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MATRIX PRODUCT-FORM SOLUTIONS OF STATIONARY PROBABILITIES IN TANDEM QUEUES

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Abstract Fujimoto *et al.*, have proved that the tail of the joint queue length distribution in a two-stage tandem queueing system has the geometric decay property. We continue to investigate the properties of stationary distributions in this tandem queueing system. Under the same conditions proposed by them, it is further shown that the stationary probability of the saturated states in the $PH/PH/c_1 \rightarrow /PH/c_2$ queue has a linear combination of product-forms. The method of linear combination of product-forms is presented in a QBD process with a countable number of phases in each level. We show that each component of these products can be expressed in terms of roots of the associated characteristic polynomials which involve only the Laplace-Stieltjes transforms of the interarrival and service time distributions.

1. Introduction

Fujimoto et al. [7] studied the asymptotic properties of the joint queue length distribution in a two-stage tandem queueing system where both interarrival times and service times are of phase type. The system is constructed with an infinite buffer and multiple servers in each stage and is denoted by $PH/PH/c_1 \rightarrow /PH/c_2$. They proved that its stationary distribution has a geometric tail, i.e., the asymptotic probability distribution of the number of customers, either that in the first queue or in the second queue goes to infinity, has a geometric decay property. Here, we provide a different approach to solve the same problem. In specific, we not only investigate the tail stationary probability but also discuss the property of the stationary distribution for saturated states (to be defined later). We prove that they can be written as a linear combination of product-forms. The method of linear combination of product-forms is presented in a quasi-birth-and-death (QBD) process with a countable number of phases in each level. To our knowledge, the result has never been reported so far in the literature.

Bertsimas [4] studied a $C_k/C_r/c$ queue. He showed that the equilibrium probabilities for saturated states are geometric in the number of waiting customers by using a generating function technique. In [3], he showed that the waiting time distribution under first-come-first-served (FCFS) discipline for the $C_k/C_r/c$ system can be expressed as a mixture of exponential distributions. Adan et al. [1] showed that in the $E_k/E_r/c$ queues the equilibrium probabilities for saturated states can be expressed as a linear combination of terms that are geometric in each of the state variables. Neuts and Takahashi [11] showed the stationary distribution of the queue length at arrivals has an exact geometric tail of rate between 0 and 1 in the $GI/PH/c$ queue with heterogeneous servers. It was further shown that the stationary waiting time distribution at arrivals has an exact exponential tail of a positive decay parameter. For a tandem queueing system $PH/PH/1 \rightarrow /PH/1$, Fujimoto and Takahashi [6] tested various types of models with various traffic intensities at stage 1 and 2.

They conjectured from all observed results that the joint queue-length probability $p(n_1, n_2)$ may be approximated by $G\eta_1^{n_1}\eta_2^{n_2}$ as $n_1, n_2 \to \infty$, where the coefficient G and the decay rates η_1, η_2 are depending on n_1, n_2. Similar models and results were investigated by Ganesh and Anantharam for $GI/M/1 \to /M/1$ in [5].

In the present paper, we show that the stationary distribution of the number of customers in the system when all servers are busy is a linear combination of product-forms. The solution technique is based on a novel approach that was taken to solve a $PH/PH/1$ system by Le Boudec [10]. He showed that all the eigenvectors used in the expression of the stationary probability of the $PH/PH/1$ system are Kronecker products and gave a simple formula for computing the stationary probability of the number of customers in the system. The essential idea of his approach is to avoid the integration of equations of saturated and unsaturated states. Because the solution of $PH/PH/1$ can be expressed in terms of roots of the associated characteristic polynomial, we may reduce the state balance equations to a vector difference equation with constant coefficients for a basis of separable solution of the equation of saturated states. We use this basis to construct a linear combination that also satisfies the conditions at boundaries of the state space. We solve a $PH/PH/c_1 \to /PH/c_2$ queueing system by showing the Matrix-geometric form solution of a QBD with a countable number of phases in each level. The result yields a new expression of the stationary distribution and may be used to compute other performance measures, such as the delay probability, the moments of the queue size distribution and the waiting distribution.

The remainder of the paper is organized as follows. In Section 2, we review some related work to this problem, especially, results in [7]. The theorems for a single-server $PH/PH/1 \to /PH/1$ system are presented in Section 3 and Section 4. In both sections, we construct the basis solutions that satisfy the state balance equations for two different cases which correspond to the traffic intensities at both stages. In Section 5, we give an outline of the proof for the multi-server $PH/PH/c_1 \to /PH/c_2$ system. All technique lemmas are proved in appendices. Finally, The paper is summarized and concluded in Section 6.

2. Model Formulation and Preliminary Results

In this section, we describe the standard form of the phase representation for queueing systems. Our goal is to outline the method in preparation for modification to be considered in the following sections. Several important theoretical results will be stated without proof. We employ notations that are consistent with [7] where there is only a few very minor exceptions.

Consider a two-stage $PH/PH/c_1 \to /PH/c_2$ system. Each stage has c_k servers and a buffer of infinite capacity, $k = 1, 2$. Service times of each server j, $j = 1, 2, \cdots, c_k$, are independent and identically distributed (i.i.d.) random variables subjecting to an irreducible phase-type distribution $PH(\boldsymbol{\beta}_{kj}, \mathbf{S}_{kj})$ with J_{kj} phases and service rate μ_{kj}. Interarrival times of customers are also i.i.d. random variables subjecting to an irreducible phase-type distribution $PH(\boldsymbol{\alpha}, \mathbf{T})$ with J_0 phases and arrival rate λ. Assume the service times and interarrival times are mutually independent. The service discipline is FCFS and to randomly choose any idle server according to state-dependent probabilities. We denote the traffic intensity at the stage k by $\rho_k = \lambda/\mu_k$, where $\mu_k = \sum_{j=1}^{c_k} \mu_{kj}$. Assume ρ_k, $k = 1, 2$, are less than 1 so that the chain is stable and has stationary state probabilities. The state of the system is represented by a vector $(n_1, n_2; i_0, i_{11}, \cdots, i_{1c_1}; i_{21}, \cdots, i_{2c_2})$, where n_k is the number of customers (including those in service) in the k-th stage, i_0 is the phase of the arrival process, and i_{kj} is the phase of the service process at the j-th server of the k-th stage,

$j = 1, 2, \cdots, c_k$, $k = 1, 2$. The index i_{kj} is interpreted to be zero if the corresponding server is idle. Under the stability condition of the system, the stationary state probabilities may be obtained and denoted by $\pi(n_1, n_2; i_0, i_{11}, \cdots, i_{1c_1}; i_{21}, \cdots, i_{2c_2})$. Sometimes for a concise representation, it is written as $\boldsymbol{\pi}_{n_1,n_2}$ in its simple expression. The saturated stationary probability is defined for $n_1 \geq c_1$ and $n_2 \geq c_2$. Otherwise, it is called a unsaturated stationary probability.

Denote by $\mathbf{I}$ the identity matrix and $\mathbf{e}'$ the column vector of all entries equal to 1. The order of them may be any positive integer according to its suffix which is defined by the corresponding $\mathbf{T}$ or $\mathbf{S}_{kj}$. Let

$$\boldsymbol{\gamma}_0 \triangleq -\mathbf{T}\mathbf{e}' \quad \text{and} \quad \boldsymbol{\gamma}_{kj} \triangleq -\mathbf{S}_{kj}\mathbf{e}'_{kj}$$

Denote by $T^*(s)$ and $S^*_{kj}(s)$ the Laplace-Stieltjes Transforms (LST) of the interarrival and service time distributions respectively. It is known that

$$T^*(s) = \boldsymbol{\alpha}(s\mathbf{I}_0 - \mathbf{T})^{-1}\boldsymbol{\gamma}_0 \quad \text{and} \quad S^*_{kj}(s) = \boldsymbol{\beta}_{kj}(s\mathbf{I}_{kj} - \mathbf{S}_{kj})^{-1}\boldsymbol{\gamma}_{kj}.$$

To solve a two-stage $PH/PH/c_1 \rightarrow /PH/c_2$ queueing system, we need to discuss the following two cases according to their intensities, ρ_1 and ρ_2.

Case I: If $\rho_1 \geq \rho_2$, we consider the system of equations (2.1) and (2.2).

$$\begin{cases} T^*(s_0) & = & h \\ S^*_{1j}(s_{1j}) & = & h^{-1} \quad j = 1, 2, ..., c_1 \\ s_0 + s_{11} + \cdots + s_{1c_1} & = & 0 \end{cases} \tag{2.1}$$

Suppose one of solutions for the system of equations (2.1) is $(h, s_0, s_{11}, ..., s_{1c_1})$=$(\eta_1$,$\sigma_0$,$\sigma_{11}$,..., $\sigma_{1c_1})$. It was proved by Fujimoto et al. [7] that $0 < \eta_1 < 1$, $\sigma_0 > 0$ and $\sigma_{1j} < 0$. Using η_1 defined above we consider another system of equations for $h, s_0, s_{11}, ..., s_{1c_1}, s_{21}, ..., s_{2c_2}$,

$$\begin{cases} T^*(s_0) = \eta_1 \\ S^*_{1j}(s_{1j}) = \eta_1^{-1}h \quad\quad j = 1, 2, ..., c_1 \\ S^*_{2j}(s_{2j}) = h^{-1} \quad\quad j = 1, 2, ..., c_2 \\ s_0 + s_{11} + \cdots + s_{1c_1} + s_{21} + \cdots + s_{2c_2} = 0 \end{cases} \tag{2.2}$$

Suppose one of the solution is $(h, s_0, s_{11}, ..., s_{1c_1}, s_{21}, ..., s_{2c_2}) = (\eta_2, \omega_0, \omega_{11}, ..., \omega_{1c_1}, \omega_{21}, ..., \omega_{2c_2})$. Notice that $\omega_0 = \sigma_0$. Based on the solution, we construct a solution basis for the stationary probabilities of saturated states when $\eta_2 < 1$. All of these solution techniques will be discussed in Section 3.

Case II: On the other hand, if $\rho_1 < \rho_2$, we consider the equations (2.3) and (2.4).

$$\begin{cases} T^*(s_0) & = & h \\ S^*_{2j}(s_{2j}) & = & h^{-1} \quad j = 1, 2, ..., c_1 \\ s_0 + s_{21} + \cdots + s_{2c_2} & = & 0 \end{cases} \tag{2.3}$$

Suppose one of solutions for the system of equations (2.3) is $(h, s_0, s_{21}, ..., s_{2c_2})$=$(\bar{\eta}_2$,$\bar{\sigma}_0$,$\bar{\sigma}_{21}$,..., $\bar{\sigma}_{2c_2})$. It was proved by Fujimoto et al. [7] that $0 < \bar{\eta}_2 < 1$, $\bar{\sigma}_0 > 0$ and $\bar{\sigma}_{2j} < 0$. Using $\bar{\eta}_2$ defined above we consider another system of equations for $(h, s_0, s_{11}, ..., s_{1c_1}, s_{21}, ..., s_{2c_2})$,

$$\begin{cases} T^*(s_0) = h \\ S^*_{1j}(s_{1j}) = \bar{\eta}_2 h^{-1} \quad\quad j = 1, 2, ..., c_1 \\ S^*_{2j}(s_{2j}) = \bar{\eta}_2^{-1} \quad\quad j = 1, 2, ..., c_2 \\ s_0 + s_{11} + \cdots + s_{1c_1} + s_{21} + \cdots + s_{2c_2} = 0 \end{cases} \tag{2.4}$$

Suppose one of the solution is $(h, s_0, s_{11}, ..., s_{1c_1}, s_{21}, ..., s_{2c_2}) = (\bar{\eta}_1, \bar{\omega}_0, \bar{\omega}_{11}, ..., \bar{\omega}_{1c_1}, \bar{\omega}_{21}, ..., \bar{\omega}_{2c_2})$. Notice that $\bar{\omega}_{2j} = \bar{\sigma}_{2j}$. Again, we may construct a solution basis for the stationary probabilities of saturated states when $\bar{\eta}_1 < 1$. All these solution techniques will be discussed in Section 4.

In order to discuss the solution technique of solving the $PH/PH/c_1 \to PH/c_2$ queueing system clearly, without been confused by massive notations, we begin by solving $PH/PH/1 \to PH/1$ first.

Thus, we shall consider case I with $c_1 = c_2 = 1$ in Section 3 and case II with $c_1 = c_2 = 1$ in Section 4. It will also be discussed in both sections that how stationary probabilities are obtained. This includes the existence of a solution basis for saturated states and how an algorithmic procedure of obtaining the stationary probabilities for unsaturated states. We also show extensions of the present approach to more complicated distributions are possible although the results are correspondingly more complex. This is shown in Section 5 by applying this method to solve the $PH/PH/c_1 \to PH/c_2$ queueing system.

3. $PH/PH/1 \to /PH/1$ Model: Case I

In this section, we consider the $PH/PH/1 \to /PH/1$ system with $\rho_1 \geq \rho_2$. The state of the system is represented by a vector $(n_1, n_2; i_0, i_1, i_2)$, where n_k is the number of customers (including the one in service) in stage $k = 1, 2$, i_0 is the phase of the arrival process, and i_k is the phase of the service process at stage k, $k = 1, 2$. To what it follows, we will show the stationary probabilities for $n_1 > 0$ and $n_2 > 0$ has a product-form. At first, we will write out all the system balance equations and check if it satisfies these equations.

3.1. Balance equations

We arrange the states $(n_1, n_2, i_0, i_1, i_2)$ in lexicographic order and partition of the state space according to n_1, i.e. ,

$$\mathcal{L}_m = \{(n_1, n_2, i_0, i_1, i_2) | n_1 = m\}, \quad m = 0, 1, 2, \cdots.$$

Define $\boldsymbol{\pi}$ the stationary vector partitioned according to $\mathcal{L}_m$'s as:

$$\boldsymbol{\pi} = (\boldsymbol{\pi}_0, \boldsymbol{\pi}_1, \cdots).$$

Let $\mathcal{L}_{mn}$ be the state space which is arranged according to n_1 and n_2, i.e. ,

$$\mathcal{L}_{mn} = \{(n_1, n_2, i_0, i_1, i_2) | n_1 = m, n_2 = n\}, \quad m, n = 0, 1, 2, \cdots$$

Thus connecting the probabilities between $\mathcal{L}_m$ and $\mathcal{L}_{mn}$ for any fixed m may be denoted as

$$\boldsymbol{\pi}_m = (\boldsymbol{\pi}_{m0}, \boldsymbol{\pi}_{m1}, \cdots),$$

where $\boldsymbol{\pi}_{00}$ is a row vector of size J_0, $\boldsymbol{\pi}_{n_1 0}$ of $J_0 J_1$, $\boldsymbol{\pi}_{0 n_2}$ of $J_0 J_2$, and $\boldsymbol{\pi}_{n_1 n_2}$ of $J_0 J_1 J_2$, if n_1, $n_2 > 0$. We denote by $\mathbf{Q}$ the transition rate matrix of the chain corresponding to the arrangement of $\mathcal{L}_m$. Then $\mathbf{Q}$ is of the block-tridiagonal form and written as

$$\mathbf{Q} = \begin{bmatrix} \mathbf{B}_0 & \mathbf{A}_0 & & & \\ \mathbf{C}_1 & \mathbf{B} & \mathbf{A} & & \\ & \mathbf{C} & \mathbf{B} & \mathbf{A} & \\ & & \mathbf{C} & \mathbf{B} & \ddots \\ & & & \ddots & \ddots & \ddots \end{bmatrix}$$

where the submatrices $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$, $\mathbf{A}_0$, $\mathbf{B}_0$, and $\mathbf{C}_1$ are given below

$$\mathbf{A} = \begin{bmatrix} \gamma_0\boldsymbol{\alpha}\otimes\mathbf{I}_1 & & & & \\ & \gamma_0\boldsymbol{\alpha}\otimes\mathbf{I}_1\otimes\mathbf{I}_2 & & & \\ & & \gamma_0\boldsymbol{\alpha}\otimes\mathbf{I}_1\otimes\mathbf{I}_2 & & \\ & & & \gamma_0\boldsymbol{\alpha}\otimes\mathbf{I}_1\otimes\mathbf{I}_2 & \\ & & & & \ddots \end{bmatrix}$$

$$\mathbf{B} = \begin{bmatrix} \mathbf{T}\oplus\mathbf{S}_1 & & & & \\ \mathbf{I}_0\otimes\mathbf{I}_1\otimes\boldsymbol{\gamma}_2 & \mathbf{T}\oplus\mathbf{S}_1\oplus\mathbf{S}_2 & & & \\ & \mathbf{I}_0\otimes\mathbf{I}_1\otimes\boldsymbol{\gamma}_2\boldsymbol{\beta}_2 & \mathbf{T}\oplus\mathbf{S}_1\oplus\mathbf{S}_2 & & \\ & & \mathbf{I}_0\otimes\mathbf{I}_1\otimes\boldsymbol{\gamma}_2\boldsymbol{\beta}_2 & \mathbf{T}\oplus\mathbf{S}_1\oplus\mathbf{S}_2 & \\ & & & \ddots & \ddots \end{bmatrix}$$

$$\mathbf{C} = \begin{bmatrix} \mathbf{0} & \mathbf{I}_0\otimes\boldsymbol{\gamma}_1\boldsymbol{\beta}_1\otimes\boldsymbol{\beta}_2 & & & \\ & \mathbf{0} & \mathbf{I}_0\otimes\boldsymbol{\gamma}_1\boldsymbol{\beta}_1\otimes\mathbf{I}_2 & & \\ & & \mathbf{0} & \mathbf{I}_0\otimes\boldsymbol{\gamma}_1\boldsymbol{\beta}_1\otimes\mathbf{I}_2 & \\ & & & \mathbf{0} & \ddots \\ & & & & \ddots \end{bmatrix}$$

$$\mathbf{A}_0 = \begin{bmatrix} \gamma_0\boldsymbol{\alpha}\otimes\boldsymbol{\beta}_1 & & & & \\ & \gamma_0\boldsymbol{\alpha}\otimes\boldsymbol{\beta}_1\otimes\mathbf{I}_2 & & & \\ & & \gamma_0\boldsymbol{\alpha}\otimes\boldsymbol{\beta}_1\otimes\mathbf{I}_2 & & \\ & & & \ddots & \\ & & & & \ddots \end{bmatrix}$$

$$\mathbf{B}_0 = \begin{bmatrix} \mathbf{T} & & & \\ \mathbf{I}_0\otimes\boldsymbol{\gamma}_2 & \mathbf{T}\oplus\mathbf{S}_2 & & \\ & \mathbf{I}_0\otimes\boldsymbol{\gamma}_2\boldsymbol{\beta}_2 & \mathbf{T}\oplus\mathbf{S}_2 & \\ & & \ddots & \ddots \\ & & & \ddots \quad \ddots \end{bmatrix}$$

and

$$\mathbf{C}_1 = \begin{bmatrix} \mathbf{0} & \mathbf{I}_0\otimes\boldsymbol{\gamma}_1\otimes\boldsymbol{\beta}_2 & & & \\ & \mathbf{0} & \mathbf{I}_0\otimes\boldsymbol{\gamma}_1\otimes\mathbf{I}_2 & & \\ & & \mathbf{0} & \mathbf{I}_0\otimes\boldsymbol{\gamma}_1\otimes\mathbf{I}_2 & \\ & & & \ddots & \ddots \\ & & & & \ddots \quad \ddots \end{bmatrix}$$

With these notations, the balance equations write

$$\begin{cases} \boldsymbol{\pi}_0\mathbf{B}_0 + \boldsymbol{\pi}_1\mathbf{C}_1 = \mathbf{0} & (3.1) \\ \boldsymbol{\pi}_0\mathbf{A}_0 + \boldsymbol{\pi}_1\mathbf{B} + \boldsymbol{\pi}_2\mathbf{C} = \mathbf{0} & (3.2) \\ \boldsymbol{\pi}_{m-1}\mathbf{A} + \boldsymbol{\pi}_m\mathbf{B} + \boldsymbol{\pi}_{m+1}\mathbf{C} = \mathbf{0} \qquad m > 1 & (3.3) \end{cases}$$

Specifically, expanding (3.3) according to $\mathcal{L}_{mn}$ with respect to $n > 1$, $n = 1$ and $n = 0$ is written as follows.

For $n > 1$, it is written as

$$\boldsymbol{\pi}_{m,n}(\mathbf{T} \oplus \mathbf{S}_1 \oplus \mathbf{S}_2) = \\ \boldsymbol{\pi}_{m-1,n}(\mathbf{T}\mathbf{e}_0'\boldsymbol{\alpha} \otimes \mathbf{I}_1 \otimes \mathbf{I}_2) + \boldsymbol{\pi}_{m,n+1}(\mathbf{I}_0 \otimes \mathbf{I}_1 \otimes \mathbf{S}_2\mathbf{e}_2'\boldsymbol{\beta}_2) + \boldsymbol{\pi}_{m+1,n-1}(\mathbf{I}_0 \otimes \mathbf{S}_1\mathbf{e}_1'\boldsymbol{\beta}_1 \otimes \mathbf{I}_2) \tag{3.4}$$

For $n = 1$, it is written as

$$\boldsymbol{\pi}_{m,1}(\mathbf{T} \oplus \mathbf{S}_1 \oplus \mathbf{S}_2) = \\ \boldsymbol{\pi}_{m-1,1}(\mathbf{T}\mathbf{e}_0'\boldsymbol{\alpha} \otimes \mathbf{I}_1 \otimes \mathbf{I}_2) + \boldsymbol{\pi}_{m,2}(\mathbf{I}_0 \otimes \mathbf{I}_1 \otimes \mathbf{S}_2\mathbf{e}_2'\boldsymbol{\beta}_2) + \boldsymbol{\pi}_{m+1,0}(\mathbf{I}_0 \otimes \mathbf{S}_1\mathbf{e}'\boldsymbol{\beta}_1 \otimes \boldsymbol{\beta}_2) \tag{3.5}$$

For $n = 0$, it is written as

$$\boldsymbol{\pi}_{m,0}(\mathbf{T} \oplus \mathbf{S}_1) = \\ \boldsymbol{\pi}_{m-1,0}(\mathbf{T}\mathbf{e}_0'\boldsymbol{\alpha} \otimes \mathbf{I}_1) + \boldsymbol{\pi}_{m,1}(\mathbf{I}_0 \otimes \mathbf{I}_1 \otimes \mathbf{S}_2\mathbf{e}_2'). \tag{3.6}$$

Expanding (3.2) according to $\mathcal{L}_{mn}$ with respect to $n > 1$, $n = 1$ and $n = 0$ is written as follows. We have
for $n > 1$,

$$\boldsymbol{\pi}_{1,n}(\mathbf{T} \oplus \mathbf{S}_1 \oplus \mathbf{S}_2) = \\ \boldsymbol{\pi}_{0,n}(\mathbf{T}\mathbf{e}_0'\boldsymbol{\alpha} \otimes \boldsymbol{\beta}_1 \otimes \mathbf{I}_2) + \boldsymbol{\pi}_{1,n+1}(\mathbf{I}_0 \otimes \mathbf{I}_1 \otimes \mathbf{S}_2\mathbf{e}_2'\boldsymbol{\beta}_2) + \boldsymbol{\pi}_{2,n-1}(\mathbf{I}_0 \otimes \mathbf{S}_1\mathbf{e}_1'\boldsymbol{\beta}_1 \otimes \mathbf{I}_2) \tag{3.7}$$

for $n = 1$,

$$\boldsymbol{\pi}_{1,1}(\mathbf{T} \oplus \mathbf{S}_1 \oplus \mathbf{S}_2) = \\ \boldsymbol{\pi}_{0,1}(\mathbf{T}\mathbf{e}_0'\boldsymbol{\alpha} \otimes \boldsymbol{\beta}_1 \otimes \mathbf{I}_2) + \boldsymbol{\pi}_{1,2}(\mathbf{I}_0 \otimes \mathbf{I}_1 \otimes \mathbf{S}_2\mathbf{e}_2'\boldsymbol{\beta}_2) + \boldsymbol{\pi}_{2,0}(\mathbf{I}_0 \otimes \mathbf{S}_1\mathbf{e}_1'\boldsymbol{\beta}_1 \otimes \boldsymbol{\beta}_2) \tag{3.8}$$

and for $n = 0$

$$\boldsymbol{\pi}_{1,0}(\mathbf{T} \oplus \mathbf{S}_1) = \\ \boldsymbol{\pi}_{0,0}(\mathbf{T}\mathbf{e}_0'\boldsymbol{\alpha} \otimes \boldsymbol{\beta}_1) + \boldsymbol{\pi}_{1,1}(\mathbf{I}_0 \otimes \mathbf{I}_1 \otimes \mathbf{S}_2\mathbf{e}_2'). \tag{3.9}$$

Expanding (3.1) according to $\mathcal{L}_{mn}$ with respect to $n > 1$, $n = 1$ and $n = 0$ is written as follows. We have
for $n > 1$,

$$\boldsymbol{\pi}_{0,n}(\mathbf{T} \oplus \mathbf{S}_2) = \boldsymbol{\pi}_{1,n-1}(\mathbf{I}_0 \otimes \mathbf{S}_1\mathbf{e}_1' \otimes \mathbf{I}_2) + \boldsymbol{\pi}_{0,n+1}(\mathbf{I}_0 \otimes \mathbf{S}_2\mathbf{e}'\boldsymbol{\beta}_2) \tag{3.10}$$

for $n = 1$,

$$\boldsymbol{\pi}_{0,1}(\mathbf{T} \oplus \mathbf{S}_2) = \boldsymbol{\pi}_{1,0}(\mathbf{I}_0 \otimes \mathbf{S}_1\mathbf{e}_1' \otimes \boldsymbol{\beta}_2) + \boldsymbol{\pi}_{0,2}(\mathbf{I}_0 \otimes \mathbf{S}_2\mathbf{e}_2'\boldsymbol{\beta}_2) \tag{3.11}$$

and for $n = 0$

$$\boldsymbol{\pi}_{0,0}\mathbf{T} = \boldsymbol{\pi}_{0,1}(\mathbf{I}_0 \otimes \mathbf{S}_2\mathbf{e}_2'). \tag{3.12}$$

The symbols $\oplus$ and $\otimes$ are algebra operators performed as a Kronecker sum and a Kronecker product respectively. They were defined in Bellman [2] and were used to simplify the representation of the system of balance equations for queues by many researchers, for example [10], [11] and [7].

3.2. Product-form solutions

To solve the $PH/PH/1 - PH/1$ queueing system, we begin by solving (2.1) for h, s_0 and s_1. It was proved by Le Boudec [9] that s_0 has J_1 solutions with positive real parts, namely $z_1, \cdots, z_{J_1}$. Then, we shall solve the system of equations (2.2) for h, s_0, s_1 and s_2. This is stated in the following lemma.

Lemma 3.1 *The polynomial equation if $\rho_1 \geq \rho_2$ and $\eta_1 < 1$*

$$S_1^*(-\omega_0 - s)S_2^*(s) = \eta_1^{-1} \tag{3.13}$$

has J_2 complex solutions with negative real parts: $z_1, \cdots, z_{J_2}$.

The proof of this lemma is provided in Appendix A.

Now, we look for a solution $\mathbf{w}_{m,n}$ of (3.4) which has the form, i.e.,

$$\mathbf{w}_{m,n} = \eta_1^m \eta_2^n (\mathbf{u}_0 \otimes \mathbf{u}_1 \otimes \mathbf{u}_2) \tag{3.14}$$

where $\mathbf{u}_0 \in C^{J_0}, \mathbf{u}_1 \in C^{J_1}, \mathbf{u}_2 \in C^{J_2}, 0 < \eta_1 < 1$ and $0 < \eta_2 < 1$.

We shall require that $\mathbf{u}_0$, $\mathbf{u}_1$ and $\mathbf{u}_2$ satisfy the normalization condition:

$$\mathbf{u}_0\mathbf{e}_0' = \mathbf{u}_1\mathbf{e}_1' = \mathbf{u}_2\mathbf{e}_2' = 1.$$

Suppose one of the solutions for (2.1) and (2.2) is

$$(\eta_1, \eta_2, \omega_0, \omega_1, \omega_2)$$

Let

$$\mathbf{u}_0 = a\boldsymbol{\alpha}(\mathbf{T} - \omega_0\mathbf{I}_0)^{-1}, \quad \mathbf{u}_1 = b_1\boldsymbol{\beta}_1(\mathbf{S}_1 - \omega_1\mathbf{I}_1)^{-1} \text{ and } \mathbf{u}_2 = b_2\boldsymbol{\beta}_2(\mathbf{S}_2 - \omega_2\mathbf{I}_2)^{-1} \tag{3.15}$$

where

$$a = \frac{\omega_0}{\eta_1 - 1}, \qquad b_1 = \frac{\omega_1\eta_1}{\eta_2 - \eta_1}, \qquad b_2 = \frac{\omega_2\eta_2}{1 - \eta_2}, \qquad \text{for } \eta_2 \neq \eta_1.$$

Thus, we have

$$\mathbf{u}_0\mathbf{T} = a\boldsymbol{\alpha} + \omega_0\mathbf{u}_0, \quad \mathbf{u}_1\mathbf{S}_1 = b_1\boldsymbol{\beta}_1 + \omega_1\mathbf{u}_1 \quad \mathbf{u}_2\mathbf{S}_2 = b_2\boldsymbol{\beta}_2 + \omega_2\mathbf{u}_2. \tag{3.16}$$

and

$$\mathbf{u}_0\mathbf{T}\mathbf{e}_0' = a\eta_1, \quad \mathbf{u}_1\mathbf{S}_1\mathbf{e}_1' = \frac{b_1\eta_2}{\eta_1}, \quad \mathbf{u}_2\mathbf{S}_2\mathbf{e}_2' = \frac{b_2}{\eta_2}. \tag{3.17}$$

Now we check (3.4) for $m > 1$ and $n > 1$. Insertion of (3.14) in (3.4) and then dividing by $\eta_1^m\eta_2^n$ yields

$$(\mathbf{u}_0 \otimes \mathbf{u}_1 \otimes \mathbf{u}_2)(\mathbf{T} \oplus \mathbf{S}_1 \oplus \mathbf{S}_2) = \frac{1}{\eta_1}(\mathbf{u}_0 \otimes \mathbf{u}_1 \otimes \mathbf{u}_2)(\mathbf{T}\mathbf{e}_0'\boldsymbol{\alpha} \otimes \mathbf{I}_1 \otimes \mathbf{I}_2) +$$

$$\eta_2(\mathbf{u}_0 \otimes \mathbf{u}_1 \otimes \mathbf{u}_2)(\mathbf{I}_0 \otimes \mathbf{I}_1 \otimes \mathbf{S}_2\mathbf{e}_2'\boldsymbol{\beta}_2) + \frac{\eta_1}{\eta_2}(\mathbf{u}_0 \otimes \mathbf{u}_1 \otimes \mathbf{u}_2)(\mathbf{I}_0 \otimes \mathbf{S}_1\mathbf{e}'\boldsymbol{\beta}_1 \otimes \mathbf{I}_2). \tag{3.18}$$

Left hand side of (3.18) becomes

$$\mathbf{u}_0\mathbf{T} \otimes \mathbf{u}_1 \otimes \mathbf{u}_2 + \mathbf{u}_0 \otimes \mathbf{u}_1\mathbf{S}_1 \otimes \mathbf{u}_2 + \mathbf{u}_0 \otimes \mathbf{u}_1 \otimes \mathbf{u}_2\mathbf{S}_2$$

Right hand side of (3.18) becomes

$$\frac{1}{\eta_1}\mathbf{u}_0\mathbf{T}\mathbf{e}_0'\boldsymbol{\alpha} \otimes \mathbf{u}_1 \otimes \mathbf{u}_2 + \eta_2\mathbf{u}_0 \otimes \mathbf{u}_1 \otimes \mathbf{u}_2\mathbf{S}_2\mathbf{e}_2'\boldsymbol{\beta}_2 + \frac{\eta_1}{\eta_2}\mathbf{u}_0 \otimes \mathbf{u}_1\mathbf{S}_1\mathbf{e}_1'\boldsymbol{\beta}_1 \otimes \mathbf{u}_2$$

Applying (3.16) in left hand side and (3.17) in right hand side will balances (3.4).

We have obtained J_1 solutions of ω_0 in (2.1) and J_2 solutions of ω_2 in (3.13) which we denote with indices $\imath$ and $\jmath$, respectively. Define

$$\mathbf{w}_{m,n}(\imath, \jmath) \triangleq \eta_1^m(\imath)\eta_2^n(\jmath)[\mathbf{u}_0(\imath, \jmath) \otimes \mathbf{u}_1(\imath, \jmath) \otimes \mathbf{u}_2(\imath, \jmath)] \tag{3.19}$$

Now any linear combination of $\mathbf{w}_{m,n}(\imath, \jmath)$ obviously satisfies the state balance equations for $m > 1$ and $n > 1$.

Lemma 3.2 $\mathbf{w}_{m,n}$ *is a complex solution of general equations (3.4) if*

$$\sum_{m,n,\imath,\jmath} |\mathbf{w}_{m,n}(\imath,\jmath)| < +\infty$$

Since (3.19) satisfies (3.4) and J_1, J_2 are finite, the proof is immediately clear by adopting the stability assumption of this system.

3.3. Algorithm for the unsaturated probabilities

In order to show the existence of a linear combination of $\mathbf{w}_{m,n}(\imath,\jmath)$, we return to the assumed form of the probabilities $\mathbf{w}_{m,n}(\imath,\jmath)$, $m > 0$, $n > 0$, $1 \le \imath \le J_1$ and $1 \le \jmath \le J_2$. We observe that the most general solution under the condition that the roots s_0 and s_2 are distinct must be

$$\sum_{\imath=1}^{J_1}\sum_{\jmath=1}^{J_2} \ell(\imath,\jmath)\mathbf{w}_{m,n}(\imath,\jmath).$$

The coefficient $\ell(\imath,\jmath)$ may be found by solving a system of linear equations in which the number of equations is greater than that of unknowns. This will be explained in following.

Observe that (3.8), (3.9), (3.11) and (3.12) has $J_0J_1J_2 + J_0J_1 + J_0J_2 + J_0$ equations with unknowns $\boldsymbol{\pi}_{11}$, $\boldsymbol{\pi}_{01}$, $\boldsymbol{\pi}_{12}$, $\boldsymbol{\pi}_{20}$, $\boldsymbol{\pi}_{10}$, $\boldsymbol{\pi}_{02}$ and $\boldsymbol{\pi}_{00}$. Since $\boldsymbol{\pi}_{11}$ and $\boldsymbol{\pi}_{12}$ are functions of $\ell(\imath,\jmath)$ and $\boldsymbol{\pi}_{20}$ may be written in terms of $\boldsymbol{\pi}_{10}$ and $\boldsymbol{\pi}_{21}$ which is also a function of $\ell(\imath,\jmath)$, the total number of unknowns are $2J_0J_2 + J_0J_1 + J_0$. Because this forms a linear homogeneous equations, one of them is necessarily substituted by the normalization equation (3.20) as written

$$\sum_{m,n,i_0,i_1,i_2} \boldsymbol{\pi}_{m,n} = 1. \tag{3.20}$$

This equation must sum up all stationary probabilities of possible states in the system to an unity. However, there are infinitely many of $\boldsymbol{\pi}_{m,n}$ for $m = 0$ or $n = 0$ in (3.20). It is not possible to take all of them in (3.20) computationally. One way to resolve this problem is to resort to the stability assumption of this system. Suppose there exists $m^* > 0$ and $n^* > 0$ such that $\boldsymbol{\pi}_{m0} \to 0$ and $\boldsymbol{\pi}_{0n} \to 0$ for all $m > m^*$ and $n > n^*$. Thus, we shall only consider $\boldsymbol{\pi}_{m0}$ for $2 \le m < m^*$ and $\boldsymbol{\pi}_{0n}$ for $2 \le n < n^*$ in (3.20). In addition, $\boldsymbol{\pi}_{m0}$ and $\boldsymbol{\pi}_{0n}$ shall satisfy (3.6) and (3.10). We will first rewrite (3.5) and (3.6) such that $\boldsymbol{\pi}_{m+1,0}$, $m > 1$, is expressed by $\boldsymbol{\pi}_{m1}$, $\boldsymbol{\pi}_{m-1,1}$ and $\boldsymbol{\pi}_{m2}$ as described in (3.21) where $(\mathbf{T} \oplus \mathbf{S}_1)^{-1}$ exists since the inverses of both $\mathbf{T}$ and $\mathbf{S}_1$ exist.

$$\begin{aligned}
&\boldsymbol{\pi}_{m,0}(\mathbf{T}\mathbf{e}_0'\boldsymbol{\alpha} \otimes \mathbf{I}_1)(\mathbf{T} \oplus \mathbf{S}_1)^{-1}(\mathbf{I}_0 \otimes \mathbf{S}_1\mathbf{e}_1'\boldsymbol{\beta}_1 \otimes \boldsymbol{\beta}_2) = \\
&\boldsymbol{\pi}_{m,1}(\mathbf{T} \oplus \mathbf{S}_1 \oplus \mathbf{S}_2) - \boldsymbol{\pi}_{m-1,1}(\mathbf{T}\mathbf{e}_0'\boldsymbol{\alpha} \otimes \mathbf{I}_1 \otimes \mathbf{I}_2) - \boldsymbol{\pi}_{m,2}(\mathbf{I}_0 \otimes \mathbf{I}_1 \otimes \mathbf{S}_2\mathbf{e}_2'\boldsymbol{\beta}_2) - \\
&\boldsymbol{\pi}_{m+1,1}(\mathbf{I}_0 \otimes \mathbf{I}_1 \otimes \mathbf{S}_2\mathbf{e}_2')(\mathbf{T} \oplus \mathbf{S}_1)^{-1}(\mathbf{I}_0 \otimes \mathbf{S}_1\mathbf{e}_2'\boldsymbol{\beta}_1 \otimes \boldsymbol{\beta}_2)
\end{aligned} \tag{3.21}$$

Similarly, $\boldsymbol{\pi}_{0,n}$, $n > 1$, can be rewritten by rearranging (3.8) and (3.9) as described in (3.22) where $(\mathbf{T} \oplus \mathbf{S}_2)^{-1}$ exists.

$$\begin{aligned}
&\boldsymbol{\pi}_{0,n+1}(\mathbf{I}_0 \otimes \mathbf{S}_2\mathbf{e}'\boldsymbol{\beta}_2)(\mathbf{T} \oplus \mathbf{S}_2)^{-1}(\mathbf{T}\mathbf{e}_0'\boldsymbol{\alpha} \otimes \boldsymbol{\beta}_1 \otimes \mathbf{I}_2) = \\
&\boldsymbol{\pi}_{1,n}(\mathbf{T} \oplus \mathbf{S}_1 \oplus \mathbf{S}_2) - \boldsymbol{\pi}_{1,n-1}(\mathbf{I}_0 \otimes \mathbf{S}_1\mathbf{e}_1' \otimes \mathbf{I}_2)(\mathbf{T} \oplus \mathbf{S}_2)^{-1}(\mathbf{T}\mathbf{e}_0'\boldsymbol{\alpha} \otimes \boldsymbol{\beta}_1 \otimes \mathbf{I}_2) - \\
&\boldsymbol{\pi}_{1,n+1}(\mathbf{I}_0 \otimes \mathbf{I}_1 \otimes \mathbf{S}_2\mathbf{e}_2'\boldsymbol{\beta}_2) - \boldsymbol{\pi}_{2,n-1}(\mathbf{I}_0 \otimes \mathbf{S}_1\mathbf{e}_1'\boldsymbol{\beta}_1 \otimes \mathbf{I}_2)
\end{aligned} \tag{3.22}$$

In (3.21) and (3.22), we observe that $\boldsymbol{\pi}_{m,0}$ and $\boldsymbol{\pi}_{0,n}$ are functions of $\ell(\imath,\jmath)$ as well as $\boldsymbol{\pi}_{m,n}$. For each fixed m and n, there are $2J_0J_1J_1$ equations but J_0J_1 and J_0J_2 unknowns with respective to $\boldsymbol{\pi}_{m,0}$ and $\boldsymbol{\pi}_{0,n}$ given $\ell(\imath,\jmath)$. Hence, considering (3.8), (3.9), (3.11), (3.12), (3.20), (3.21) and (3.22) together for fixed m^* and n^*, we have a system of linear nonhomogeneous equations where the number of equations is much greater than that of unknowns. Thus the solution of this problem may not be unique. The uniqueness property is guaranteed only when the steady state probability of this system exists. For the present model, there maybe involves a more complicated procedure needed to select the appropriate basis solutions from the class of available solutions.

There are greatly many literature concerning the solution strategies of this type of problems. Since its topic is not centered to our purpose, we shall only suggest this system of equations is possible to be solved by some popular numerical methods, e.g., the least square algorithm (see [8]). What we want to show is that the solution of such a system of equations does exist since the system is stable. Therefore, a linear combination of product-forms exists if and only if $\ell(\imath,\jmath)$ is determined by a system of linear equations. We present our main result in the following.

Theorem 3.1 *There exists coefficients* $\ell(\imath,\jmath)$ *such that*

$$\boldsymbol{\pi}_{m,n}=\sum_{\imath=1}^{J_1}\sum_{\jmath=1}^{J_2}\ell(\imath,\jmath)\mathbf{w}_{m,n}(\imath,\jmath).$$

The proof is given in Appendix B. Note that a real $\boldsymbol{\pi}$ is not guaranteed since (3.13) may have no real solution.

We will write the algorithm for adjusting the coefficients $\ell(\imath,\jmath)$ as follows:

Step 1 Write $\boldsymbol{\pi}_{m,0}$ and $\boldsymbol{\pi}_{0,n}$ in terms of $\ell(\imath,\jmath)$ by (3.21) and (3.22).

Step 2 Set a linear nonhomogeneous system consisting of equations (3.8), (3.9), (3.11), (3.12), (3.20), (3.21) and (3.22).

Step 3 Solve it by the least square method and obtain $\ell(\imath,\jmath)$ and the unsaturated probabilities.

The cost of computing $\boldsymbol{\pi}$ is briefly discussed here. By presented previously, the cost of computing $\boldsymbol{\pi}$ involves solving two different systems of equations: one is nonlinear for roots of the associated characteristic polynomials in (3.13); the other is linear for unsaturated probabilities and $\ell(\imath,\jmath)$ in Step 2.

In general, to solve (2.1) for $c_1>1$ one may first write s_{1j} in terms of s_0 for each j, in (3.23).

$$T^*(s_0)S^*_{1j}(s_{1j})=1,\qquad j=1,2,\cdots,c_1. \tag{3.23}$$

Afterwards, s_0 can be obtained by applying any method of nonlinear equations, e.g., Newton's method, to solve (3.24).

$$s_0+s_{11}+\cdots+s_{1c_1}=0. \tag{3.24}$$

After s_0 is solved, s_{1j}, $\mathbf{u}_0$, $\mathbf{u}_1$, $\mathbf{u}_2$, and $\mathbf{w}_{m,n}$ in (3.19) can be calculated directly. The computational cost of attaining product-forms $\mathbf{w}_{m,n}$ depends on the method we adopt to solve the nonlinear equation (3.24). The other part is to solve a linear nonhomogeneous system in Step 2. Compared with (3.1), (3.2) and (3.3), its size has been greatly reduced. Moreover, if Newton's method is used to solve the nonlinear equation, its convergent rate is quadratic, which speeds up the solution procedure.

4. $PH/PH/1 \to /PH/1$ Model: Case II

Consider the $PH/PH/1 \to /PH/1$ system with $\rho_1<\rho_2$. We will follow almost the same line of arguments in Section 3 to discuss the existence of a linear combination of product-forms for saturated probabilities in this case. The state of the system is represented by a vector $(n_1, n_2; i_0, i_1, i_2)$, where n_k is the number of customers (including the one in service) in stage $k = 1, 2$, i_0 is the phase of the arrival process, and i_k is the phase of the service process at stage k, $k = 1, 2$.

4.1. Balance equations

We rearrange the states $(n_1, n_2, i_0, i_1, i_2)$ and partition the state space according to n_2, i.e.,

$$\bar{\mathcal{L}}_m = \{(n_1, n_2, i_0, i_1, i_2) | n_2 = m\}, \quad n = 0, 1, 2, \cdots.$$

Define $\bar{\boldsymbol{\pi}}$ the stationary vector partitioned according to $\bar{\mathcal{L}}_m$'s as:

$$\bar{\boldsymbol{\pi}} = (\bar{\boldsymbol{\pi}}_0, \bar{\boldsymbol{\pi}}_1, \cdots)$$

Thus connecting the probabilities between $\bar{\mathcal{L}}_m$ and $\mathcal{L}_{mn}$ for any fixed m may be denoted as

$$\bar{\boldsymbol{\pi}}_m = (\bar{\boldsymbol{\pi}}_{m0}, \bar{\boldsymbol{\pi}}_{m1}, \cdots),$$

where $\bar{\boldsymbol{\pi}}_{00}$ is a row vector of size J_0, $\bar{\boldsymbol{\pi}}_{n_1 0}$ of $J_0 J_2$, $\bar{\boldsymbol{\pi}}_{0 n_2}$ of $J_0 J_1$, and $\bar{\boldsymbol{\pi}}_{n_1 n_2}$ of $J_0 J_1 J_2$, if $n_1, n_2 > 0$. We denote by $\bar{\mathbf{Q}}$ the transition rate matrix of the chain corresponding to the arrangement of $\bar{\mathcal{L}}_m$. Then $\bar{\mathbf{Q}}$ is of the block-tridiagonal form and written as

$$\bar{\mathbf{Q}} = \begin{bmatrix} \bar{\mathbf{B}}_0 & \bar{\mathbf{A}}_0 & & & \\ \bar{\mathbf{C}}_1 & \bar{\mathbf{B}} & \bar{\mathbf{A}} & & \\ & \bar{\mathbf{C}} & \bar{\mathbf{B}} & \bar{\mathbf{A}} & \\ & & \bar{\mathbf{C}} & \bar{\mathbf{B}} & \ddots \\ & & & \ddots & \ddots & \ddots \end{bmatrix}$$

where the submatrices $\bar{\mathbf{A}}$, $\bar{\mathbf{B}}$, $\bar{\mathbf{C}}$ are given below

$$\bar{\mathbf{A}} = \begin{bmatrix} \mathbf{0} & & & & \\ \mathbf{I}_0 \otimes \boldsymbol{\gamma}_1 \otimes \mathbf{I}_2 & \mathbf{0} & & & \\ & \mathbf{I}_0 \otimes \boldsymbol{\gamma}_1 \boldsymbol{\beta}_1 \otimes \mathbf{I}_2 & \mathbf{0} & & \\ & & \mathbf{I}_0 \otimes \boldsymbol{\gamma}_1 \boldsymbol{\beta}_1 \otimes \mathbf{I}_2 & \mathbf{0} & \\ & & & \ddots & \ddots \end{bmatrix}$$

$$\bar{\mathbf{B}} = \begin{bmatrix} \mathbf{T} \oplus \mathbf{S}_2 & \gamma_0 \boldsymbol{\alpha} \otimes \boldsymbol{\beta}_1 \otimes \mathbf{I}_2 & & & \\ & \mathbf{T} \oplus \mathbf{S}_1 \oplus \mathbf{S}_2 & \gamma_0 \boldsymbol{\alpha} \otimes \mathbf{I}_1 \otimes \mathbf{I}_2 & & \\ & & \mathbf{T} \oplus \mathbf{S}_1 \oplus \mathbf{S}_2 & \gamma_0 \boldsymbol{\alpha} \otimes \mathbf{I}_1 \otimes \mathbf{I}_2 & \\ & & & \mathbf{T} \oplus \mathbf{S}_1 \oplus \mathbf{S}_2 & \ddots \\ & & & & \ddots \end{bmatrix}$$

$$\bar{\mathbf{C}} = \begin{bmatrix} \mathbf{I}_0 \otimes \boldsymbol{\gamma}_2 \boldsymbol{\beta}_2 & & & & \\ & \mathbf{I}_0 \otimes \mathbf{I}_1 \otimes \boldsymbol{\gamma}_2 \boldsymbol{\beta}_2 & & & \\ & & \mathbf{I}_0 \otimes \mathbf{I}_1 \otimes \boldsymbol{\gamma}_2 \boldsymbol{\beta}_2 & & \\ & & & \mathbf{I}_0 \otimes \mathbf{I}_1 \otimes \boldsymbol{\gamma}_2 \boldsymbol{\beta}_2 & \\ & & & & \ddots \end{bmatrix}$$

$$\bar{\mathbf{A}}_0 = \begin{bmatrix} \mathbf{0} & & & & \\ \mathbf{I}_0 \otimes \boldsymbol{\gamma}_1 \otimes \boldsymbol{\beta}_2 & \mathbf{0} & & & \\ & \mathbf{I}_0 \otimes \boldsymbol{\gamma}_1 \boldsymbol{\beta}_1 \otimes \boldsymbol{\beta}_2 & \mathbf{0} & & \\ & & \mathbf{I}_0 \otimes \boldsymbol{\gamma}_1 \boldsymbol{\beta}_1 \otimes \boldsymbol{\beta}_2 & \mathbf{0} & \\ & & & \ddots & \ddots \end{bmatrix} \tag{4.1}$$

$$\bar{\mathbf{B}}_0 = \begin{bmatrix} \mathbf{T} & \gamma_0\boldsymbol{\alpha}\otimes\boldsymbol{\beta}_1 & & & \\ & \mathbf{T}\oplus\mathbf{S}_1 & \gamma_0\boldsymbol{\alpha}\otimes\mathbf{I}_1 & & \\ & & \mathbf{T}\oplus\mathbf{S}_1 & \gamma_0\boldsymbol{\alpha}\otimes\mathbf{I}_1 & \\ & & & \ddots & \ddots \end{bmatrix} \tag{4.2}$$

and

$$\bar{\mathbf{C}}_1 = \begin{bmatrix} \mathbf{I}_0\otimes\boldsymbol{\gamma}_2 & & & \\ & \mathbf{I}_0\otimes\mathbf{I}_1\otimes\boldsymbol{\gamma}_2 & & \\ & & \mathbf{I}_0\otimes\mathbf{I}_1\otimes\boldsymbol{\gamma}_2 & \\ & & & \ddots \end{bmatrix} \tag{4.3}$$

With these notations, the state balance equations write

$$\bar{\boldsymbol{\pi}}\bar{Q} = \mathbf{0}.$$

Specifically, for $m > 1$ with respect to $n > 1$, $n = 1$ and $n = 0$ it is written as follows. For $n > 1$, it is

$$\begin{aligned} &\bar{\boldsymbol{\pi}}_{m,n}(\mathbf{T}\oplus\mathbf{S}_1\oplus\mathbf{S}_2) = \\ &\bar{\boldsymbol{\pi}}_{m-1,n+1}(\mathbf{I}_0\otimes\mathbf{S}_1\mathbf{e}_1'\boldsymbol{\beta}_1\otimes\mathbf{I}_2) + \bar{\boldsymbol{\pi}}_{m,n-1}(\mathbf{T}\mathbf{e}_0'\boldsymbol{\alpha}\otimes\mathbf{I}_1\otimes\mathbf{I}_2) + \bar{\boldsymbol{\pi}}_{m+1,n}(\mathbf{I}_0\otimes\mathbf{I}_1\otimes\mathbf{S}_2\mathbf{e}_2'\boldsymbol{\beta}_2). \end{aligned} \tag{4.4}$$

For $n = 1$, it is

$$\begin{aligned} &\bar{\boldsymbol{\pi}}_{m,1}(\mathbf{T}\oplus\mathbf{S}_1\oplus\mathbf{S}_2) = \\ &\bar{\boldsymbol{\pi}}_{m-1,2}(\mathbf{I}_0\otimes\mathbf{S}_1\mathbf{e}_1'\boldsymbol{\beta}_1\otimes\mathbf{I}_2) + \bar{\boldsymbol{\pi}}_{m,0}(\mathbf{T}\mathbf{e}_0'\boldsymbol{\alpha}\otimes\boldsymbol{\beta}_1\otimes\mathbf{I}_2) + \bar{\boldsymbol{\pi}}_{m+1,1}(\mathbf{I}_0\otimes\mathbf{I}_1\otimes\mathbf{S}_2\mathbf{e}_2'\boldsymbol{\beta}_2). \end{aligned} \tag{4.5}$$

For $n = 0$, it is

$$\begin{aligned} &\bar{\boldsymbol{\pi}}_{m,0}(\mathbf{T}\oplus\mathbf{S}_2) = \\ &\bar{\boldsymbol{\pi}}_{m-1,1}(\mathbf{I}_0\otimes\mathbf{S}_1\mathbf{e}_1'\otimes\mathbf{I}_2) + \bar{\boldsymbol{\pi}}_{m+1,0}(\mathbf{I}_0\otimes\mathbf{S}_2\mathbf{e}_2'\boldsymbol{\beta}_2). \end{aligned} \tag{4.6}$$

For $m = 1$ with respect to $n > 1$, $n = 1$ and $n = 0$ it is written as follows. For $n > 1$, it becomes

$$\begin{aligned} &\bar{\boldsymbol{\pi}}_{1,n}(\mathbf{T}\oplus\mathbf{S}_1\oplus\mathbf{S}_2) = \\ &\bar{\boldsymbol{\pi}}_{0,n+1}(\mathbf{I}_0\otimes\mathbf{S}_1\mathbf{e}_1'\boldsymbol{\beta}_1\otimes\boldsymbol{\beta}_2) + \bar{\boldsymbol{\pi}}_{1,n-1}(\mathbf{T}\mathbf{e}_0'\boldsymbol{\alpha}\otimes\mathbf{I}_1\otimes\mathbf{I}_2) + \bar{\boldsymbol{\pi}}_{2,n}(\mathbf{I}_0\otimes\mathbf{I}_1\otimes\mathbf{S}_2\mathbf{e}_2'\boldsymbol{\beta}_2). \end{aligned} \tag{4.7}$$

For $n = 1$, it becomes

$$\begin{aligned} &\bar{\boldsymbol{\pi}}_{1,1}(\mathbf{T}\oplus\mathbf{S}_1\oplus\mathbf{S}_2) = \\ &\bar{\boldsymbol{\pi}}_{0,2}(\mathbf{I}_0\otimes\mathbf{S}_1\mathbf{e}_1'\boldsymbol{\beta}_1\otimes\boldsymbol{\beta}_2) + \bar{\boldsymbol{\pi}}_{1,0}(\mathbf{T}\mathbf{e}_0'\boldsymbol{\alpha}\otimes\boldsymbol{\beta}_1\otimes\mathbf{I}_2) + \bar{\boldsymbol{\pi}}_{2,1}(\mathbf{I}_0\otimes\mathbf{I}_1\otimes\mathbf{S}_2\mathbf{e}_2'\boldsymbol{\beta}_2). \end{aligned} \tag{4.8}$$

For $n = 0$, it becomes

$$\begin{aligned} &\bar{\boldsymbol{\pi}}_{1,0}(\mathbf{T}\oplus\mathbf{S}_2) = \\ &\bar{\boldsymbol{\pi}}_{0,1}(\mathbf{I}_0\otimes\mathbf{S}_1\mathbf{e}_1'\otimes\boldsymbol{\beta}_2) + \bar{\boldsymbol{\pi}}_{2,0}(\mathbf{I}_0\otimes\mathbf{S}_2\mathbf{e}_2'\boldsymbol{\beta}_2). \end{aligned} \tag{4.9}$$

For $m = 0$ with respect to $n > 1$, $n = 1$ and $n = 0$ it is written as follows. For $n > 1$, it becomes

$$\bar{\boldsymbol{\pi}}_{0,n}(\mathbf{T}\oplus\mathbf{S}_1) = \bar{\boldsymbol{\pi}}_{1,n}(\mathbf{I}_0\otimes\mathbf{I}_1\otimes\mathbf{S}_2\mathbf{e}_2') + \bar{\boldsymbol{\pi}}_{0,n-1}(\mathbf{T}\mathbf{e}_0'\boldsymbol{\alpha}\otimes\mathbf{I}_1) \tag{4.10}$$

For $n = 1$, it becomes

$$\bar{\boldsymbol{\pi}}_{0,1}(\mathbf{T}\oplus\mathbf{S}_1) = \bar{\boldsymbol{\pi}}_{1,1}(\mathbf{I}_0\otimes\mathbf{I}_1\otimes\mathbf{S}_2\mathbf{e}_2') + \bar{\boldsymbol{\pi}}_{0,0}(\mathbf{T}\mathbf{e}_0'\boldsymbol{\alpha}\otimes\boldsymbol{\beta}_1) \tag{4.11}$$

For $n = 0$, it becomes

$$\bar{\boldsymbol{\pi}}_{0,0}\mathbf{T} = \bar{\boldsymbol{\pi}}_{1,0}(\mathbf{I}_0\otimes\mathbf{S}_2\mathbf{e}_2') \tag{4.12}$$

4.2. Product-form solutions

To solve the $PH/PH/1-PH/1$ queueing system with $\rho_1<\rho_2$, we begin by considering $PH/PH/1$. First, we shall show $\eta_1<\bar{\eta}_2$ where η_1 and $\bar{\eta}_2$ are solutions with respective to (2.1) and (2.3). Consider $T^*(-x)S^*(x)=1$ in which $S^*(x)$ denotes the LST of a service time distribution with service rate μ. Notice x is a function of μ. It is easy to check that x increases as μ decreases. When σ_1 and $\bar{\sigma}_2$ are solutions defined by (2.1) and (2.3) for $\mu_1>\mu_2$ respectively, we have $\sigma_1<\bar{\sigma}_2$, which implies $-\sigma_1>-\bar{\sigma}_2$. Hence, $T^*(-\sigma_1)<T^*(-\bar{\sigma}_2)$, that is $\eta_1<\bar{\eta}_2$. Given $\bar{\eta}_2$ obtained by (2.3), we solve the system of equations (2.4) which results in $(h,s_0,s_1)=(\bar{\eta}_1,\bar{\omega}_0,\bar{\omega}_1)$. Then we have the following lemma.

Lemma 4.1 *The polynomial equation if $\rho_1<\rho_2$ and $\bar{\eta}_1<\bar{\eta}_2<1$,*

$$T^*(-\bar{\omega}_2-s)S_1^*(s)=\bar{\eta}_2, \tag{4.13}$$

has J_1 complex solutions with negative real parts: $z_1,\cdots,z_{J_1}$.

The proof of this lemma is provided in Appendix C.

Now, we look for a solution $\mathbf{w}_{m,n}$ which has the form, i.e.,

$$\mathbf{w}_{m,n}=\bar{\eta}_1^n\bar{\eta}_2^m(\mathbf{u}_0\otimes\mathbf{u}_1\otimes\mathbf{u}_2), \tag{4.14}$$

where $\mathbf{u}_0\in C^{J_0},\mathbf{u}_1\in C^{J_1},\mathbf{u}_2\in C^{J_2}, 0<\bar{\eta}_1<1$ and $0<\bar{\eta}_2<1$.

We shall require that $\mathbf{u}_0$, $\mathbf{u}_1$ and $\mathbf{u}_2$ satisfy the normalization condition:

$$\mathbf{u}_0\mathbf{e}_0'=\mathbf{u}_1\mathbf{e}_1'=\mathbf{u}_2\mathbf{e}_2'=1.$$

Suppose one of the solutions of (2.3) and (2.4) as described before is

$$(\bar{\eta}_1,\bar{\eta}_2,\bar{\omega}_0,\bar{\omega}_1,\bar{\omega}_2)$$

Let

$$\mathbf{u}_0=a\boldsymbol{\alpha}(\mathbf{T}-\bar{\omega}_0\mathbf{I}_0)^{-1},\quad \mathbf{u}_1=b_1\boldsymbol{\beta}_1(\mathbf{S}_1-\bar{\omega}_1\mathbf{I}_1)^{-1}\ \text{ and }\ \mathbf{u}_2=b_2\boldsymbol{\beta}_2(\mathbf{S}_2-\bar{\omega}_2\mathbf{I}_2)^{-1},$$

where

$$a=\frac{\bar{\omega}_0}{\bar{\eta}_2-1},\qquad b_1=\frac{\bar{\omega}_1\bar{\eta}_2}{\bar{\eta}_1-\bar{\eta}_2},\quad \text{and}\quad b_2=\frac{\bar{\omega}_2\bar{\eta}_1}{1-\bar{\eta}_1},\qquad \text{for } \bar{\eta}_2\neq\bar{\eta}_1.$$

Thus, we have

$$\mathbf{u}_0\mathbf{T}=a\boldsymbol{\alpha}+\omega_0\mathbf{u}_0,\quad \mathbf{u}_1\mathbf{S}_1=b_1\boldsymbol{\beta}_1+\omega_1\mathbf{u}_1,\quad \mathbf{u}_2\mathbf{S}_2=b_2\boldsymbol{\beta}_2+\omega_2\mathbf{u}_2, \tag{4.15}$$

and

$$\mathbf{u}_0\mathbf{T}\mathbf{e}_0'=a\bar{\eta}_2,\quad \mathbf{u}_1\mathbf{S}_1\mathbf{e}_1'=\frac{b_1\bar{\eta}_1}{\bar{\eta}_2},\quad \mathbf{u}_2\mathbf{S}_2\mathbf{e}_2'=\frac{b_2}{\bar{\eta}_1}. \tag{4.16}$$

Now we check (4.14) inserted in (4.4) for $m>1$ and $n>1$. Like that has been done in Section 3, we have

$$(\mathbf{u}_0\otimes\mathbf{u}_1\otimes\mathbf{u}_2)(\mathbf{T}\oplus\mathbf{S}_1\oplus\mathbf{S}_2)=\frac{1}{\bar{\eta}_2}(\mathbf{u}_0\otimes\mathbf{u}_1\otimes\mathbf{u}_2)(\mathbf{T}\mathbf{e}_0'\boldsymbol{\alpha}\otimes\mathbf{I}_1\otimes\mathbf{I}_2)+$$

$$\bar{\eta}_1(\mathbf{u}_0\otimes\mathbf{u}_1\otimes\mathbf{u}_2)(\mathbf{I}_0\otimes\mathbf{I}_1\otimes\mathbf{S}_2\mathbf{e}_2'\boldsymbol{\beta}_2)+\frac{\bar{\eta}_2}{\bar{\eta}_1}(\mathbf{u}_0\otimes\mathbf{u}_1\otimes\mathbf{u}_2)(\mathbf{I}_0\otimes\mathbf{S}_1\mathbf{e}_1'\boldsymbol{\beta}_1\otimes\mathbf{I}_2). \tag{4.17}$$

Left hand side of (4.17) becomes

$$\mathbf{u}_0\mathbf{T}\otimes\mathbf{u}_1\otimes\mathbf{u}_2+\mathbf{u}_0\otimes\mathbf{u}_1\mathbf{S}_1\otimes\mathbf{u}_2+\mathbf{u}_0\otimes\mathbf{u}_1\otimes\mathbf{u}_2\mathbf{S}_2$$

and right hand side of (4.17) becomes

$$\frac{1}{\bar{\eta}_2}\mathbf{u}_0\mathbf{T}\mathbf{e}_0'\boldsymbol{\alpha}\otimes\mathbf{u}_1\otimes\mathbf{u}_2+\bar{\eta}_1\mathbf{u}_0\otimes\mathbf{u}_1\otimes\mathbf{u}_2\mathbf{S}_2\mathbf{e}_2'\boldsymbol{\beta}_2+\frac{\bar{\eta}_2}{\bar{\eta}_1}\mathbf{u}_0\otimes\mathbf{u}_1\mathbf{S}_1\mathbf{e}_1'\boldsymbol{\beta}_1\otimes\mathbf{u}_2.$$

They are equivalent after a little algebra on applying (4.15) to the left hand side and (4.16) to the right hand side.

We have obtained J_2 solutions of $\bar{\omega}_2$ in (2.3) and J_1 solutions of $\bar{\omega}_1$ in (4.13) which we denote with indices $\imath$ and $\jmath$, respectively. Define for each $\imath$ and $\jmath$,

$$\mathbf{w}_{m,n}(\imath,\jmath)\triangleq\bar{\eta}_2^m(\imath)\bar{\eta}_1^n(\jmath)[\mathbf{u}_0(\imath,\jmath)\otimes\mathbf{u}_1(\imath,\jmath)\otimes\mathbf{u}_2(\imath,\jmath)] \tag{4.18}$$

Now any linear combination of $\mathbf{w}_{m,n}(\imath,\jmath)$ obviously satisfies the balance equations for $m>1$ and $n>1$.

Lemma 4.2 $\mathbf{w}_{m,n}$ *is a complex solution of general equations (4.4) if*

$$\sum_{m,n,\imath,\jmath}|w_{m,n}(\imath,\jmath)|<+\infty$$

The proof is omitted for the same reason in Lemma 3.2.

4.3. Algorithm for the unsaturated probabilities

Returning to the assumed form of the probabilities $\mathbf{w}_{m,n}(\imath,\jmath)$, $m>0$, $n>0$, $1\le\imath\le J_2$ and $1\le\jmath\le J_1$. We observe that the most general solution under the condition that the roots s_2 and s_1 are distinct must be

$$\sum_{\imath=1}^{J_2}\sum_{\jmath=1}^{J_1}\ell(\imath,\jmath)\mathbf{w}_{m,n}(\imath,\jmath). \tag{4.19}$$

Adjusting the coefficients $\ell(\imath,\jmath)$ are according to the following two system of equations (4.20) and (4.21) plus the normalization equation (3.20).

$$\begin{aligned}&\bar{\boldsymbol{\pi}}_{m+1,0}(\mathbf{I}_0\otimes\mathbf{S}_2\mathbf{e}_2'\boldsymbol{\beta}_2)(\mathbf{T}\oplus\mathbf{S}_2)^{-1}(\mathbf{T}\mathbf{e}_0'\boldsymbol{\alpha}\otimes\boldsymbol{\beta}_1\otimes\mathbf{I}_2)=\\&\bar{\boldsymbol{\pi}}_{m,1}(\mathbf{T}\oplus\mathbf{S}_1\oplus\mathbf{S}_2)-\bar{\boldsymbol{\pi}}_{m-1,2}(\mathbf{I}_0\otimes\mathbf{S}_1\mathbf{e}_1'\boldsymbol{\beta}_1\otimes\mathbf{I}_2)-\bar{\boldsymbol{\pi}}_{m+1,1}(\mathbf{I}_0\otimes\mathbf{I}_1\otimes\mathbf{S}_2\mathbf{e}_2'\boldsymbol{\beta}_2)-\\&\bar{\boldsymbol{\pi}}_{m-1,1}(\mathbf{I}_0\otimes\mathbf{S}_1\mathbf{e}_1'\otimes\mathbf{I}_2)(\mathbf{T}\oplus\mathbf{S}_2)^{-1}(\mathbf{T}\mathbf{e}_0'\boldsymbol{\alpha}\otimes\boldsymbol{\beta}_1\otimes\mathbf{I}_2)\end{aligned} \tag{4.20}$$

and

$$\begin{aligned}&\bar{\boldsymbol{\pi}}_{0,n}(\mathbf{T}\mathbf{e}_0'\boldsymbol{\alpha}\otimes\mathbf{I}_1)(\mathbf{T}\oplus\mathbf{S}_1)^{-1}(\mathbf{I}_0\otimes\mathbf{S}_1\mathbf{e}_1'\boldsymbol{\beta}_1\otimes\boldsymbol{\beta}_2)=\\&\bar{\boldsymbol{\pi}}_{1,n}(\mathbf{T}\oplus\mathbf{S}_1\oplus\mathbf{S}_2)-\bar{\boldsymbol{\pi}}_{1,n-1}(\mathbf{T}\mathbf{e}_0'\boldsymbol{\alpha}\otimes\mathbf{I}_1\otimes\mathbf{I}_2)-\bar{\boldsymbol{\pi}}_{2,n}(\mathbf{I}_0\otimes\mathbf{I}_1\otimes\mathbf{S}_2\mathbf{e}_2'\boldsymbol{\beta}_2)-\\&\bar{\boldsymbol{\pi}}_{1,n+1}(\mathbf{I}_0\otimes\mathbf{I}_1\otimes\mathbf{S}_2\mathbf{e}_2')(\mathbf{T}\oplus\mathbf{S}_1)^{-1}(\mathbf{I}_0\otimes\mathbf{S}_1\mathbf{e}_1'\boldsymbol{\beta}_1\otimes\boldsymbol{\beta}_2)\end{aligned} \tag{4.21}$$

Thus we may present the second result as the following theorem.

Theorem 4.1 *There exist coefficients* $\ell(\imath,\jmath)$ *such that*

$$\bar{\boldsymbol{\pi}}_{m,n}=\sum_{\imath=1}^{J_2}\sum_{\jmath=1}^{J_1}\ell(\imath,\jmath)\mathbf{w}_{m,n}(\imath,\jmath).$$

The proof is omitted here.

We will write the algorithm for adjusting the coefficients $\ell(\imath,\jmath)$ as follows:

Step 1 Write $\bar{\boldsymbol{\pi}}_{m,0}$ and $\bar{\boldsymbol{\pi}}_{0,n}$ in terms of $\ell(\imath,\jmath)$ by (4.20) and (4.21).

Step 2 Set a linear nonhomogeneous system consisting of equations (3.20), (4.8), (4.9), (4.11), (4.12), (4.20) and (4.21).

Step 3 Solve it by the least square method and obtain $\ell(\imath,\jmath)$ and the unsaturated probabilities.

5. $PH/PH/c_1 \to /PH/c_2$ Model

Since the methodology described in Section 3 and 4 can be applied explicitly to the $PH/PH/c_1 \to /PH/c_2$ system. We shall only give an outline of proofs to the following theorem.

Theorem 5.1 *There exists coefficients $l(i,j)$ such that for $m > 0$, $n > 0$,*

$$\boldsymbol{\pi}_{m,n} = \sum_{i=1}^{\mathrm{J}_1} \sum_{j=1}^{\mathrm{J}_2} l(i,j)\mathbf{w}_{m,n}(i,j)$$

where $\mathrm{J}_1 = \prod_{\mathrm{k}=1}^{c_1} \mathrm{J}_{2\mathrm{k}}$, $\mathrm{J}_2 = \prod_{\mathrm{k}=1}^{c_2} \mathrm{J}_{2\mathrm{k}}$, $\mathbf{w}_{m,n}(i,j)$ is a product-form solution in terms of distributions at each phase, and $l(i,j)$ is the coefficient determined by its state balance equations.

Depending on ρ_1 and ρ_2, there are two cases to discuss for this theorem just like that in Sections 3 and 4. Since these proofs will be derived in the similar manner, we omit the proof for Case II.

Consider the system $PH/PH/c_1 \to /PH/c_2$ for Case I where $\rho_1 \geq \rho_2$. In order to compare the derivation of this section with that in Section 3, we do not change some notations which were used for $c_1 = c_2 = 1$. We believe it shall be self-evident as described in equations. The state of the system is represented by a vector $(n_1, n_2; i_0, \mathbf{i}_1, \mathbf{i}_2)$, where $\mathbf{i}_k$ is a vector which is equal to $(i_{k1}, ..., i_{kc_k})$, where i_{kq} is the phase of the service process at server q in stage k, $q = 1, 2, ..., c_k$ and $k = 1, 2$.

5.1. Balance equations

We arrange the states $(n_1, n_2, i_0, \mathbf{i}_1, \mathbf{i}_2)$ in lexicographic order and partition the state space according to n_1, i.e. ,

$$\mathcal{L}_m = \{(n_1, n_2, i_0, \mathbf{i}_1, \mathbf{i}_2) | n_1 = m\}, \quad m = 0, 1, 2, \cdots.$$

Define $\boldsymbol{\pi}$ the stationary vector partitioned according to $\mathcal{L}_m$'s as:

$$\boldsymbol{\pi} = (\boldsymbol{\pi}_0, \boldsymbol{\pi}_1, \cdots)$$

Let $\mathcal{L}_{mn}$ be the state space which is arranged according to n_1 and n_2, i.e. ,

$$\mathcal{L}_{mn} = \{(n_1, n_2, i_0, \mathbf{i}_1, \mathbf{i}_2) | n_1 = m, n_2 = n\}, \quad m, n = 0, 1, 2, \cdots$$

In this case, $\mathbf{A}$, $\mathbf{B}$ are $\mathbf{C}$ in Section 3 are of the form:

$$\mathbf{A} = \begin{bmatrix} \mathbf{A}_{00} & & & & & & \\ & \mathbf{A}_{11} & & & & & \\ & & \ddots & & & & \\ & & & \mathbf{A}_{c_1c_1} & & & \\ & & & & \mathbf{A}_{c_1c_1} & & \\ & & & & & \mathbf{A}_{c_1c_1} & \\ & & & & & & \ddots \end{bmatrix}$$

where

$$\mathbf{A}_{00} = \boldsymbol{\gamma}_0 \boldsymbol{\alpha} \otimes (\mathbf{I}_{11} \otimes \cdots \otimes \mathbf{I}_{1c_1}),$$
$$\mathbf{A}_{c_1c_1} = \boldsymbol{\gamma}_0 \boldsymbol{\alpha} \otimes (\mathbf{I}_{11} \otimes \cdots \otimes \mathbf{I}_{1c_1}) \otimes (\mathbf{I}_{21} \otimes \cdots \otimes \mathbf{I}_{2c_2});$$

$$\mathbf{B} = \begin{bmatrix} \mathbf{B}_{00} & & & & & & \\ \mathbf{B}_{10} & \mathbf{B}_{11} & & & & & \\ & \ddots & \ddots & & & & \\ & & \mathbf{B}_{c_1c_1-1} & \mathbf{B}_{c_1c_1} & & & \\ & & & \mathbf{B}_{c_1+1c_1} & \mathbf{B}_{c_1c_1} & & \\ & & & & \mathbf{B}_{c_1+1c_1} & \mathbf{B}_{c_1c_1} & \\ & & & & & \ddots & \ddots \end{bmatrix}$$

where

$$\mathbf{B}_{00} = \mathbf{T} \oplus (\mathbf{S}_{11} \oplus \cdots \oplus \mathbf{S}_{1c_1}),$$

$$\mathbf{B}_{c_1c_1} = \mathbf{T} \oplus (\mathbf{S}_{11} \oplus \cdots \oplus \mathbf{S}_{1c_1}) \oplus (\mathbf{S}_{21} \oplus \cdots \oplus \mathbf{S}_{2c_2});$$

$$\mathbf{B}_{c_1+1c_1} = \mathbf{I}_0 \otimes (\mathbf{I}_{11} \otimes \cdots \otimes \mathbf{I}_{1c_1}) \otimes (\boldsymbol{\gamma}_{21}\boldsymbol{\beta}_{21} \oplus \cdots \oplus \boldsymbol{\gamma}_{2c_2}\boldsymbol{\beta}_{2c_2});$$

$$\mathbf{C} = \begin{bmatrix} 0 & \mathbf{C}_{01} & & & & & \\ & 0 & \mathbf{C}_{12} & & & & \\ & & \ddots & \ddots & & & \\ & & & 0 & \mathbf{C}_{c_1c_1+1} & & \\ & & & & 0 & \mathbf{C}_{c_1c_1+1} & \\ & & & & & 0 & \ddots \\ & & & & & & \ddots \end{bmatrix}$$

$$\mathbf{C}_{c_1+1c_1} = \mathbf{I}_0 \otimes ((\boldsymbol{\gamma}_{11}\boldsymbol{\beta}_{11} \oplus \cdots \oplus \boldsymbol{\gamma}_{1c_1}\boldsymbol{\beta}_{1c_1}) \otimes (\mathbf{I}_{21} \otimes \cdots \otimes \mathbf{I}_{2c_2});$$

All these matrices are defined in [7] as well. We do not bother to write out $\mathbf{A}_0$, $\mathbf{B}_0$ and $\mathbf{C}_1$ since we know they are only matrices regulating those unsaturated probabilities. According to observations in Section 3.3, all unsaturated probabilities may be expressed in term of $l(i,j)$ and they can be obtained as well as $l(i,j)$ by the least square method. In the subsequent section, we shall concentrate on the product-form solution of the saturated stationary probabilities.

5.2. Product-form solutions

To solve the $PH/PH/c_1 \to PH/c_2$ queueing system, we begin by considering the system of equations (2.1) for h, s_0, s_{11}, $\cdots$,s_{1c_1}. For each service distribution S_{1j}, $j = 1, 2, .., c_1$, there are J_{1j} complex solutions with positive real parts. This proof is straightforward from [9]. Then (2.2) shall be brought into consider with the solutions of (2.1). Like Lemma 3.1, we shall prove the following lemma as we solve (2.2).

Lemma 5.1 *For each j and k, $j = 1, 2, .., c_1$, $k = 1, 2, .., c_2$, if $\rho_1 \geq \rho_2$ and $\eta_1 < 1$, the polynomial equation*

$$S_{1j}^*(-\omega_0 - s)S_{2k}^*(s) = \eta_1^{-1} \tag{5.1}$$

has J_{2k} complex solutions with negative real parts: $z_1, \cdots, z_{J_{2k}}$.
The proof of this lemma is omitted here.

Suppose $s = \eta_2 < 1$ is attained after solving (5.1). Now, we look for a solution $\mathbf{w}_{m,n}$ which has the form, namely,

$$\mathbf{w}_{m,n} = \eta_1^m \eta_2^n (\mathbf{u}_0 \otimes \mathbf{u}_1 \otimes \mathbf{u}_2) \tag{5.2}$$

where $\mathbf{u}_1 = (\mathbf{u}_{11} \otimes \cdots \otimes \mathbf{u}_{1c_1})$

$$\mathbf{u}_2 = (\mathbf{u}_{21} \otimes \cdots \otimes \mathbf{u}_{2c_2})$$
$$\mathbf{u}_0 \in C^{J_0}, \mathbf{u}_{1j} \in C^{J_{1j}}, \mathbf{u}_{2j} \in C^{J_{2j}}, 0 < \eta_1 < 1, \text{and } 0 < \eta_2 < 1.$$

We shall require that $\mathbf{u}_0$, $\mathbf{u}_{1j}$ and $\mathbf{u}_{2j}$ satisfy the normalization condition:

$$\mathbf{u}_0\mathbf{e}_0' = \mathbf{u}_{1j}\mathbf{e}_{1j}' = \mathbf{u}_{2j}\mathbf{e}_{2j}' = 1, \quad \text{for all } j. \tag{5.3}$$

Suppose one of the solutions of (2.1) and (2.2) is

$$(\eta_1, \eta_2, \omega_0, \omega_{11}, ..., \omega_{1c_1}, \omega_{21}, ..., \omega_{2c_2})$$

Let

$$\mathbf{u}_0 = a\boldsymbol{\alpha}(\mathbf{T}-\omega_0\mathbf{I}_0)^{-1}, \ \ \mathbf{u}_{1j} = b_{1j}\boldsymbol{\beta}_{1j}(\mathbf{S}_{1j}-\omega_{1j}\mathbf{I}_{1j})^{-1} \text{ and } \mathbf{u}_{2j} = b_{2j}\boldsymbol{\beta}_{2j}(\mathbf{S}_{2j}-\omega_{2j}\mathbf{I}_{2j})^{-1} \tag{5.4}$$

where

$$a = \frac{\omega_0}{\eta_1 - 1}, \qquad b_{1j} = \frac{\omega_{1j}\eta_1}{\eta_2 - \eta_1}, \quad \text{and } b_{2j} = \frac{\omega_{2j}\eta_2}{1-\eta_2}, \quad \eta_2 \neq \eta_1.$$

Thus, we have

$$\mathbf{u}_0\mathbf{T} = a\boldsymbol{\alpha} + \omega_0\mathbf{u}_0, \quad \mathbf{u}_{1j}\mathbf{S}_{1j} = b_{1j}\boldsymbol{\beta}_{1j} + \omega_{1j}\mathbf{u}_{1j} \quad \mathbf{u}_{2j}\mathbf{S}_{2j} = b_{2j}\boldsymbol{\beta}_{2j} + \omega_{2j}\mathbf{u}_{2j} \tag{5.5}$$

and

$$\mathbf{u}_0\mathbf{T}\mathbf{e}' = a\eta_1, \quad \mathbf{u}_{1j}\mathbf{S}_{1j}\mathbf{e}' = \frac{b_{1j}\eta_2}{\eta_1}, \quad \mathbf{u}_{2j}\mathbf{S}_{2j}\mathbf{e}' = \frac{b_{2j}}{\eta_2}. \tag{5.6}$$

Let

$$\begin{gathered}\mathbf{S}_1 = \mathbf{S}_{11} \oplus \cdots \oplus \mathbf{S}_{1c_1}, \quad \mathbf{S}_2 = \mathbf{S}_{21} \oplus \cdots \oplus \mathbf{S}_{2c_2};\\ \tilde{\mathbf{S}}_1 = \mathbf{S}_{11}\mathbf{e}_{11}'\boldsymbol{\beta}_{11} \oplus \cdots \oplus \mathbf{S}_{1c_1}\mathbf{e}_{1c_1}'\boldsymbol{\beta}_{1c_1} \quad \tilde{\mathbf{S}}_2 = \mathbf{S}_{21}\mathbf{e}_{21}'\boldsymbol{\beta}_{21} \oplus \cdots \oplus \mathbf{S}_{2c_2}\mathbf{e}_{2c_2}'\boldsymbol{\beta}_{2c_2};\\ \mathbf{I}_1 = \mathbf{I}_{11} \otimes \cdots \otimes \mathbf{I}_{1c_1} \quad \mathbf{I}_2 = \mathbf{I}_{21} \otimes \cdots \otimes \mathbf{I}_{2c_2}.\end{gathered} \tag{5.7}$$

Now we check (5.2) inserted in the state balance for $m > 1$ and $n > 1$. It results in

$$\begin{gathered}(\mathbf{u}_0 \otimes \mathbf{u}_1 \otimes \mathbf{u}_2)(\mathbf{T} \oplus \mathbf{S}_1 \oplus \mathbf{S}_2) = \frac{1}{\eta_1}(\mathbf{u}_0 \otimes \mathbf{u}_1 \otimes \mathbf{u}_2)(\mathbf{T}\mathbf{e}'\boldsymbol{\alpha} \otimes \mathbf{I}_1 \otimes \mathbf{I}_2) +\\ \eta_2(\mathbf{u}_0 \otimes \mathbf{u}_1 \otimes \mathbf{u}_2)(\mathbf{I}_0 \otimes \mathbf{I}_1 \otimes \tilde{\mathbf{S}}_2) + \frac{\eta_1}{\eta_2}(\mathbf{u}_0 \otimes \mathbf{u}_1 \otimes \mathbf{u}_2)(\mathbf{I}_0 \otimes \tilde{\mathbf{S}}_1 \otimes \mathbf{I}_2).\end{gathered} \tag{5.8}$$

From left hand side of (5.8), we have

$$\mathbf{u}_0\mathbf{T} \otimes \mathbf{u}_1 \otimes \mathbf{u}_2 + \mathbf{u}_0 \otimes \mathbf{u}_1\mathbf{S}_1 \otimes \mathbf{u}_2 + \mathbf{u}_0 \otimes \mathbf{u}_1 \otimes \mathbf{u}_2\mathbf{S}_2.$$

From right hand side of (5.8), we have

$$\frac{1}{\eta_1}\mathbf{u}_0\mathbf{T}\mathbf{e}'\boldsymbol{\alpha} \otimes \mathbf{u}_1 \otimes \mathbf{u}_2 + \eta_2\mathbf{u}_0 \otimes \mathbf{u}_1 \otimes \mathbf{u}_2\tilde{\mathbf{S}}_2 + \frac{\eta_1}{\eta_2}\mathbf{u}_0 \otimes \mathbf{u}_1\tilde{\mathbf{S}}_1 \otimes \mathbf{u}_2.$$

After a little algebra from applying (5.5) and (5.6) to both sides respectively, we have the equivalence.

We have obtained J_1 solutions of ω_0 in (2.1) and J_2 solutions of ω_2 in (5.1) which we denote with indices $\imath$ and $\jmath$, respectively. Define

$$\mathbf{w}_{m,n}(\imath, \jmath) \triangleq \eta_1^m(\imath)\eta_2^n(\jmath)[\mathbf{u}_0(\imath, \jmath) \otimes \mathbf{u}_1(\imath, \jmath) \otimes \mathbf{u}_2(\imath, \jmath)] \tag{5.9}$$

where $\imath = (j_{11}, ..., j_{1c_1})$, $\jmath = (j_{21}, ..., j_{2c_2})$ and $j_{..}$ are indices associated with solutions ω_0 and ω_2 corresponding to (2.1) and (2.2) respectively. Now any linear combination of $\mathbf{w}_{m,n}(\imath, \jmath)$ obviously satisfies the balance equations for $m > 1$ and $n > 1$.

Lemma 5.2 $\mathbf{w}_{m,n}$ *is a complex solution of general state balance equations for* $m > 0$ *and* $n > 0$ *if*

$$\sum_{m,n,\imath,\jmath} |\mathbf{w}_{m,n}(\imath,\jmath)| < +\infty \tag{5.10}$$

The proof is immediate from Lemma 3.2.

We observe that the most general solution under the condition that the roots s_0 and s_{2j} are distinct must be

$$\sum_{j_{11}=1}^{J_{11}} \cdots \sum_{j_{1c_1}=1}^{J_{1c_1}} \sum_{j_{21}=1}^{J_{21}} \cdots \sum_{j_{2c_2}=1}^{J_{2c_2}} \ell(\imath,\jmath)\mathbf{w}_{m,n}(\imath,\jmath). \tag{5.11}$$

The algorithm for attaining unsaturated probabilities and $\ell(\imath,\jmath)$ is not discussed here since it is not our focus in this paper.

6. Summary and Conclusions

The method of linear combination of product-forms has been used to solve the $PH/PH/c_1 \to /PH/c_2$ stationary system. The computational complexity for solving (3.1), (3.2) and (3.3) is apparently reduced since all the stationary probabilities for saturated states are expressed in terms of the product-forms and are only the functions of $\ell(\imath,\jmath)$ whose dimension is $\mathrm{J}_1\mathrm{J}_2$. Writing the stationary state probability in matrix geometric form we find that each component of these products can be expressed in terms of roots of the associated characteristic polynomials which involve only the Laplace-Stieltjes transforms of the interarrival and service time distributions.

As proved in [13], these roots are Perron-Frobenius eigenvalues of some non-negative matrices that solve (3.3). Although Fujimoto et al. [7] have not examined the property of the probability of saturated states, it is apparent that the vectors with associated those Perron-Frobenius eigenvalues form the solution basis for determining all stationary probabilities in the system as shown in this paper. We, thus, conjecture that some non-negative matrices can be also determined by the Perron-Frobenius eigenvalues as well as their eigenvectors so that the computational complexity of the algorithm for unsaturated probabilities can be even reduced. These results are easily exploited to develop an efficient and stable numerical algorithm, which is expected to work well for relatively large systems with high traffic intensities.

The expression for the stationary probabilities leads to similar expressions for measures of system performance such as the moments of the queue and the waiting time. Although the analysis has been worked out for the main results which can easily be extended to more stages, the problem of extending them to an arbitrary number of queues in tandem remains open. This is because a more complicated procedure is needed to select the appropriate basis solutions from the even larger class of available solutions. It would also be of interest to extend the result to more general service distributions.

Appendix A

The polynomial equation

$$S_1^*(-\omega_0 - s)S_2^*(s) = \eta_1^{-1}$$

has J_2 complex solutions with negative real parts: $s_1, \cdots, s_{J_2}$. Clearly, one root of this equation occurs at $s = 0$. In order to find the remaining roots, we make use of Rouche's theorem. Let $\ddot{T}^*$ be denominator of T^* with its denominant coefficient equal to 1, $\dot{T}^*$ the corresponding numerator. $\ddot{S}^*$ and $\dot{S}^*$ are defined in the same manner for S^*. Let

$$f(s) = \dot{S}_1^*(-\omega_0 - s)\dot{S}_2^*(s)\eta_1 - \ddot{S}_1^*(-\omega_0 - s)\ddot{S}_2^*(s)$$
$$g(s) = -\ddot{S}_1^*(-\omega_0 - s)\ddot{S}_2^*(s)$$

We now choose $D_{-c,r}$ to be the contour that runs up the imaginary axis and then forms a r-radius semicircle moving counterclockwise and surrounding the left half of s-plane as shown in Figure 1. Consider this contour since we are concerned about all the poles and zeroes in $\mathrm{Re}(s) < 0$ so that we may properly include them in (2.2). We now show that $|f(s) - g(s)| < |g(s)|$ for a suitable choice of $D_{-c,r}$. Note that

$$|\frac{f(s) - g(s)}{g(s)}| = |S_1^*(-\omega_0 - s)S_2^*(s)\eta_1|$$

for all s such that $g(s) \neq 0$. Since $\rho_1 \geq \rho_2$, i.e., $\lambda < \mu_1 \leq \mu_2$, $D_{-c,r}$ is decided by (2.1), (A.1) and (A.2) such that for $\mathrm{Re}(s) < 0$,

$$\int_0^\infty |\exp(\omega_0 + s)t| \text{ d } S_1(t) < \infty \tag{A.1}$$

and

$$\int_0^\infty |\exp(-st)| \text{ d } S_2(t) < \infty. \tag{A.2}$$

Thus, we have

$$\begin{aligned} |S_1^*(-\omega_0 - s)S_2^*(s)| &< |\int_0^\infty \exp(\omega_0 + x)t \text{ d } S_1(t) \int_0^\infty \exp(-xt) \text{ d } S_2(t)| \\ &= |S_1^*(-\omega_0 - x)S_2^*(x)|. \end{aligned}$$

Let $h(x) = S_1^*(-\omega_0 - x)S_2^*(x)$. Note that $h(x)$ is convex as shown in [7]. Since $S_2^*(x)$ is monotone decreasing of x, if $\eta_2 < 1$ then $\omega_2 < 0$. Moreover, we know $h(0) = \eta_1^{-1}$ and $h(\omega_2) = \eta_1^{-1}$. Thus, we have $h'(0) > 0$, namely,

$$S_1^*(-\omega_0)S_2^{*\prime}(0) - S_1^{*\prime}(-\omega_0) > 0.$$

This may be checked by the assumption $\mu_1 \leq \mu_2$ as follows. Because of $S_1^{*\prime}(0) = -\mu_1^{-1}$, $S_2^{*\prime}(0) = -\mu_2^{-1}$, $S_1^*(-w_0) > 0$, $S_1^*(x)$ is convex and decreasing, we have

$$S_1^{*\prime}(-w_0) < S_1^{*\prime}(0) \leq S_2^{*\prime}(0)S_1^*(-w_0).$$

Therefore for $c > 0$ small enough such that and $-r < x < -c$, we have

$$|S_1^*(-\omega_0 - x)S_2^*(x)| < \eta_1^{-1}$$

which implies

$$|S_1^*(-\omega_0 - s)S_2^*(s)| < \eta_1^{-1}.$$

Now for $\mathrm{Re}(s) < 0$ and for large enough values of r, we have $|s| = r$ and

$$|S_1^*(-\omega_0 - s)S_2^*(s)| < \eta_1^{-1}$$

It is thus proven that f and g have the same number of negative real parts. Since $\ddot{S}_2^*(s) = det(sI_2 - S_2)$, i.e., the characteristic polynomial of S_2, and all eigenvalues of S_2 have a negative real part (see Seneta [12]), $g(s)$ has J_2 complex solutions with negative real parts which is the number of eigenvalues of S_2. That ends the proof.

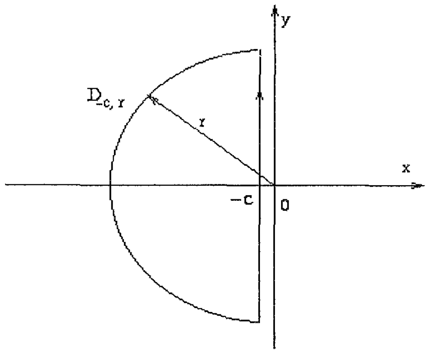

Figure 1:

Appendix B

For each choice of the coefficients $\ell(\imath, \jmath)$ the sequence $\pi_{m,n}$ given by (3.19) satisfies the equations (3.4). The remaining equations (3.5)–(3.12) are the equations for states $m = 0$ or $n = 0$. Consider equations (3.8), (3.9), (3.11) and (3.12). These equations form a linear homogeneous system for the unknowns $\ell(\imath, \jmath)$ and the unknown quantities $\pi_{0,0}$, $\pi_{1,0}$, $\pi_{0,1}$, and $\pi_{0,2}$. The number of equations is equal to $J_0J_1+J_0J_2+J_0+J_0J_1J_2$ but the number of unknowns is equal to $J_0+J_0J_1+2J_0J_2+J_1J_2$. Since $J_2 \geq 2$ and $J_1 \geq 2$, we have $J_0J_1+J_0J_2 \leq J_0J_1J_2$. Here, by first omitting $J_0J_2(J_1-1)-J_1J_2$ equations in (3.8), the reduced system of equations including (3.8), (3.9), (3.11) and (3.12) with $J_0+J_0J_1+2J_0J_2+J_1J_2$ equations and unknowns has a nonnull solution. If all these quantities are null, then at least one of the coefficient $\ell(\imath, \jmath)$ must be nonnull since the Markov process is ergodic and $\pi_{0,0}$ is nonnull. It implies that $\pi_{1,0}$, $\pi_{0,1}$, $\pi_{0,2}$, $\pi_{1,2}$ and $\pi_{2,1}$ are nonnull solutions. Then starting with $\pi_{2,1}$, $\pi_{1,1}$ and $\pi_{2,2}$, we can find $\pi_{m,0}$, $m \geq 2$, by (3.21). Similarly, starting with $\pi_{1,1}$, $\pi_{1,2}$, $\pi_{2,1}$ and $\pi_{1,3}$, we can find $\pi_{0,n}$, $n \geq 3$, by (3.22). From Lemma 3.2, we know the sum of $\pi_{m,n}$ over all states converges absolutely which implies summing over these equations and changing summations is allowed. Thus, by dropping one of the equations in the linear homogeneous system formed (8), (9), (11) and (12) and using the normalization equation (20), we find a linear nonhomogeneous system. Since $\pi_{m,n}$ is an absolutely convergent solution of all equilibrium equations, the coefficients $\ell(\imath, \jmath)$ can be determined by these equations.

Appendix C

The polynomial equation

$$T^*(-\bar{\omega}_2 - s)S_1^*(s) = \bar{\eta}_2$$

has J_1 complex solutions with negative real parts: $s_1, \cdots, s_{J_1}$. Clearly, one root of this equation occurs at $s = 0$. In order to find the remaining roots, we make use of Rouche's theorem again, following the same arguments in Appendix A. Let

$$f(s) = \dot{T}^*(-\bar{\omega}_2 - s)\dot{S}_1^*(s)\bar{\eta}_2^{-1} - \ddot{T}^*(-\bar{\omega}_2 - s)\ddot{S}_1^*(s)$$

$$g(s) = -\ddot{T}^*(-\bar{\omega}_2 - s)\ddot{S}_1^*(s)$$

We now choose $\bar{D}_{-c,r}$ to be the contour that runs up the imaginary axis and then forms a r-radius semicircle moving counterclockwise and surrounding the left half of s-plane. Consider this contour since we are concerned about all the poles and zeroes in $\mathrm{Re}(s) < 0$ so that we may properly include them in (2.4). We now show that $|f(s) - g(s)| < |g(s)|$ for a suitable choice of $\bar{D}_{-c,r}$. Note that

$$|\frac{f(s) - g(s)}{g(s)}| = |T^*(-\bar{\omega}_2 - s)S_1^*(s)\bar{\eta}_2^{-1}|$$

for all s such that $g(s) \neq 0$. Since $\rho_1 < \rho_2$, i.e., $\lambda < \mu_2 < \mu_1$, $\bar{D}_{-c,r}$ is decided by (2.3), (C.1) and (C.2) such that for $\mathrm{Re}(s) < 0$,

$$\int_0^\infty |\exp(\bar{\omega}_2 + s)t| \mathrm{~d}~T(t) < \infty \tag{C.1}$$

and

$$\int_0^\infty |\exp(-st)| \mathrm{~d}~S_1(t) < \infty. \tag{C.2}$$

Thus, we have

$$\begin{aligned} |T^*(-\bar{\omega}_2 - s)S_1^*(s)| &< |\int_0^\infty \exp(\bar{\omega}_2 + x)t \mathrm{~d}~T(t) \int_0^\infty \exp(-xt) \mathrm{~d}~S_1(t)| \\ &= |T^*(-\bar{\omega}_2 - x)S_1^*(x)|. \end{aligned}$$

Let $h(x) = T^*(-\bar{\omega}_2 - x)S_1^*(x)$. Note that $h(x)$ is logarithmic-convex. Since $S_1^*(x)$ is monotone decreasing of x, if $\bar{\eta}_1 < \bar{\eta}_2 < 1$ then $\bar{\omega}_1 < 0$. Moreover, we know $h(0) = \bar{\eta}_2$ and $h(\bar{\omega}_1) = \bar{\eta}_2$. Thus, we have $h'(0) > 0$, namely,

$$T^*(-\bar{\omega}_2)S_1^{*\prime}(0) - T^{*\prime}(-\bar{\omega}_2) > 0$$

Therefore for $c > 0$ small enough such that and $-r < x < -c$, we have

$$|T^*(-\bar{\omega}_2 - x)S_1^*(x)| < \bar{\eta}_2$$

which implies

$$|T^*(-\bar{\omega}_0 - s)S_1^*(s)| < \bar{\eta}_2.$$

Now for $\mathrm{Re}(s) < 0$ and for large enough values of r, we have $|s| = r$ and

$$|T^*(-\bar{\omega}_2 - s)S_1^*(s)| < \bar{\eta}_2$$

It is thus proven that f and g have the same number of negative real parts. Since $\ddot{S}_1^*(s) = det(sI_1 - S_1)$, i.e., the characteristic polynomial of S_1, and all eigenvalues of S_1 have a negative real part (see Seneta [12]), $g(s)$ has J_1 complex solutions with negative real parts which is the number of eigenvalues of S_1. That ends the proof.

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