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ON $M/G/1$ QUEUES WITH THE FIRST N CUSTOMERS OF EACH BUSY PERIOD RECEIVING EXCEPTIONAL SERVICES

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Abstract This paper studies generalized $M/G/1$ queues in which the first N customers of each busy period receive exceptional services. Applying the supplementary variable approach, we derive the recursion formulas to obtain the generating function of the stationary queue length distribution given that n customers have been served since the beginning of current busy period. Furthermore, we present a computationally tractable scheme which recursively determines the moments of the queue length distribution and the sojourn time distribution. Special cases are treated in detail. Numerical examples are also provided.

1. Introduction

In this paper, we consider a single server queueing system where the service time of each customer depends on the number of customers served before him/her in the current busy period. Customers arrive according to a Poisson process with rate λ. The service discipline is FCFS (first come first served). In each busy period, let B_n $(n \geq 0)$ be the service time when n customers have been served since the beginning of current busy period. We assume that B_n $(n \geq 0)$ are mutually independent but may have different distribution function $B_n(x)$ $(n \geq 0)$. Furthermore, we assume that $B_n(x) = B_N(x)$ for $n \geq N$. Such queueing systems are called the first N exceptional services model.

Such queueing systems are immediately applicable to many fields such as computer systems, telecommunication systems and production systems. For example, nice applicable examples in the modern computer systems are introduced by Li et al. [2].

Several researchers have studied queueing systems in which the service time of a customer depends on the number of customers served in the current busy period. Welch [4] studied a generalized $M/G/1$ queueing system in which the first customer of each busy period receives exceptional service. If we set $N = 1$, Welch's model is considered as a special case of our model. Li et al. [2] studied an $M/M/1$ queueing system in which the service rates depend on the number of customers served since the beginning of current busy period. They obtained the Laplace transform of busy period and some performance measures. For the first N exceptional service model, they also derived a closed formula for the generating function of the stationary queue length distribution. Recently Igaki et al. [1] studied an $M/G/1$ queueing system in which the service time distribution depends on the number of customers served since the beginning of current busy period for the case that N may be infinity. They obtained the transform results for the system idle probability at time t, the busy period, and the number of customers at time t given that n customers have left the system at time t since the beginning of current busy period. They also analyzed the virtual waiting time at time t. However, they did not obtain the definite scheme for computing the moments of the stationary queue length distribution and the sojourn time distribution.

In this paper, by restricting N to finite, we derive the recursion formulas to obtain the generating function of the stationary queue length distribution given that n customers have been served since the beginning of current busy period by applying the supplementary variable approach. Furthermore, we present a computationally tractable scheme which recursively determines the moments of the queue length distribution and the sojourn time distribution.

This paper is constructed as follows. We describe the model and introduce notation in Section 2. In Section 3, we derive a set of Laplace-Stieltjes transform equations using supplementary variable approach. In Section 4, we give some system state probabilities to obtain the moments of the queue length distribution and the sojourn time distribution. In Section 5, we derive the numerical algorithm procedure for the generating function of the stationary queue length distribution. Special cases for $N = 1, 2, 3$ are treated in detail, yielding explicit formulas for the generating functions of the stationary queue length distribution in Section 6. Numerical examples are also provided for some special cases in Section 7.

2. Model and Notation

Consider the generalized $M/G/1$ queues in which the service time distribution depends on the number of customers served since the beginning of current busy period. We assume that customers arrive at a single server queueing system according to a Poisson process with intensity λ. These arriving customers are served under the FCFS discipline, where the customers will be served in order of their arrivals to the system. Let $B_n(x)$ denote the service time distribution function when n customers have been served since the beginning of current busy period. The Laplace-Stieltjes transform (LST) of $B_n(x)$ is defined by $B_n^*(\theta) \equiv \int_0^\infty e^{-\theta x} B_n(dx)$. Furthermore, we assume that the service time distribution becomes stable after some customers have been served in the current busy period. That is, there is a positive integer $N \geq 1$ such that $B_n(x) = B_N(x)$ for $n \geq N$. Let $L(t)$ be the number of customers in the system including the one in the server at time t. Let $M(t)$ be the number of customers served since the beginning of current busy period at time t. If the server is idle at time t, $M(t)$ is defined to be 0. Furthermore, let $\hat{B}(t)$ be the remaining service time if there are some customers in system at time t. We further need the following notation for our subsequent analysis.

$$
\begin{aligned}
&p_0(t) \equiv P(L(t) = 0) \\
&p_{i,j}(x,t)dx \equiv P(L(t) = i, M(t) = j, x < \hat{B}(t) \leq x + dx) \\
&\qquad\qquad\qquad\qquad\qquad (i = 1, 2, \ldots; j = 0, 1, \ldots, N-1) \\
&p_{i,N}(x,t)dx \equiv P(L(t) = i, M(t) \geq N, x < \hat{B}(t) \leq x + dx) \qquad (i = 1, 2, \ldots) \\
&P_{i,j}^*(\theta, t) \equiv \int_0^\infty e^{-\theta x} p_{i,j}(x,t)dx \qquad (i = 1, 2, \ldots; j = 0, 1, \ldots, N)
\end{aligned}
$$

3. Supplementary Variable Approach

In this section, we obtain Laplace-Stieltjes transforms (LST), $P_{i,j}^*(\theta) \equiv \lim_{t\to\infty} P_{i,j}^*(\theta, t)$ $(i \geq 1; 0 \leq j \leq N)$, which are the bases of analysis in the following section.

Suppose that there are no customers in system at $t = 0$, that is, $L(0) = 0$ and $M(0) = 0$. Since the service times are generally distributed, the process $\{L(t), M(t)\}$ does not form a Markov process. To make our system Markovian, we use the supplementary variable approach. Using the remaining service time as a supplementary variable, the joint distribution of the queue length, the number of customers served since the beginning of current busy period and the remaining service time at time t, that is, $\{L(t), M(t), \hat{B}(t)\}$ forms a Markov process. Observing the system state at time t and $t + \Delta t$ and taking the limit of $\Delta t \to 0$, we have the following partial differential difference equations.

$$\frac{dp_0(t)}{dt} = -\lambda p_0(t) + \sum_{j=0}^{N} p_{1,j}(0,t) \tag{1}$$

$$\frac{\partial p_{1,0}(x,t)}{\partial t} - \frac{\partial p_{1,0}(x,t)}{\partial x} = -\lambda p_{1,0}(x,t) + \lambda p_0(t)\frac{B_0(dx)}{dx} \tag{2}$$

$$\frac{\partial p_{i,0}(x,t)}{\partial t} - \frac{\partial p_{i,0}(x,t)}{\partial x} = -\lambda p_{i,0}(x,t) + \lambda p_{i-1,0}(x,t) \qquad (i = 2, 3, \dots) \tag{3}$$

$$\frac{\partial p_{1,j}(x,t)}{\partial t} - \frac{\partial p_{1,j}(x,t)}{\partial x} = -\lambda p_{1,j}(x,t) + p_{2,j-1}(0,t)\frac{B_j(dx)}{dx} \qquad (j = 1, \dots, N-1) \tag{4}$$

$$\frac{\partial p_{i,j}(x,t)}{\partial t} - \frac{\partial p_{i,j}(x,t)}{\partial x} = -\lambda p_{i,j}(x,t) + \lambda p_{i-1,j}(x,t) + p_{i+1,j-1}(0,t)\frac{B_j(dx)}{dx} \tag{5}$$
$$(i = 2, 3, \dots; j = 1, 2, \dots, N-1)$$

$$\frac{\partial p_{1,N}(x,t)}{\partial t} - \frac{\partial p_{1,N}(x,t)}{\partial x} = -\lambda p_{1,N}(x,t) + \{p_{2,N-1}(0,t) + p_{2,N}(0,t)\}\frac{B_N(dx)}{dx} \tag{6}$$

$$\frac{\partial p_{i,N}(x,t)}{\partial t} - \frac{\partial p_{i,N}(x,t)}{\partial x} = -\lambda p_{i,N}(x,t) + \lambda p_{i-1,N}(x,t) + \{p_{i+1,N-1}(0,t) + p_{i+1,N}(0,t)\}\frac{B_N(dx)}{dx} \qquad (i = 2, 3, \dots) \tag{7}$$

Assume that the system is stable. We discuss the stability condition later. Let $p_0 \equiv \lim_{t\to\infty} p_0(t)$ and $p_{i,j}(x) \equiv \lim_{t\to\infty} p_{i,j}(x,t)$ $(i = 1, 2, \dots; j = 0, \dots, N)$. Taking the limit of $t \to \infty$, we obtain the following equilibrium results from (1)–(7) using $\lim_{t\to\infty} \frac{dp_0(t)}{dt} = 0$ and $\lim_{t\to\infty} \frac{\partial p_{i,j}(x,t)}{\partial t} = 0$ $(i = 1, 2, \dots; j = 0, \dots, N)$.

$$\lambda p_0 = \sum_{j=0}^{N} p_{1,j}(0) \tag{8}$$

$$-\frac{dp_{1,0}(x)}{dx} = -\lambda p_{1,0}(x) + \lambda p_0 \frac{B_0(dx)}{dx} \tag{9}$$

$$-\frac{dp_{i,0}(x)}{dx} = -\lambda p_{i,0}(x) + \lambda p_{i-1,0}(x) \qquad (i = 2, 3, \dots) \tag{10}$$

$$-\frac{dp_{1,j}(x)}{dx} = -\lambda p_{1,j}(x) + p_{2,j-1}(0)\frac{B_j(dx)}{dx} \qquad (j = 1, \dots, N-1) \tag{11}$$

$$-\frac{dp_{i,j}(x)}{dx} = -\lambda p_{i,j}(x) + \lambda p_{i-1,j}(x) + p_{i+1,j-1}(0)\frac{B_j(dx)}{dx} \tag{12}$$
$$(i = 2, 3, \dots; j = 1, 2, \dots, N-1)$$

$$-\frac{dp_{1,N}(x)}{dx} = -\lambda p_{1,N}(x) + \{p_{2,N-1}(0) + p_{2,N}(0)\}\frac{B_N(dx)}{dx} \tag{13}$$

$$-\frac{dp_{i,N}(x)}{dx} = -\lambda p_{i,N}(x) + \lambda p_{i-1,N}(x) + \{p_{i+1,N-1}(0) + p_{i+1,N}(0)\}\frac{B_N(dx)}{dx} \quad (i = 2, 3, \dots) \tag{14}$$

Taking the LST's of (9)–(14), we have

$$(\lambda - \theta)P^*_{1,0}(\theta) = \lambda p_0 B^*_0(\theta) - p_{1,0}(0) \tag{15}$$

$$(\lambda - \theta)P^*_{i,0}(\theta) = \lambda P^*_{i-1,0}(\theta) - p_{i,0}(0) \qquad (i = 2, 3, \dots) \tag{16}$$

$$(\lambda - \theta)P^*_{1,j}(\theta) = p_{2,j-1}(0)B^*_j(\theta) - p_{1,j}(0) \qquad (j = 1, 2, \dots, N-1) \tag{17}$$

$$(\lambda - \theta)P^*_{i,j}(\theta) = \lambda P^*_{i-1,j}(\theta) + p_{i+1,j-1}(0)B^*_j(\theta) - p_{i,j}(0) \quad (i = 2, 3, \dots; j = 1, 2, \dots, N-1) \tag{18}$$

$$(\lambda - \theta)P^*_{1,N}(\theta) = \{p_{2,N-1}(0) + p_{2,N}(0)\}B^*_N(\theta) - p_{1,N}(0) \tag{19}$$

$$(\lambda - \theta)P^*_{i,N}(\theta) = \lambda P^*_{i-1,N}(\theta) + \{p_{i+1,N-1}(0) + p_{i+1,N}(0)\}B^*_N(\theta) - p_{i,N}(0) \quad (i = 2, 3, \dots). \tag{20}$$

4. System State Probability

In this section, we give $p_0, p_{i,0}(0)$ $(1 \le i \le N)$ and $p_{1,j}(0)$ $(1 \le i \le N)$ to obtain the moments of the queue length distribution and the sojourn time distribution in Section 5.

4.1. $p_{i,0}$ $(1 \le i \le N)$ in terms of p_0

We express $p_{i,0}(0)$ $(1 \le i \le N)$ in terms of p_0. Substituting $\theta = \lambda$ into (15) and (16), we have

$$p_{1,0}(0) = \lambda p_0 B^*_0(\lambda) \tag{21}$$

$$p_{i,0}(0) = \lambda P^*_{i-1,0}(\lambda) \qquad (i = 2, 3, \dots). \tag{22}$$

Differentiating (15) and (16) $n+1$ times and inserting $\theta = \lambda$, we have

$$-(n+1)P^{*(n)}_{1,0}(\lambda) = \lambda p_0 B^{*(n+1)}_0(\lambda) \tag{23}$$

$$-(n+1)P^{*(n)}_{i,0}(\lambda) = \lambda P^{*(n+1)}_{i-1,0}(\lambda) \qquad (i = 2, 3, \dots). \tag{24}$$

Using (23) and (24), we have

$$P^*_{i,0}(\lambda) = (-1)^i \frac{\lambda^i}{i!} p_0 B^{*(i)}_0(\lambda) \qquad (i = 1, \dots, N). \tag{25}$$

From (21), (22) and (25), we can express p_{i0} $(1 \le i \le N)$ in terms of p_0 as

$$p_{i,0}(0) = (-1)^{i-1} \frac{\lambda^i}{(i-1)!} p_0 B^{*(i-1)}_0(\lambda) \qquad (i = 1, \dots, N). \tag{26}$$

4.2. $p_{1,j}(0)$ $(1 \le j \le N)$ in terms of p_0

We now express $p_{1,j}(0)$ $(1 \le j \le N)$ in terms of p_0. Substituting $\theta = \lambda$ into (17) and (18), we have

$$p_{1,j}(0) = p_{2,j-1}(0)B^*_j(\lambda) \qquad (j = 1, 2, \dots, N-1) \tag{27}$$

$$p_{i,j}(0) = \lambda P^*_{i-1,j}(\lambda) + p_{i+1,j-1}(0)B^*_j(\lambda) \qquad (i = 2, 3, \dots; j = 1, 2, \dots, N-1). \tag{28}$$

Differentiating (17) and (18) $n+1$ times and substituting $\theta = \lambda$, we have

$$-(n+1)P_{1,j}^{*(n)}(\lambda) = p_{2,j-1}(0)B_j^{*(n+1)}(\lambda) \qquad (j = 1,2,\dots,N-1) \tag{29}$$

$$-(n+1)P_{i,j}^{*(n)}(\lambda) = \lambda P_{i-1,j}^{*(n+1)}(\lambda) + p_{i+1,j-1}(0)B_j^{*(n+1)}(\lambda) \tag{30}$$

$$(i = 2,3,\dots\,; j = 1,2,\dots,N-1).$$

Using (26)–(30), we can calculate $p_{i,j}(0)$ $(j = 1,2,\dots,N-1; i = 1,2,\dots,N-j)$ in terms of p_0 by the following numerical algorithm.

Algorithm

for $j = 1$ **to** $N-1$ **do**
  for $i = 1$ to $N-j$ **do**

$$p_{i,j}(0) = \sum_{k=0}^{i-1}(-1)^k \frac{\lambda^k}{k!} p_{i-k+1,j-1}(0)B_j^{*(k)}(\lambda) \tag{31}$$

Hence, $p_{1,j}(0)$ $(j = 1,2,\dots,N-1)$ can be obtained from (31). Finally, we have

$$p_{1,N}(0) = \lambda p_0 - \sum_{j=0}^{N-1} p_{1,j}(0), \tag{32}$$

from (8). It immediately follows that we can express $p_{1,j}(0)$ $(j = 1,2,\dots,N)$ in terms of p_0 from (26), (31) and (32).

4.3. Generating functions

We define the following generating functions.

$$q_j(z) \equiv \sum_{i=1}^{\infty} p_{i,j}(0)z^i \qquad (j = 0,\dots,N) \tag{33}$$

$$Q_j^*(z,\theta) \equiv \sum_{i=1}^{\infty} P_{i,j}^*(\theta)z^i \qquad (j = 0,\dots,N) \tag{34}$$

Using (15)–(20), we have

$$(\lambda - \lambda z - \theta)Q_0^*(z,\theta) = \lambda p_0 B_0^*(\theta)z - q_0(z) \tag{35}$$

$$(\lambda - \lambda z - \theta)Q_j^*(z,\theta) = \left[\frac{q_{j-1}(z)}{z} - p_{1,j-1}(0)\right] B_j^*(\theta) - q_j(z) \qquad (j = 1,2,\dots,N-1) \tag{36}$$

$$(\lambda - \lambda z - \theta)Q_N^*(z,\theta) = \left[\frac{q_{N-1}(z) + q_N(z)}{z} - p_{1,N-1}(0) - p_{1,N}(0)\right] B_N^*(\theta) - q_N(z). \tag{37}$$

Substituting $\theta = \lambda - \lambda z$ into (35)–(37), we have

$$q_0(z) = \lambda p_0 B_0^*(\lambda - \lambda z)z \tag{38}$$

$$q_j(z) = \left[\frac{q_{j-1}(z)}{z} - p_{1,j-1}(0)\right] B_j^*(\lambda - \lambda z) \qquad (j = 1,2,\dots,N-1) \tag{39}$$

$$q_N(z) = \left[\frac{q_{N-1}(z) + q_N(z)}{z} - p_{1,N-1}(0) - p_{1,N}(0)\right] B_N^*(\lambda - \lambda z). \tag{40}$$

Rearranging (40), we have

$$q_N(z) = \frac{\{q_{N-1}(z) - p_{1,N-1}(0)z - p_{1,N}(0)z\}B_N^*(\lambda - \lambda z)}{z - B_N^*(\lambda - \lambda z)}. \tag{41}$$

Furthermore, substituting $\theta = 0$ into (35)–(37), we obtain

$$Q_0^*(z,0) = \frac{p_0 z\{1 - B_0^*(\lambda - \lambda z)\}}{1 - z} \tag{42}$$

$$Q_j^*(z,0) = \frac{\{q_{j-1}(z) - p_{1,j-1}(0)z\}\{1 - B_j^*(\lambda - \lambda z)\}}{\lambda z(1 - z)} \qquad (j = 1, 2, \ldots, N-1) \tag{43}$$

$$Q_N^*(z,0) = \frac{\{q_{N-1}(z) + q_N(z) - p_{1,N-1}(0)z - p_{1,N}(0)z\}\{1 - B_N^*(\lambda - \lambda z)\}}{\lambda z(1 - z)}. \tag{44}$$

4.4. Determination of p_0

Substituting $z = 1$ into (38) and (39), we have

$$q_0(1) = \lambda p_0 \tag{45}$$

$$q_j(1) = q_{j-1}(1) - p_{1,j-1}(0) \qquad (j = 1, 2, \ldots, N-1). \tag{46}$$

Next, differentiating (38) and (39), and substituting $z = 1$, we have

$$q_0^{(1)}(1) = \lambda p_0\{1 - \lambda B_0^{*(1)}(0)\} \tag{47}$$

$$q_j^{(1)}(1) = q_{j-1}^{(1)}(1) - q_{j-1}(1) - \lambda B_j^{*(1)}(0)\{q_{j-1}(1) - p_{1,j-1}(0)\} \qquad (j = 1, 2, \ldots, N-1). \tag{48}$$

Using L'Hospital's rule in (41), we have

$$q_N(1) = \lim_{z \to 1} q_N(z) = \frac{q_{N-1}^{(1)}(1) - p_{1,N-1}(0) - p_{1,N}(0)}{1 + \lambda B_N^{*(1)}(0)}. \tag{49}$$

Furthermore, using L'Hospital's rule in (42)–(44), we finally obtain

$$Q_0^*(1,0) = -\lambda p_0 B_0^{*(1)}(0) \tag{50}$$

$$Q_j^*(1,0) = -\{q_{j-1}(1) - p_{1,j-1}(0)\}B_j^{*(1)}(0) \qquad (j = 1, 2, \ldots, N-1) \tag{51}$$

$$Q_N^*(1,0) = -\{q_{N-1}(1) + q_N(1) - p_{1,N-1}(0) - p_{1,N}(0)\}B_N^{*(1)}(0). \tag{52}$$

Since $p_{1,j}(0)$ $(j = 0, \ldots, N-1)$ are expressed in terms of p_0, we can express $Q_j^*(1,0)$ $(j = 0, \ldots, N)$ in terms of p_0 using (45)–(52). From the normalization condition,

$$p_0 + \sum_{j=0}^{N} Q_j^*(1,0) = 1, \tag{53}$$

we have p_0. Therefore, the system state probabilities immediately follow.

Remark 4.1 It follows from (49) that this queueing system is stable if and only if $1 + \lambda B_N^{*(1)}(0) > 0$, that is, $\lambda \mathbf{E}(B_N) < 1$.

5. Moments of Queue Length and Sojourn Time

In this section we derive a computationally tractable scheme which determines the queue length distribution and the sojourn time distribution in steady state. We assume in this paper that the sojourn time is defined as the time between the arrival epoch and the end of the service time of an arbitrary customer.

5.1. Moments of queue length

We define the generating function of the steady state queue length distribution as

$$R(z) \equiv \mathbf{E}(z^L) = p_0 + \sum_{j=0}^{N} Q_j^*(z,0), \tag{54}$$

where L denotes the steady state queue length. Differentiating (54) n times with respect to z and substituting $z=1$, it is shown that nth factorial moment of L is given as

$$\mathbf{E}[L(L-1)\cdots(L-n+1)] \equiv R^{(n)}(1) = \sum_{j=0}^{N} Q_j^{*(n)}(1,0). \tag{55}$$

Hence, to obtain the nth factorial moment of L, it suffices to calculate $Q_j^{*(n)}(1,0)$ $(j=0,\ldots,N)$. Since $p_0, p_{1,j}(0)$ $(j=0,\ldots,N)$ and $q_j(1)$ $(j=0,\ldots,N)$ explicitly follow from the previous section, we have the following tractable numerical algorithm to calculate $Q_j^{*(n)}(1,0)$ $(j=0,\ldots,N)$ by differentiating (38)–(44) and substituting $z=1$.

Algorithm

for $k=0$ **to** n **do**

begin

$$q_0^{(k+1)}(1) = \lambda p_0\{(-\lambda)^{k+1}B_0^{*(k+1)}(0) + (k+1)(-\lambda)^k B_0^{*(k)}(0)\} \tag{56}$$

$$\begin{aligned} q_j^{(k+1)}(1) &= -(k+1)q_j^{(k)}(1) + [q_{j-1}(1) - p_{1,j-1}(0)](-\lambda)^{k+1}B_j^{*(k+1)}(0) \\ &\quad + (k+1)[q_{j-1}^{(1)}(1) - p_{1,j-1}(0)](-\lambda)^k B_j^{*(k)}(0) \\ &\quad + \sum_{m=2}^{k+1}\binom{k+1}{m} q_{j-1}^{(m)}(1)(-\lambda)^{k+1-m}B_j^{*(k+1-m)}(0) \qquad (j=1,2,\ldots,N-1) \end{aligned} \tag{57}$$

$$\begin{aligned} q_N^{(k)}(1) &= \frac{1}{(k+1)\{1+\lambda B_N^{*(1)}(0)\}}\left[\sum_{m=2}^{k+1}\binom{k+1}{m}(-\lambda)^m B_N^{*(m)}(0) q_N^{(k+1-m)}(1)\right. \\ &\quad + \sum_{m=2}^{k+1}\binom{k+1}{m} q_{N-1}^{(m)}(1)(-\lambda)^{k+1-m}B_N^{*(k+1-m)}(0) \\ &\quad \left. +(k+1)\{q_{N-1}^{(1)} - p_{1,N-1}(0) - p_{1,N}(0)\}(-\lambda)^k B_N^{*(k)}(0)\right] \end{aligned} \tag{58}$$

end

$$Q_0^{*(n)}(1,0) = \begin{cases} \dfrac{p_0}{n+1}[(-\lambda)^{n+1}B_0^{*(n+1)}(0) + (n+1)(-\lambda)^n B_0^{*(n)}(0)] & (n \geq 1) \\ -\lambda p_0 B_0^{*(1)}(0) & (n=0) \end{cases} \tag{59}$$

for $k=1$ **to** n **do**

begin

$$
\begin{aligned}
Q_j^{*(k)}(1,0) = & -kQ_j^{*(k-1)}(1,0) - \frac{1}{k+1}[q_{j-1}(1) - p_{1,j-1}(0)](-\lambda)^k B_j^{*(k+1)}(0) \\
& - [q_{j-1}^{(1)}(1) - p_{1,j-1}(0)](-\lambda)^{k-1} B_j^{*(k)}(0) \\
& - \frac{1}{k+1}\sum_{m=2}^{k}\binom{k+1}{m} q_{j-1}^{(m)}(1)(-\lambda)^{k-m} B_j^{*(k+1-m)}(0) \qquad (j = 1, 2, \ldots, N-1)
\end{aligned} \tag{60}
$$

$$
\begin{aligned}
Q_N^{*(k)}(1,0) = & -kQ_N^{*(k-1)}(1,0) \\
& - \frac{1}{k+1}[q_{N-1}(1) + q_N(1) - p_{1,N-1}(0) - p_{1,N}(0)](-\lambda)^k B_N^{*(k+1)}(0) \\
& - [q_{N-1}^{(1)}(1) + q_N^{(1)}(1) - p_{1,N-1}(0) - p_{1,N}(0)](-\lambda)^{k-1} B_N^{*(k)}(0) \\
& - \frac{1}{k+1}\sum_{m=2}^{k}\binom{k+1}{m}[q_{N-1}^{(m)}(1) + q_N^{(m)}(1)](-\lambda)^{k-m} B_N^{*(k+1-m)}(0)
\end{aligned} \tag{61}
$$

end

5.2. Moments of sojourn time

Let S and $S^*(\theta)$ be the steady state sojourn time of an arbitrary customer and its LST, respectively. Since the arrival process is Poissonian and the service discipline is FCFS, the distribution of L and S are related by

$$R(z) = S^*(\lambda - \lambda z). \tag{62}$$

Hence, we have

$$\mathbf{E}[L(L-1)\ldots(L-n+1)] = R^{(n)}(1) = (-1)^n \lambda^n S^{*(n)}(0) = \lambda^n \mathbf{E}(S^n). \tag{63}$$

We obtain from (63) that the nth moment of S can be calculated from the nth factorial moment of L.

6. Special Cases

In this section we derive explicit formulas for the generating function of the stationary queue length distribution, $R(z)$, and the LST of sojourn time distribution, $S^*(\theta)$, for the special cases that N equals 1, 2 and 3. Substituting $z = 1 - \lambda/\theta$ in (62), we have

$$S^*(\theta) = R(1 - \theta/\lambda). \tag{64}$$

Hence, if the explicit formula for $R(z)$ is found, then we have the explicit formula for $S^*(\theta)$ from (64).

(1) $N = 1$

$$
\begin{aligned}
R(z) & \equiv p_0 + Q_0^*(z,0) + Q_1^*(z,0) \\
& = \frac{p_0\{zB_0^*(\lambda - \lambda z) - B_1^*(\lambda - \lambda z)\}}{z - B_1^*(\lambda - \lambda z)}
\end{aligned} \tag{65}
$$

where

$$p_0 = \frac{1 - \lambda \mathbf{E}(B_1)}{1 + \lambda\{\mathbf{E}(B_0) - \mathbf{E}(B_1)\}}$$

$$S^*(\theta) = \frac{p_0\{(\lambda-\theta)B_0^*(\theta) - \lambda B_1^*(\theta)\}}{\lambda - \theta - \lambda B_1^*(\theta)} \tag{66}$$

Remark 6.1 The queueing system for $N = 1$ coincides with the queueing system studied by Welch [4] and the explicit formulas of $R(z)$ and $S^*(\theta)$ coincide with the results in Welch [4] and Takagi [3].

(2) $N = 2$

$$\begin{aligned} R(z) &\equiv p_0 + Q_0^*(z,0) + Q_1^*(z,0) + Q_2^*(z,0) \\ &= \frac{p_0}{z - B_2^*(\lambda - \lambda z)} [zB_0^*(\lambda - \lambda z) - B_2^*(\lambda - \lambda z) \\ &+\{B_0^*(\lambda - \lambda z) - B_0^*(\lambda)\}\{B_1^*(\lambda - \lambda z) - B_2^*(\lambda - \lambda z)\}] \end{aligned} \tag{67}$$

where

$$p_0 = \frac{1 - \lambda \mathbf{E}(B_2)}{1 + \lambda\{\mathbf{E}(B_0) - \mathbf{E}(B_2)\} + \lambda\{1 - B_0^*(\lambda)\}\{\mathbf{E}(B_1) - \mathbf{E}(B_2)\}}$$

$$\begin{aligned} S^*(\theta) &= \frac{p_0}{\lambda - \theta - \lambda B_2^*(\theta)} [(\lambda - \theta)B_0^*(\theta) - \lambda B_2^*(\theta) \\ &+\lambda\{B_0^*(\theta) - B_0^*(\lambda)\}\{B_1^*(\theta) - B_2^*(\theta)\}] \end{aligned} \tag{68}$$

(3) $N = 3$

$$\begin{aligned} R(z) &\equiv p_0 + Q_0^*(z,0) + Q_1^*(z,0) + Q_2^*(z,0) + Q_3^*(z,0) \\ &= \frac{p_0}{z - B_3^*(\lambda - \lambda z)} [zB_0^*(\lambda - \lambda z) - B_3^*(\lambda - \lambda z) \\ &+ \{B_0^*(\lambda - \lambda z) - B_0^*(\lambda)\}\{B_1^*(\lambda - \lambda z) - B_3^*(\lambda - \lambda z)\} \\ &+ B_0^{*(1)}(\lambda)B_1^*(\lambda)\{B_2^*(\lambda - \lambda z) - B_3^*(\lambda - \lambda z)\} \\ &+B_1^*(\lambda - \lambda z)\{B_0^*(\lambda - \lambda z) - B_0^*(\lambda)\}\{B_2^*(\lambda - \lambda z) - B_3^*(\lambda - \lambda z)\}/z] \end{aligned} \tag{69}$$

where

$$p_0 = \frac{1 - \lambda \mathbf{E}(B_3)}{A},$$

and

$$\begin{aligned} A = 1 &+ \lambda\{\mathbf{E}(B_0) - \mathbf{E}(B_3)\} + \lambda\{1 - B_0^*(\lambda)\}\{\mathbf{E}(B_1) - \mathbf{E}(B_3)\} \\ &+ \lambda\{1 - B_0^*(\lambda) + B_0^{*(1)}(\lambda)B_1^*(\lambda)\}\{\mathbf{E}(B_2) - \mathbf{E}(B_3)\} \end{aligned}$$

$$\begin{aligned} S^*(\theta) &= \frac{p_0}{\lambda - \theta - \lambda B_3^*(\theta)} [(\lambda - \theta)B_0^*(\theta) - \lambda B_3^*(\theta) \\ &+ \lambda\{B_0^*(\theta) - B_0^*(\lambda)\}\{B_1^*(\theta) - B_3^*(\theta)\} \\ &+ \lambda B_0^{*(1)}(\lambda)B_1^*(\lambda)\{B_2^*(\theta) - B_3^*(\theta)\} \\ &+\lambda^2 B_1^*(\theta)\{B_0^*(\theta) - B_0^*(\lambda)\}\{B_2^*(\theta) - B_3^*(\theta)\}/(\lambda - \theta)] \end{aligned} \tag{70}$$

7. Numerical Examples

In this section, we present numerical examples to demonstrate the numerical algorithm derived in Section 5.

In Table 1 and Table 2, we show the system idle probability, p_0, the mean queue length, $\mathbf{E}(L)$, and the variance of queue length, $\mathbf{V}(L)$, for an $M/M/1$ queue and an $M/D/1$ queue with $N = 3$, respectively. For both queueing systems, we assume that the arrival rate is $\lambda = 1$. We also assume that the relations among mean service times given that n customers have been served since the beginning of current busy period are

$$\mathbf{E}(B_n) = \frac{1}{n+1}\mathbf{E}(B_0) \qquad (n = 1, 2, 3). \tag{71}$$

Furthermore, we define $\rho_N \equiv \lambda \mathbf{E}(B_N)$. From Remark 4.1, ρ_N is less than unity if and only if the queueing system is stable.

Table 1. System idle probability p_0, mean queue length $\mathbf{E}(L)$ and variance of queue length $\mathbf{V}(L)$ for an $M/M/1$ queue with $N = 3$

ρ_N	p_0	$\mathbf{E}(L)$	$\mathbf{V}(L)$
0.1	0.6755	0.4653	0.7987
0.2	0.4687	1.0210	2.3346
0.3	0.3329	1.6422	4.5140
0.4	0.2390	2.3348	7.3039
0.5	0.1709	3.1287	10.790
0.6	0.1196	4.0926	15.346
0.7	0.0796	5.3918	22.296
0.8	0.0477	7.5258	37.464
0.9	0.0216	12.994	110.34

Table 2. System idle probability p_0, mean queue length $\mathbf{E}(L)$ and variance of queue length $\mathbf{V}(L)$ for an $M/D/1$ queue with $N = 3$

ρ_N	p_0	$\mathbf{E}(L)$	$\mathbf{V}(L)$
0.1	0.6733	0.3983	0.4648
0.2	0.4622	0.7930	1.0223
0.3	0.3241	1.1936	1.6542
0.4	0.2302	1.6160	2.3709
0.5	0.1634	2.0824	3.2129
0.6	0.1138	2.6312	4.2928
0.7	0.0756	3.3470	5.9660
0.8	0.0453	4.4803	9.7399
0.9	0.0206	7.2813	28.204

Numerical examples in Table 1 and Table 2 show the following interesting result. If ρ_N is same for both an $M/M/1$ queue and an $M/D/1$ queue, $\mathbf{E}(L)$ and $\mathbf{V}(L)$ for an $M/M/1$ queue are larger than those for an $M/D/1$ queue. However, for same ρ_N, the system idle probability p_0 for an $M/M/1$ queue is larger than that for an $M/D/1$ queue. This is a remarkable phenomenon.

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