all positive numbers Γ.

Proof: By the definition of H_0 and the following inequalities

$$\|\min(x, F(x))\| \leq \|\min(x, y)\| + \|F(x) - y\| \leq 2\|H_0(z)\|,$$

Assumption A1' implies A1.

For $y = F(x)$,

$$\|H_0(z)\| = \|\min(x, y)\| = \|\min(x, F(x))\|.$$

Hence Assumption A1 implies A1'.

Moreover, by the relation of the "min" function and the Fischer-Burmeister function (3), Assumptions A1' and A1" are equivalent, and hence A1" is also equivalent to A1. ∎

Assumptions A1' and A1" are used in many papers, and sufficient conditions that ensure $\hat{D}(\Gamma)$ or $\tilde{D}(\Gamma)$ are bounded have been studied. Using Lemma 3, we can apply these sufficient conditions to $D(\Gamma)$ as follows.

Proposition 2 *1. If F is a uniform P function, $D(\Gamma)$ is bounded for every $\Gamma > 0$ [38].*

2. If $F(x) = Mx + q$ and M is an R_0 matrix, $D(\Gamma)$ is bounded for every $\Gamma > 0$ [10].

3. If F is a monotone function and CP(F) has a strictly feasible point, $D(\Gamma)$ is bounded for every $\Gamma < \Gamma_0/2$ [19], where

$$\Gamma_0 = \sup\{\min_{1\le i\le n} \min(x_i, y_i),\ z \in \text{int}S\}.$$

4. If F is a P_0 function and the solution set of CP(F) is nonempty and bounded, $D(\Gamma)$ is bounded for every positive Γ sufficiently small [23].

Remark 1. The condition in result 3 of Proposition 2 can be replaced by the following assumption.

AM. F is a monotone function and the solution set of CP(F) is nonempty and bounded.

This is due to the following result for monotone functions presented in [7] :
CP(F) has a nonempty and bounded solution set if and only if CP(F) has a strictly feasible point.

However, in the case where F is P_0 function, the existence of a strictly feasible point does not imply that the solution set is bounded. See Example 1.1 in [20].

Remark 2. The condition AM does not guarantee either $\hat{D}(\Gamma)$ or $\tilde{D}(\Gamma)$ to be bounded for every $\Gamma > 0$. For example, consider CP(F) with $F(x) \equiv 1$. This lacks the flexibility in choosing an initial point. Theoretically, we can choose a strictly feasible point $z \in \text{int}S$ as an initial point. (See result 3 of Proposition 2.) However, finding a strictly feasible point is generally as difficult as solving the problem. Some modifications have been proposed to improve the limitations of smoothing methods in dealing with monotone CP(F) [5, 35]. A simple way is to use a new NCP function proposed by Chen, Chen and Kanzow [7]

$$\phi_0(a, b) = \lambda(a + b - \sqrt{a^2 + b^2}) + (1 - \lambda)a_+b_+,$$

where $\lambda \in (0, 1)$. Using this NCP function, $D(\Gamma)$ is bounded for every $\Gamma > 0$ under the condition AM. A version of smoothing functions of this NCP function is given by Jiang and Ralph [36],

$$\phi(a, b, \epsilon) = \lambda(a + b - \sqrt{a^2 + b^2 + 2\epsilon^2}) + \frac{1-\lambda}{4}(a + \sqrt{a^2 + 2\epsilon^2})(b + \sqrt{b^2 + 2\epsilon^2}).$$

Early, Yamashita and Fukushima [61] gave the following NCP function

$$\frac{1}{4}(ab)_+^4 + \frac{1}{2}(a + b - \sqrt{a^2 + b^2})^2.$$

By this NCP function, $D(\Gamma)$ is also bounded for every $\Gamma > 0$ under the condition AM. Also see [44, 40].

Remark 3. Result 4 of Proposition 2 implies that a short central path exists [7] under the following condition:

AP0. F is a P_0 function and the solution set of CP(F) is nonempty and bounded.

Condition AP0 is weaker than assumption AM and the condition of uniform P-function as well as conditions used by Kojima, Megiddo and Noma in [42]. However, Example 2.1 in

[20] shows that the central path can be very short under AP0, and strictly feasible points may not exist if the solution set degenerates to be unbounded. This suggests that interior point methods and smoothing methods are not readily applicable for solving P_0 function CP(F). Recently, Gowda and Sznajder [33] and Facchinei [23] gave a notable result that the solution set of CP(F) is connected under condition AP0. Based on this result, global convergence of Big-M smoothing methods and regularized smoothing methods is established under condition AP0 [20, 25, 48, 54].

4. Applications

We present a brief view of smoothing methods for some optimization problems.

4.1 Mathematical programs with equilibrium constraints (MPEC)

Consider the following mathematical programming problem with equilibrium constraints

$$\begin{array}{ll} \min\limits_{x,u} & f(x,u) \\ \text{s.t} & g(x,u) \geq 0 \\ & F(x,u) \geq 0,\ x \geq 0,\ F(x,u)^T x = 0, \end{array}$$

where $f : R^{n+m} \to R$, $g : R^{n+m} \to R^l$, $F : R^{n+m} \to R^n$ are continuously differentiable. The feasible set is generally non-convex even if the usual inequality constraints $g(x,u) \geq 0$ are omitted and F is strongly monotone with respect to x. For example, consider $n = m = 1$ and $F(x,u) = 2x + u + 1$. The feasible set is $\{(0,u),\ u \geq -1\} \cup \{(x, -2x-1), x > 0\}$. The Mangasarian-Fromovitz Constraint Qualification does not hold at $(0,-1)$. This suggests that the feasible set is numerically ill-posed.

The basic idea of smoothing methods for solving the MPEC is to substitute a nonsmooth equations for the CP(F) constraints and then approximately solve the following continuously differentiable programming problem for a sequence of values $\epsilon = \epsilon_k \to 0$,

$$\begin{array}{ll} \min\limits_{x,u} & f(x,u) \\ \text{s.t} & g(x,u) \geq 0 \\ & F(x,u) - y = 0 \\ & \phi(x_i, y_i, \epsilon) = 0, i = 1, 2, \ldots, n. \end{array}$$

Numerical algorithms based on this smooth approximation have been studied in [24, 29, 30, 36].

4.2 Variational inequalities with box constraints (VIB)

Let $l \in \{R \cup -\infty\}^n$ and $u \in \{R \cup \infty\}^n$ such that $l < u$. The variational inequality problem with box constraints, denoted by VIB(F, l, u), is to find $x \in [l, u]$ such that

$$(v - x)^T F(x) \geq 0, \quad \text{for all } v \in [l, u].$$

It is easy to see that VIB(F, l, u) reduces to CP(F) when $l = 0$ and $u = \infty$.

This problem can be reformulated as the system of nonsmooth equations

$$H_0(z) = \begin{pmatrix} F(x) - y \\ x - \text{mid}(l, u, x - y) \end{pmatrix} = 0, \tag{11}$$

where mid($\cdot$) denotes the componentwise median operator. Gabriel and Moré [32] extended the Chen-Mangasarian family of smoothing functions (4) for CP(F) to VIB(F, l, u). By the

Gabriel-Moré family, we can construct a smoothing function of $H_0(z)$ as follows

$$H(z,\epsilon)=\begin{pmatrix} F(x)-y \\ x_1-\int_{-\infty}^{\infty}\text{mid}(l_1,u_1,x_1-y_1-\epsilon s)\rho(s)ds \\ \vdots \\ x_n-\int_{-\infty}^{\infty}\text{mid}(l_n,u_n,x_n-y_n-\epsilon s)\rho(s)ds \end{pmatrix}.$$

Algorithm 1 and Theorems 1 and 2 are applicable for solving (11).

4.3 Semi-infinite programming

Consider the following semi-infinite programming problem:

$$\begin{array}{ll} \min & f(x) \\ \text{s.t} & x \in X \\ & g_i(x) = \max_{\omega\in\Omega} h_i(x,\omega) \le 0, \qquad i=1,\dots m, \end{array}$$

where $f: R^n \to R, h_i: R^n \times \Omega \to R$ are continuously differentiable functions, Ω is a closed interval in R and X is a convex set in R^n.

In general, g_i is a nonsmooth function. The constraint $g_i(x) \le 0$ is equivalent to

$$G_i(x) = \int_\Omega \max(h_i(x,\omega),0)d\omega = 0.$$

The smoothing technique used in [57] is to approximate $\max(h_i(x,\omega),0)$ by the following smoothing function

$$h_i(x,\omega,\epsilon)=\begin{cases} \dfrac{(h_i(x,\omega)+\epsilon)^2}{4\epsilon} & \text{if } |h_i(x,\omega)|\le\epsilon \\ \max(h_i(x,\omega),0) & \text{otherwise,} \end{cases} \qquad (12)$$

and to replace $G_i(x)$ by

$$G_i(x,\epsilon) = \int_\Omega h_i(x,\omega,\epsilon)d\omega.$$

This smoothing function is related to the Chen-Mangasarian smoothing function (4). Indeed,

$$h_i(x,\omega,\epsilon) = \int_{-\infty}^{\infty} \max(h_i(x,\omega)+2\epsilon s,0)\rho(s)ds,$$

where

$$\rho(s)=\begin{cases} 1 & \text{if } |s|\le\frac{1}{2} \\ 0 & \text{otherwise.} \end{cases}$$

Teo, Rehbock and Jennings have successfully used the smoothing function (12) to solve some electric engineering problems. We expect other smoothing functions to work well for these problems.

4.4 Constrained optimization problems

Consider the constrained optimization problem

$$\begin{array}{ll} \min & f(x) \\ \text{s.t} & g_i(x) \le 0, \qquad i=1,2,\dots,m, \end{array}$$

where $f, g_i : R^n \to R$ are proper convex functions. A class of penalty and barrier methods approximate this problem by solving a family of unconstrained minimization problems of the form

$$\min f(x) + \alpha(r) \sum_{i=1}^{m} \theta(g_i(x)/r),$$

where $r > 0$ is a penalty parameter which will eventually go to 0, and the function $\alpha : R_+ \to R_+$ satisfies

$$\lim_{r \to 0^+} \alpha(r)/r = \infty.$$

The class of functions θ used in [1] is defined by

$$\theta(u) = \int_{-\infty}^{\infty} \max(u - s, 0)\rho(s)ds,$$

i.e., set $\epsilon = 1$ in the Chen-Mangasarian smoothing function for $\max(0, u)$. It is easy to see

$$\lim_{u \to \infty} \theta(u) = \infty, \qquad \lim_{u \to -\infty} \theta(u) = 0, \quad \text{and} \quad \max(0, u) \le \theta(u).$$

Hence, functions θ defined by smoothing functions satisfy the convergence conditions of many penalty and barrier methods.

5. Concluding Remarks

Various smoothing Newton methods for complementarity problems have been tested on a number of problems [2, 13, 19, 39, 63]. Reported numerical results demonstrate that smoothing methods are extremely promising. Furthermore, notable efforts have been made to overcome some difficulties in using smoothing Newton methods.

- When the function F is only defined in the nonnegative orthant R_+, the iterates must remain in R_+. To get rid of the restriction, Qi, Sun and Zhou [51] proposed a smoothing Newton method for solving nonsmooth equations reformulating the CP(F) by Robinson's normal equation [52].
- The nonsingularity of $H_z(z, \epsilon)$ for all $z \in R^{2n}$ and $\epsilon \in R_{++}$ can not be ensured without the assumption that F is a P_0 function. Kanzow and Pieper [39] suggested to use a gradient step when the system of linear equations in the smoothing Newton methods is not solvable.
- When the derivative of F is too complicated to compute, calculating the derivative of a smoothing function is even more difficult. To avoid computation of the derivatives, smoothing quasi-Newton methods are studied in [15, 43]. Moreover, Li and Fukushima [43] established some properties for global convergence of smoothing quasi-Newton methods.

This paper illustrates basic approaches and review some recent results of smoothing Newton methods for complementary problems. A short paper cannot completely cover all aspects of smoothing methods and all of the more recent developments. We are willing to admit that some important subjects are omitted.

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