

MEAN-ABSOLUTE DEVIATION PORTFOLIO OPTIMIZATION MODEL UNDER TRANSACTION COSTS

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Abstract We will propose a branch and bound algorithm for solving a portfolio optimization model under nonconvex transaction costs. It is well known that the unit transaction cost is larger when the amount of transaction is small while it remains stable up to a certain point and then increases due to illiquidity effects. Therefore, the transaction cost function is typically nonconvex. The existence of nonconvex transaction costs very much affects the optimal portfolio particularly when the amount of fund is small. However, the portfolio optimization problem under nonconvex transaction cost are largely set aside due to its computational difficulty. In fact, there are only a few studies which treated nonconvex costs in a rigorous manner. In this paper, we will propose a branch and bound algorithm for solving a mean-absolute deviation portfolio optimization model assuming that the cost function is concave. We will use a linear underestimating function for a concave cost function to calculate a good bound, and demonstrate that a fairly large scale problem can be solved in an efficient manner using the real stock data and transaction cost table in the Tokyo Stock Exchange. Finally, extension of our algorithm to rebalancing will be briefly touched upon.

1. Introduction

Practitioners are very much concerned about transaction costs since it has significant effects on the investment strategy. In particular, when the amount of fund is relatively small, diversification of fund may incur an unacceptably large transaction costs. Unfortunately, however transaction costs are often ignored or treated in an ad-hoc manner because the precise treatment of transaction costs leads to a nonconvex minimization problem for which there exists no efficient method for calculating its exact optimal solution, at least until recently. As a result, there are very few serious works in this field. The only exception, to the authors knowledge, are the works by A. Perold [13] and J. Mulvey [10] in which transaction cost function is approximated by a piecewise linear convex function. However, this approach is not valid for the more important nonconvex transaction cost function.

The primary purpose of this paper is to develop a computational scheme which can generate an optimal solution of a mean- absolute deviation model under concave transaction costs. We will propose a branch and bound method by exploiting the special structure, namely the low rank nonconvex structure of this problem and show that this approach can calculate a globally optimal solution in an efficient manner.

Figure 1 shows a typical form of the transaction cost $c(x)$ as a function of the amount of transaction x . As shown in this figure, the unit transaction cost is larger when x is small, while it gradually decreases as x increases. Therefore, the cost function $c(x)$ is concave up to certain point, say point A. When x passes this point, the unit transaction cost becomes constant and hence $c(x)$ increases linearly up to some point, say B. When the amount of transaction x passes this point, the unit price of stock increases due to illiquidity, namely due to the shortage of supply. Therefore, the function $c(x)$ becomes convex beyond point B.

Unfortunately, however we do not know in advance where the point B is located. Only a small amount of purchase may increase the unit cost significantly if many investors want to purchase the same stock. Also, the effect of a large amount of purchase may be canceled by a large amount of sale of some other investors so that the unit price may remain stable.

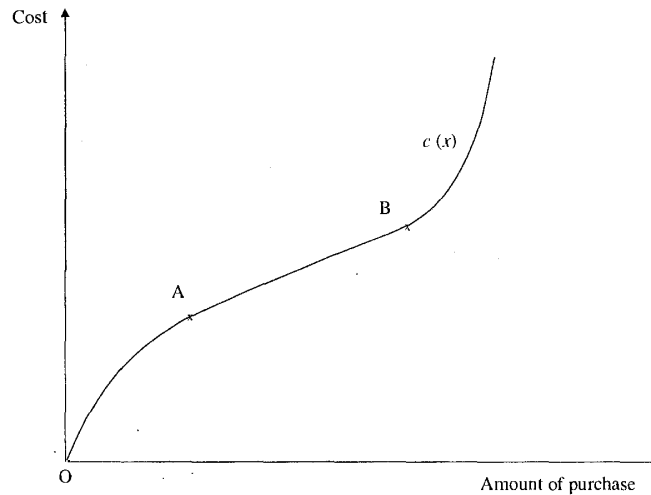


Figure 1: Transaction Cost Function

Therefore, we restrict our discussion to the situation where the amount of fund is within the range where we can calculate the cost function exactly, i.e., within the range where the cost is concave.

In the next section, we will formulate the problem as a convex maximization problem. We assume that the risk function is given by the absolute deviation of the rate of return instead of the standard deviation (or variance).

The mean-absolute deviation (MAD) approach is proposed by one of the authors in [5] now widely used by practitioners to solve a large scale portfolio optimization problem with more than 1000 assets because it can be reduced to a linear programming problem, instead of a quadratic programming problem in the case of mean-variance (MV) model [9]. In section 3, we will propose a branch and bound algorithm for solving a linearly constrained convex maximization problem by adapting the algorithm proposed by Phong et al [14]. We will show that the subproblem to be solved in the branch and bound method becomes a linear programming problem which can be solved very fast. Section 4 will be devoted to the results of numerical experiments of the algorithm proposed in Section 3 using real stock data and real transaction cost data in the Tokyo Stock Exchange. We will show that our algorithm can in fact generate a (nearly) optimal solution in a very efficient manner. In Section 5, we will extend this algorithm to a more complicated rebalancing problem.

2. Mean-Absolute Deviation Model with Transaction Costs

Let there be n assets S_j ($j = 1, \dots, n$) in the market and let R_j be the random variable representing the rate of return of S_j without transaction cost. We will assume that the vector of random variables (R_1, \dots, R_n) is distributed over a finitely many set of points $\{(r_{1t}, \dots, r_{nt}), t = 1, \dots, T\}$ and that the probability

$$f_t = P_r\{(R_1, \dots, R_n) = (r_{1t}, \dots, r_{nt})\}, \quad t = 1, \dots, T, \quad (2.1)$$

is known. Then the expected rate of return r_j without transaction cost of S_j is given by

$$r_j = \sum_{t=1}^T f_t r_{jt}. \quad (2.2)$$

Let x_j be the proportion of fund to be allocated to S_j . Then the expected rate of return of the portfolio $\mathbf{x}=(x_1, \dots, x_n)$ is given by $\sum_{j=1}^n r_j x_j$. The actual expected rate of return under transaction costs is therefore, given by

$$r(\mathbf{x}) = \sum_{j=1}^n \{r_j x_j - c_j(x_j)\}, \quad (2.3)$$

where $c_j(\cdot)$ is a concave transaction cost associated with stock S_j . As in [5], we will employ the absolute deviation

$$\begin{aligned} W[\mathbf{x}] &= E[|R(\mathbf{x}) - E[R(\mathbf{x})]|] \\ &= \sum_{t=1}^T f_t \left| \sum_{j=1}^n (r_{jt} - r_j) x_j \right|, \end{aligned} \quad (2.4)$$

as the measure of risk. The mean-absolute deviation (MAD) efficient frontier under transaction cost can be calculated by solving a following convex maximization problem.

$$\left\{ \begin{array}{ll} \text{maximize} & f(\mathbf{x}) \equiv \sum_{j=1}^n \{r_j x_j - c_j(x_j)\} \\ \text{subject to} & \sum_{t=1}^T f_t \left| \sum_{j=1}^n (r_{jt} - r_j) x_j \right| \leq w, \\ & \sum_{j=1}^n x_j = 1, \\ & 0 \leq x_j \leq \alpha_j, \quad j = 1, \dots, n. \end{array} \right. \quad (2.5)$$

The MAD model (without transaction cost) was first proposed by one of the authors as an alternative to the standard mean-variance (MV) model [9]. It has been demonstrated in [5] that MAD model can generate an optimal portfolio much faster than the MV model since it can be reduced to a linear programming problem instead of quadratic programming problem. Also, it is shown in Konno et al.[6] that MAD model shares the same properties as the MV model including well known CAPM type relations. Further, it is proved by Ogryczak and Ruszczyński [12] that those portfolios on the MAD efficient frontier corresponds to efficient portfolios in terms of the second degree stochastic dominance.

Let us now introduce an alternative representation of the problem (2.5) which is more suitable to construct a branch and bound algorithm. First let us introduce a set of nonnegative variables y_t, z_t , $t = 1, \dots, T$ satisfying the following conditions.

$$\begin{aligned} y_t - z_t &= f_t \sum_{j=1}^n (r_{jt} - r_j) x_j, & t = 1, \dots, T, \\ y_t z_t &= 0; y_t \geq 0; z_t \geq 0, & t = 1, \dots, T. \end{aligned}$$

Then we have the following representation

$$\left| f_t \sum_{j=1}^n (r_{jt} - r_j) x_j \right| = y_t + z_t, \quad t = 1, \dots, T. \quad (2.6)$$

Therefore, the problem (2.5) can be rewritten as follows.

$$\begin{aligned}
 & \text{maximize } f(\mathbf{x}) \equiv \sum_{j=1}^n \{r_j x_j - c_j(x_j)\} \\
 & \text{subject to } \sum_{t=1}^T (y_t + z_t) \leq w \\
 & \qquad y_t - z_t = f_t \sum_{j=1}^n (r_{jt} - r_j) x_j, \quad t = 1, \dots, T, \\
 & \qquad y_t z_t = 0, \quad t = 1, \dots, T, \\
 & \qquad \sum_{j=1}^n x_j = 1, \\
 & \qquad y_t \geq 0, \quad z_t \geq 0, \quad t = 1, \dots, T, \\
 & \qquad 0 \leq x_j \leq \alpha_j, \quad j = 1, \dots, n.
 \end{aligned} \tag{2.7}$$

Theorem 2.1 : The complementarity constraint $y_t z_t = 0$, $t = 1, \dots, T$ can be eliminated from (2.7).

Proof : Let $(x_1^*, \dots, x_n^*, y_1^*, \dots, y_T^*, z_1^*, \dots, z_T^*)$ be an optimal solution of equation (2.7) without complementarity condition and let us assume that $y_t^* z_t^* > 0, t \in I \subset \{1, \dots, T\}$. For $t \in I$, let

$$\begin{aligned}
 \begin{pmatrix} \hat{y}_t \\ \hat{z}_t \end{pmatrix} &= \begin{pmatrix} y_t^* - z_t^* \\ 0 \end{pmatrix} \text{ if } y_t^* \geq z_t^* \geq 0, \\
 \begin{pmatrix} \hat{y}_t \\ \hat{z}_t \end{pmatrix} &= \begin{pmatrix} 0 \\ z_t^* - y_t^* \end{pmatrix} \text{ if } z_t^* \geq y_t^* \geq 0.
 \end{aligned}$$

Then $(x_1^*, \dots, x_n^*, \hat{y}_1, \dots, \hat{y}_T, \hat{z}_1, \dots, \hat{z}_T)$ satisfies all the constraints of (2.7). Also it has the same objective values as $(x_1^*, \dots, x_n^*, y_1^*, \dots, y_T^*, z_1^*, \dots, z_T^*)$. In view of the relation

$$\begin{aligned}
 \sum_{t=1}^T (y_t - z_t) &= \sum_{t=1}^T f_t \sum_{j=1}^n (r_{jt} - r_j) x_j \\
 &= \sum_{j=1}^n \left(\sum_{t=1}^T f_t r_{jt} - r_j \right) x_j = 0,
 \end{aligned}$$

we can eliminate (z_1, \dots, z_T) from (2.7) to obtain an alternative representation.

$$\begin{aligned}
 & \text{maximize } f(\mathbf{x}) \equiv \sum_{j=1}^n \{r_j x_j - c_j(x_j)\} \\
 & \text{subject to } \sum_{t=1}^T y_t \leq w/2, \\
 & \qquad y_t \geq f_t \sum_{j=1}^n (r_{jt} - r_j) x_j, \quad t = 1, \dots, T, \\
 & \qquad \sum_{j=1}^n x_j = 1, \\
 & \qquad y_t \geq 0, \quad t = 1, \dots, T, \\
 & \qquad 0 \leq x_j \leq \alpha_j, \quad j = 1, \dots, n.
 \end{aligned} \tag{2.8}$$

3. A Branch and Bound Algorithm

Let us describe a branch and bound algorithm for solving a linearly constrained convex maximization problem (2.8). Let $\mathbf{x} = (x_1, \dots, x_n)$, $\mathbf{y} = (y_1, \dots, y_T)$ and let

$$F = \left\{ (\mathbf{x}, \mathbf{y}) \mid \sum_{t=1}^T y_t \leq w/2, \quad y_t - f_t \sum_{j=1}^n (r_{jt} - r_j)x_j \geq 0, \right. \\ \left. \sum_{j=1}^n x_j = 1, \quad y_t \geq 0, \quad t = 1, \dots, T \right\}. \quad (3.1)$$

The problem (2.8) can be denoted as follows

$$(P_0) \quad \begin{cases} \text{maximize} & f(\mathbf{x}) = \sum_{j=1}^n \{r_j x_j - c_j(x_j)\} \\ \text{subject to} & (\mathbf{x}, \mathbf{y}) \in F, \\ & \mathbf{0} \leq \mathbf{x} \leq \boldsymbol{\alpha}. \end{cases} \quad (3.2)$$

where $\boldsymbol{\alpha} = (\alpha_1, \dots, \alpha_n)$. Let $(\mathbf{x}^*, \mathbf{y}^*)$ be an optimal solution of (3.2) and let $f^* = f(\mathbf{x}^*)$. Let us replace $c_j(x_j)$ by a linear underestimating function $\delta_j(x_j)$

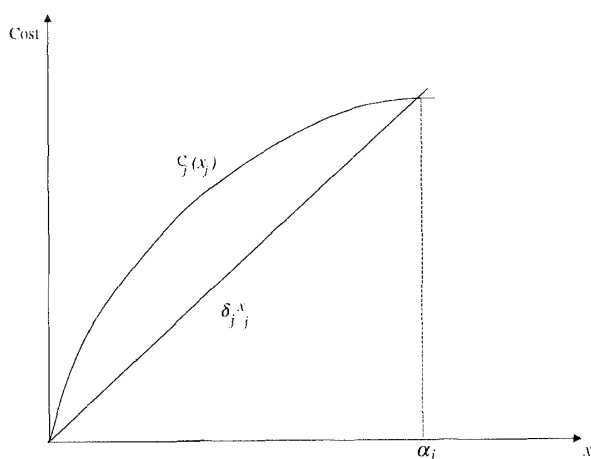


Figure 2: A Linear Underestimating Function

and denote

$$g_0(\mathbf{x}) = \sum_{j=1}^n (r_j - \delta_j)x_j, \quad (3.3)$$

and define a linear programming problem

$$(Q_0) \quad \begin{cases} \text{maximize} & g_0(\mathbf{x}) = \sum_{j=1}^n (r_j - \delta_j)x_j \\ \text{subject to} & (\mathbf{x}, \mathbf{y}) \in F, \\ & \mathbf{0} \leq \mathbf{x} \leq \boldsymbol{\alpha}. \end{cases} \quad (3.4)$$

Let us note that

$$g_0(\mathbf{x}) \geq f(\mathbf{x}), \quad \mathbf{0} \leq \mathbf{x} \leq \boldsymbol{\alpha}. \quad (3.5)$$

Let \mathbf{x}^0 be an optimal solution of (3.4). If

$$\sum_{j=1}^n \{c_j(x_j^0) - \delta_j x_j^0\} \leq \varepsilon, \quad (3.6)$$

then $(\mathbf{x}^0, \mathbf{y}^0)$ is an approximate optimal solution of (P_0) with error less than ε

Theorem 3.1:

$$g_0(\mathbf{x}^0) \geq f^* \geq f(\mathbf{x}^0). \tag{3.7}$$

Proof: It follows from (3.5) that

$$\begin{aligned} g_0(\mathbf{x}^0) &= \max \{g_0(\mathbf{x}) \mid (\mathbf{x}, \mathbf{y}) \in F, \mathbf{0} \leq \mathbf{x} \leq \boldsymbol{\alpha}\} \\ &\geq \max \{f(\mathbf{x}) \mid (\mathbf{x}, \mathbf{y}) \in F, \mathbf{0} \leq \mathbf{x} \leq \boldsymbol{\alpha}\} = f^*. \end{aligned}$$

The relation $f^* \geq f(\mathbf{x}^0)$ follows from the fact that \mathbf{x}^0 is a feasible solution of (3.4). Let us consider the case when (3.6) does not hold. Let

$$c_s(x_s^0) - \delta_s x_s^0 = \max\{c_j(x_j^0) - \delta_j x_j^0 \mid j = 1, \dots, n\}, \tag{3.8}$$

and let

$$X_1 = \{x \mid 0 \leq x_s \leq \alpha_s/2, 0 \leq x_j \leq \alpha_j, j \neq s\}, \tag{3.9}$$

$$X_2 = \{x \mid \alpha_s/2 \leq x_s \leq \alpha_s, 0 \leq x_j \leq \alpha_j; j \neq s\}, \tag{3.10}$$

and define two subproblems

$$(P_1) \begin{cases} \text{maximize} & f(\mathbf{x}) \\ \text{subject to} & (\mathbf{x}, \mathbf{y}) \in F, \\ & \mathbf{x} \in X_1. \end{cases} \tag{3.11}$$

$$(P_2) \begin{cases} \text{maximize} & f(\mathbf{x}) \\ \text{subject to} & (\mathbf{x}, \mathbf{y}) \in F, \\ & \mathbf{x} \in X_2. \end{cases} \tag{3.12}$$

Let $\mathbf{x}^1, \mathbf{x}^2$ be, respectively an optimal solution of (P_1) and (P_2) . Then either \mathbf{x}^1 , or \mathbf{x}^2 is an optimal solution of (P_0) . Corresponding to the subproblem (P_1) , let us define a linear function $g_1(\mathbf{x})$ underestimating $f(\mathbf{x})$ as shown in Figure 3.

$$g_1(\mathbf{x}) = \sum_{j \neq s} (r_j - \delta_j)x_j + (r_s x_s - c_{s1}(x_s)) \tag{3.13}$$

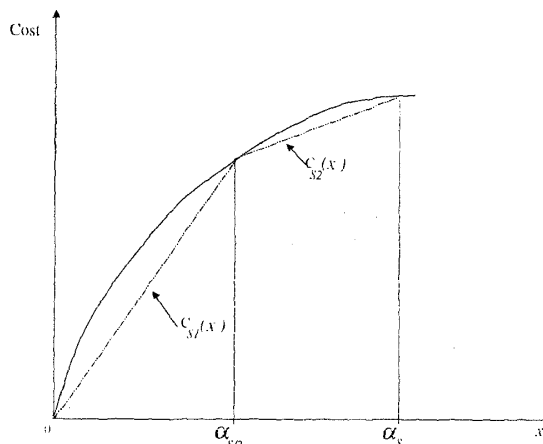


Figure 3: Bisection Scheme

and define a relaxed linear programming problem

$$(Q_1) \left\{ \begin{array}{l} \text{maximize } g_1(\mathbf{x}) \\ \text{subject to } (\mathbf{x}, \mathbf{y}) \in F, \\ \mathbf{x} \in X_1. \end{array} \right. \quad (3.14)$$

If (Q_1) is infeasible, then (Q_1) can be fathomed. Otherwise, let $\hat{\mathbf{x}}^1$ be an optimal solution of (Q_1) . If

$$|g_1(\hat{\mathbf{x}}^1) - f(\hat{\mathbf{x}}^1)| < \varepsilon, \quad (3.15)$$

then the problem has been solved with an approximate optimal solution $\hat{\mathbf{x}}^1$. Since $g_1(\mathbf{x}) \geq f(\mathbf{x})$, $\forall \mathbf{x} \in X_1$, we have

$$g_1(\hat{\mathbf{x}}^1) \geq f(\hat{\mathbf{x}}^1) \geq f(\hat{\mathbf{x}}^1). \quad (3.16)$$

Therefore, if

$$g_1(\hat{\mathbf{x}}^1) \leq f(\mathbf{x}^0),$$

then (P_1) can be fathomed since $f(\mathbf{x}^1) \leq f(\mathbf{x}^0)$.

Algorithm (Branch and Bound Method)

- 1° $\mathbf{P} = \{(P_0)\}$, $\hat{f} = -\infty$, $k = 0$.
 $\beta^0 = 0$, $\alpha^0 = \alpha$; $X_0 = \{x \mid \beta^0 \leq x \leq \alpha^0\}$.
- 2° If $\mathbf{P} = \{\phi\}$, then goto 9°; Otherwise goto 3°.
- 3° Choose a problem $(P_k) \in \mathbf{P}$:

$$(P_k) \left\{ \begin{array}{l} \text{maximize } f(\mathbf{x}) = \sum_{j=1}^n \{r_j x_j - c_j(x_j)\} \\ \text{subject to } (\mathbf{x}, \mathbf{y}) \in F, \\ \mathbf{x} \in X_k, \end{array} \right.$$

$$\mathbf{P} = \mathbf{P} \setminus \{(P_k)\}.$$

- 4° Let $c_j^k(x_j)$ be a linear underestimating function of $c_j(x_j)$ over the interval $\beta_j^k \leq x_j \leq \alpha_j^k$, $j = 1, \dots, n$ and define a linear programming problem

$$(Q_k) \left\{ \begin{array}{l} \text{maximize } g_k(\mathbf{x}) = \sum_{j=1}^n \{r_j x_j - c_j^k(x_j)\} \\ \text{subject to } (\mathbf{x}, \mathbf{y}) \in F, \\ \mathbf{x} \in X_k, \end{array} \right.$$

If (Q_k) is infeasible then go to 2°. Otherwise let \mathbf{x}^k be an optimal solution of (Q_k) .

If $|g_k(\mathbf{x}^k) - f(\mathbf{x}^k)| > \varepsilon$ then goto 8°. Otherwise let $f_k = f(\mathbf{x}^k)$.

- 5° If $f_k < \hat{f}$ then goto 7°; Otherwise goto 6°.
- 6° If $\hat{f} = f_k$; $\hat{\mathbf{x}} = \hat{\mathbf{x}}^k$ and eliminate all the subproblems (P_t) for which $g_t(\mathbf{x}^t) \leq \hat{f}$.
- 7° If $g_k(\mathbf{x}^k) \leq \hat{f}$ then goto 2°. Otherwise goto 8°.

8° Let $c_s(x_s^k) - c_s^k(x_s^k) = \max\{c_j(x_j^k) - c_j^k(x_j^k) \mid j = 1, \dots, n\}$,

$$\begin{aligned} X_{l+1} &= X_k \cap \{\beta_s^k \leq x_s \leq (\beta_s^k + \alpha_s^k)/2\}, \\ X_{l+2} &= X_k \cap \{(\beta_s^k + \alpha_s^k)/2 \leq x_s \leq \alpha_s^k\}, \end{aligned}$$

and define two subproblems :

$$(P_{l+1}) \left\{ \begin{array}{l} \text{maximize } f(\mathbf{x}) \\ \text{subject to } (\mathbf{x}, \mathbf{y}) \in F, \\ \mathbf{x} \in X_{l+1}. \end{array} \right.$$

$$(P_{l+2}) \left\{ \begin{array}{l} \text{maximize } f(\mathbf{x}) \\ \text{subject to } (\mathbf{x}, \mathbf{y}) \in F, \\ \mathbf{x} \in X_{l+2}. \end{array} \right.$$

$\mathbf{P} = \mathbf{P} \cup \{P_{l+1}, P_{l+2}\}$, $k = k + 1$ and goto 3°.

where (P_l) represents the problem in \mathbf{P} with the largest subscript.

9° Stop : $(\hat{\mathbf{x}})$ is an approximate optimal solution of (P_0)

Theorem 3.2 : $\hat{\mathbf{x}}$ converges to an ε - optimal solution of (P_0) as $k \rightarrow \infty$.

Proof : See e.g. [7,15]

The problem (P_0) can, in principle be solved by this algorithm. To accelerate convergence, we may replace the bisection scheme by a so-called ω - subdivision scheme [15] in which the interval $[\beta_s^k, \alpha_s^k]$ is divided into two sub intervals $[\beta_s^k, x_s^k]$ and $[x_s^k, \alpha_s^k]$ where x_s^k is the s th component of the optimal solution \mathbf{x}^k of (Q_k) as depicted in Figure 4.

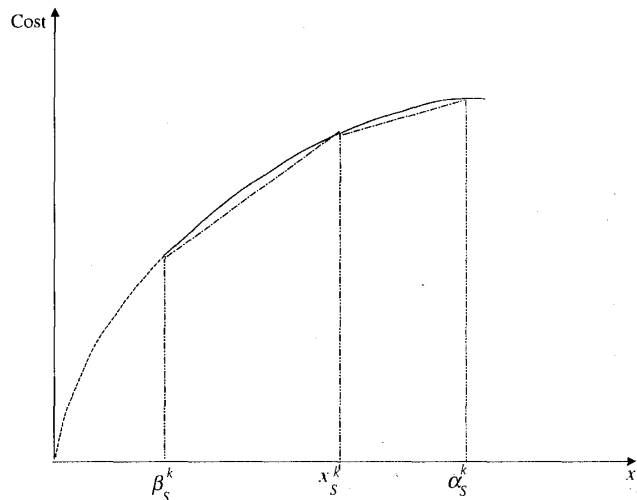


Figure 4: ω - subdivision

This subdivision scheme usually accelerate convergence, though it is not theoretically guaranteed. An alternative scheme is to reduce the size of the problem by using the following theorem.

Theorem 3.3: There exists an optimal solution \mathbf{x}^0 of (Q_0) , at most $T+1$ components of which satisfy $0 < x_j^0 < \alpha_j$.

Proof : Let $\mathbf{x}^*, \mathbf{y}^*$ be an optimal solution of (Q_0) . Then \mathbf{x}^* is an optimal solution of the linear programming problem :

$$\left. \begin{array}{l} \text{maximize } g_0(\mathbf{x}) = \sum_{j=1}^n (r_j - \delta_j)x_j \\ \text{subject to } f_t \sum_{j=1}^n (r_{jt} - r_j)x_j \leq y_t^*, \quad t = 1, \dots, T, \\ \sum_{j=1}^n x_j = 1, \\ 0 \leq x_j \leq \alpha_j, \quad j = 1, \dots, n. \end{array} \right\} \quad (3.17)$$

By the fundamental theorem of linear programming, this problem has a basic optimal solution. It is straightforward to see that any basic solution has at most $T+1$ components with the property $0 < x_j < \alpha_j$. This theorem implies that at most $T+1$ assets are subject to approximation error (Note that $c_j(x_j) = \delta_j x_j$ if $x_j = 0$ or $x_j = \alpha_j$). Also those assets with smaller fraction of investment are subject to relatively large transaction cost since the cost function is concave. Therefore, most of those assets with $x_j^0 = 0$ are expected to satisfy the property $x_j^* = 0$, where x^* is the optimal solution of (P_0) .

This observation leads us to another approximation of the original problem (P_0) . Let us assume without loss of generality that the first J ($\leq T+1$) components of x^0 are positive. We will apply the branch and bound algorithm to the reduced problem:

$$(P'_0) \left\{ \begin{array}{l} \text{maximize } g_0(x) = \sum_{j=1}^J \{r_j x_j - c_j(x_j)\} \\ \text{subject to } (x_1, \dots, x_J, 0, \dots, 0, \mathbf{y}) \in F, \\ \sum_{j=1}^J x_j = 1, \\ 0 \leq x_j \leq \alpha_j, \quad j = 1, \dots, J. \end{array} \right. \quad (3.18)$$

It is expected that the optimal solution of (P'_0) is a good approximation of the optimal solution \mathbf{x}^* of (P_0) . In fact, our experiments show that the relative difference of the optimal value of (P_0) and (P'_0) is usually no more than 1.5%.

4. Results of Numerical Simulation

We tested the algorithm proposed in Section 3 using 10 sets of 36 monthly data of 200 stocks using the real transaction cost data. These stocks are chosen from the set of stocks included in NIKKEI 225 Index. The program was coded in C^{++} and was tested on SPARC II workstation.

Figure 5 and 6 show the average and the standard deviation of the computation time required to calculate an ε -optimal solution where $\varepsilon = 10^{-5}$. Also we employed breadth first rule and ω -subdivision strategy throughout the test.

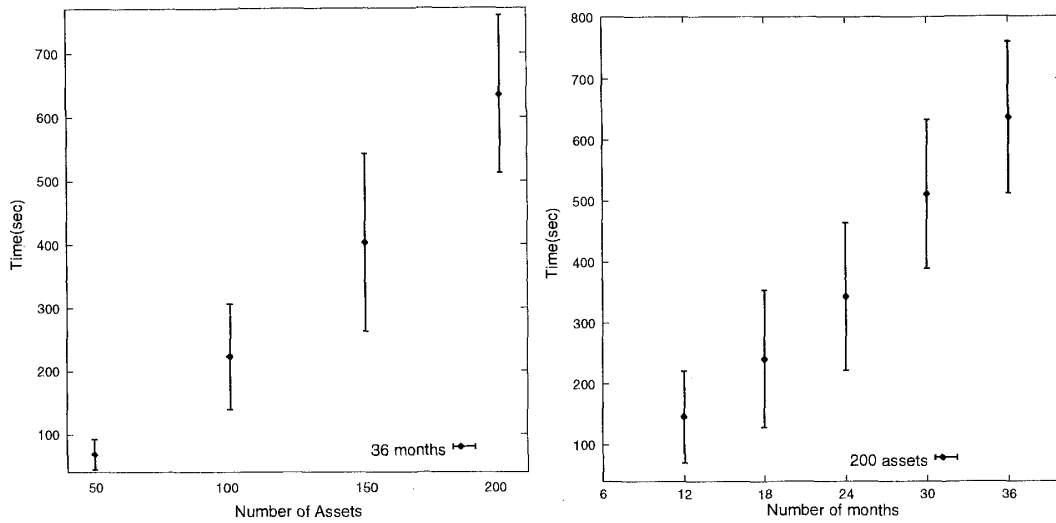


Figure 5: CPU Time for Different n Figure 6: CPU Time for Different T

We see from these figures that the computation time increases almost linearly with respect to T and n . Common observations in nonconvex minimization is a sharp increase of computation time with respect to the rank of nonconvexity, namely T in this case. This remarkable efficiency is partly due to the fact that linear underestimating function is a very good approximation to the concave cost function. Also, the breadth first search resulted in a good feasible solution in the earlier stage and many subproblems have been eliminated by bounding procedure. The average number of subproblems solved was 12. By improving the implementation, problems with over $t = 60$ and $n = 1000$ (without variable reduction) would be solved in a practical amount of time.

Figure 7 shows the transaction cost where the amount of investment is 1 billion .

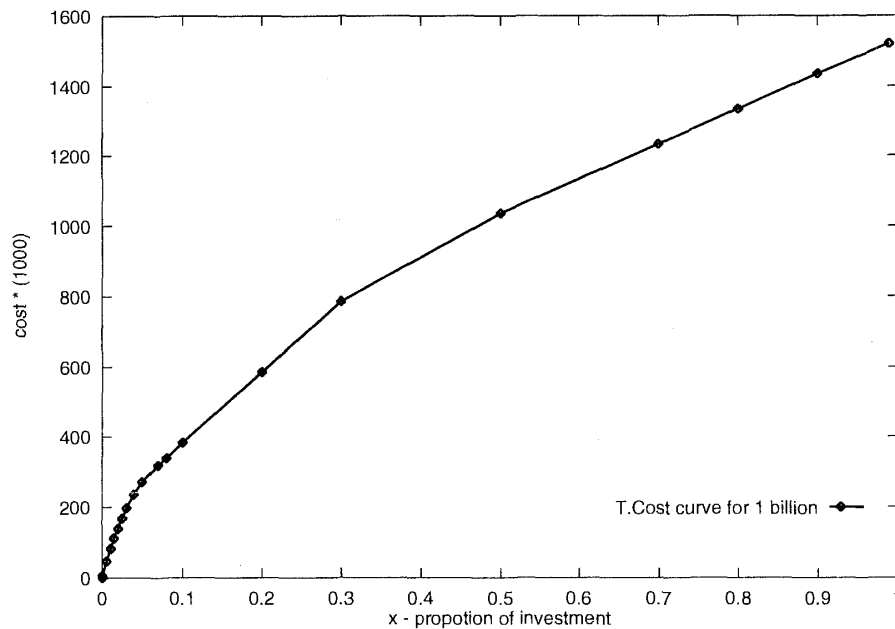


Figure 7: Transaction cost function

Figure 8 shows the efficient frontier for different amount of fund. Also Figure 9 is the graph of the maximal expected rate of return with and without transaction costs.

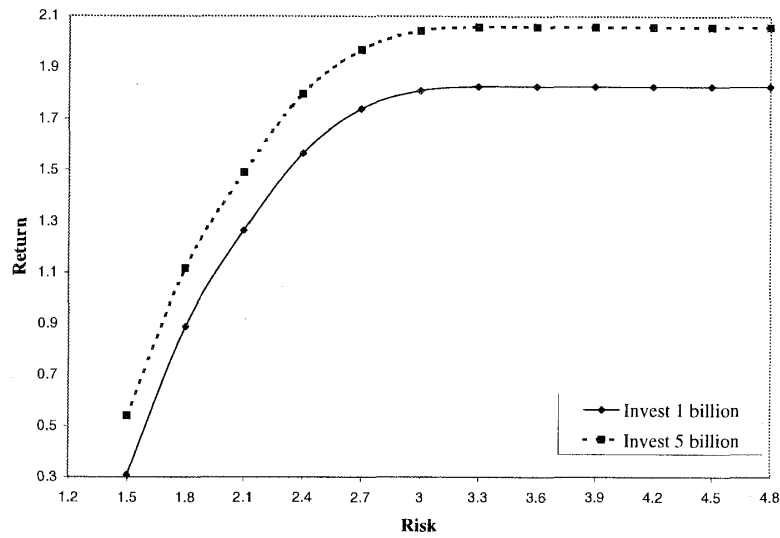


Figure 8: Efficient Frontier for Different Amount of Funds ($\alpha_j = 0.06$)

The dotted graph in Figure 8, represents the actual expected rate of return calculated by MAD model without transaction cost and then adjusted by the transaction cost for purchasing the portfolio is showed by the broken lines in Figure 9. We see from this figure that the exact treatment of transaction cost leads to much better results.

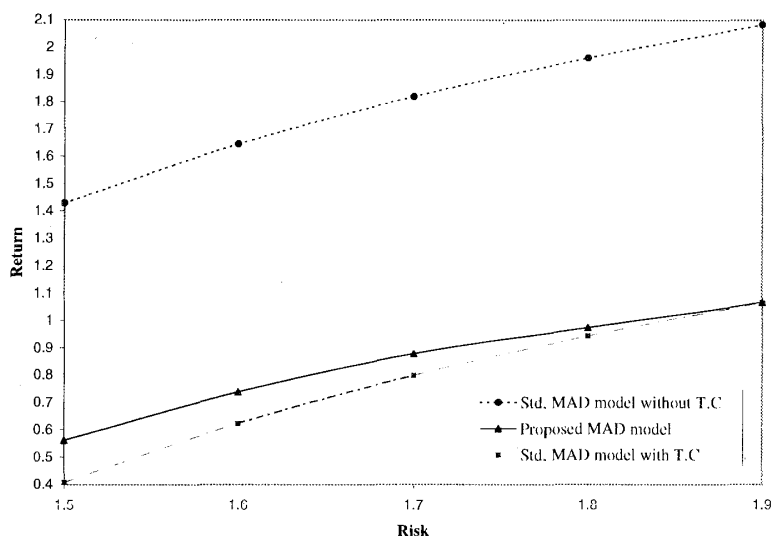


Figure 9: Efficient Frontiers for Standard and proposed (MAD) model

The difference of the number of assets included in the optimal portfolio when $n=200$, and $T=36$, are between one to three. For example, when $\alpha_j = 0.06$, $w = 1.5$, the number of assets in the portfolio is 21 when there is no transaction cost, while it is 18 when there is transaction cost.

5. Extension to Minimum Cost Rebalancing

The algorithm developed in Section 3 can be extended to rebalancing situation. Let \mathbf{x}^0 be the portfolio at hand and assume that an investor wants to rebalance a portfolio in such a way that the rate of return $E[R(\mathbf{x})]$ is greater than some constant ρ and that the risk is $W[R(\mathbf{x})]$ is smaller than some constant w . Let X be an investable set. Then the problem can be formulated as follows.

$$\left\{ \begin{array}{l} \text{minimize } c(\mathbf{x}) \\ \text{subject to } E[R(\mathbf{x})] \geq \rho, \\ \quad \quad \quad W[R(\mathbf{x})] \leq w, \\ \quad \quad \quad \mathbf{x} \in X, \end{array} \right. \tag{5.1}$$

where $c(\mathbf{x})$ is the cost function.

Let us introduce a new set of variables

$$\mathbf{v} = \mathbf{x} - \mathbf{x}^0,$$

Then the problem (5.1) can be represented as follows

$$(H) \left\{ \begin{array}{l} \text{minimize } \sum_{j=1}^n c_j(v_j) \\ \text{subject to } \sum_{j=1}^n r_j v_j \geq \rho - \sum_{j=1}^n r_j x_j^0, \\ \quad \quad \quad \sum_{t=1}^T |z_t| \leq w, \\ \quad \quad \quad z_t = f_t \sum_{j=1}^n [(r_{jt} - r_j)v_j + (r_{jt} - r_j)x_j^0], \quad t = 1, \dots, T, \\ \quad \quad \quad -x_j^0 \leq v_j \leq \alpha_j - x_j^0, \quad j = 1, \dots, n, \\ \quad \quad \quad v \in V, \end{array} \right.$$

where V is the set act of feasible v' s corresponding to X , and $c_j(v_j)$ is the cost associated with purchasing v_j units (if $v_j > 0$) and selling v_j units (if $v_j < 0$) of j th asset. Let us assume again that $c_j(v_j)$ is piecewise concave and that $c_j(0) = 0$ for all j (Figure 10)

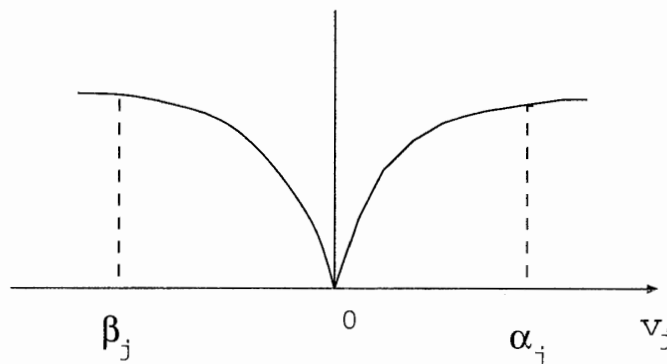


Figure 10: Piecewise Concave Cost Function

We can construct a branch and bound method similar to the one discussed in Section 3. Let (H_k) be a subproblem

$$(H_k) \left\{ \begin{array}{l} \text{minimize } \sum_{j=1}^n c_j v_j \\ \text{subject to } (\mathbf{v}, \mathbf{z}) \in F, \\ \quad \quad \quad \beta_j^k \leq v_j \leq \alpha_j^k, \quad j = 1, \dots, n, \end{array} \right.$$

where

$$F = \left\{ (\mathbf{v}, \mathbf{z}) \mid \sum_{j=1}^n r_j v_j \geq \rho - \sum_{j=1}^n r_j x_j^0, \sum_{t=1}^T |z_t| \leq w, \right. \\ \left. z_t = f_t \sum_{j=1}^n [(r_{jt} - r_j)v_j + (r_{jt} - r_j)x_j^0], t = 1, \dots, T, \mathbf{v} \in V \right\}.$$

We will approximate the function $c_j(v_j)$ in the interval $[\alpha_j^k, \beta_j^k]$ by a piecewise-linear convex function $c_j^k(v_j)$ as depicted in Figure 11.

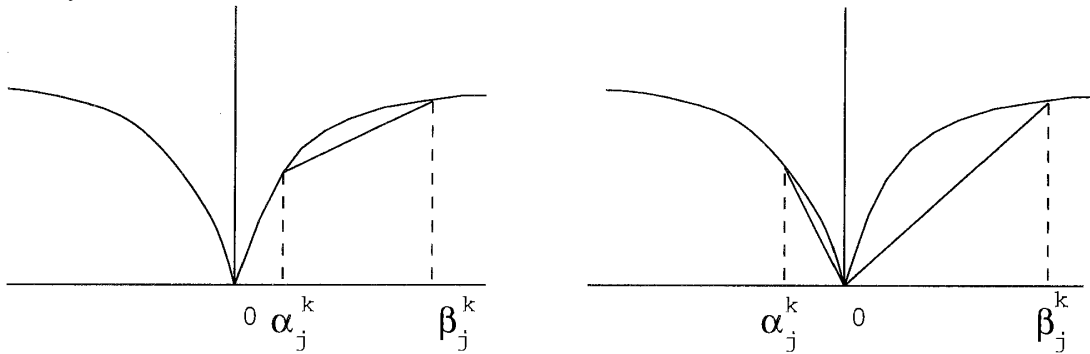


Figure 11: Piecewise Linear Underestimating Function and define a relaxed subproblem

$$(\bar{H}_k) \begin{cases} \text{minimize} & \sum_{j=1}^n c_j^k(v_j) \\ \text{subject to} & (\mathbf{v}, \mathbf{z}) \in F, \\ & \beta_j^k \leq v_j \leq \alpha_j^k, \quad j = 1, \dots, n, \end{cases}$$

which can be reduced to a linear programming problem by using a standard method.

6. Conclusion and Future Direction of Research

In this paper, we proposed a branch and bound method for solving a portfolio optimization problem under concave transaction costs. It has been demonstrated that practical problems with over 200 assets using upto 36 historical or scenario data can be successfully solved in a practical amount of time. It is also expected that the problem with over one thousand assets with 36 data can be solved by the same approach. This means that we can now handle the transaction costs in a rigorous way as long as it is a concave function of the amount of transaction. When the number of data is over 60 or when the transaction cost is neither convex nor concave as depicted in Figure 1, the problem is much more difficult and needs further research.

Another difficult problem associated with portfolio optimization is the handling of minimal transaction unit constraint. When the amount of investment is large enough, rounding the amount of transaction to the closest integer multiple of minimal transaction unit would have negligible effects. However, when the amount of investment is relatively small, above rounding method would not work. We are now extending the branch and bound method proposed in this paper to the problem with concave transaction costs and minimal transaction unit constraints, whose results will be reported subsequently.

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