

## CHARACTERISTICS ON STOCHASTIC DEA EFFICIENCY —RELIABILITY AND PROBABILITY BEING EFFICIENT—

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*Abstract* Data envelopment analysis (DEA) is a useful non-parametric method to evaluate a relative efficiency of multi-input and multi-output units based on observed data. In general, observed data have inherent uncertainty, however, it is difficult to treat the stochastic data in the conventional DEA model. It is required the development of stochastic DEA model, where the uncertainty like a measurement error should be incorporated. We consider in this paper an efficiency analysis of decision making units (DMUs) by using inputs and outputs data with stochastic variations, and discuss some stochastic measures of efficiency taking into account of the measurement error.

The most interesting characteristic is reliability and robustness of the efficiency result. We propose a measure for reliability of efficient DMUs as the amount of stochastic variations that remain the efficient DMU being efficient. A minimum efficiency score at a specified probability level is also used as a robustness measure. Moreover, we discuss some stochastic measures such as an expected efficiency score, a probability being efficient, an  $\alpha$ -percentile of efficiency score.

### 1. Introduction

Data envelopment analysis (DEA) is a useful non-parametric method to evaluate a relative efficiency of multi-input and multi-output units based on observed data, however, it is difficult to treat the stochastic data. In general, the uncertainty is inherent in observed data. The importance of stochastic DEA is evident (see e.g. Seiford [5]), where the uncertainty like a measurement error should be incorporated. Olesen and Petersen [4] proposed the chance constrained DEA model which provided the chance constrained efficiency index. However, the formulated problem is a non-convex programming problem, so they give the lower and upper bounds on the chance constrained efficiency measures. Sengupta [7] also proposed the chance constrained model for single output units. Statistical analysis given by Banker [3] shows the efficient frontier as a maximum likelihood estimates. A bootstrapping method is proposed to estimate an efficient frontier using sample data [2, 8].

We consider in this paper an efficiency analysis of decision making units (DMUs) by using inputs and outputs data with stochastic variations, and discuss some stochastic measures of efficiency taking into account of the measurement error. The most interesting characteristic is reliability and robustness of the efficiency result. We propose a measure for reliability of efficient units as the amount of stochastic variations that remain the efficient DMU being efficient. A minimum efficiency score at a specified probability level is also used as a robustness measure. Moreover, we discuss some stochastic measures such as an expected efficiency score, a probability being efficient, an  $\alpha$ -percentile of efficiency score.

We consider  $n$  DMUs. Each DMU consumes varying amounts of  $m$  different inputs to produce  $s$  different outputs. Specially, DMU <sub>$j$</sub>  consumes amounts  $X_j = \{x_{ij}\}$  of inputs

( $i = 1, \dots, m$ ) and produces amounts  $Y_j = \{y_{rj}\}$  of outputs ( $r = 1, \dots, s$ ). Here we mainly consider the CCR input oriented model. The conventional DEA model determines the relative efficiency without stochastic variations by the following linear programming problem.

$$\text{CCR-D} \left\{ \begin{array}{l} \max_{u,v} \quad w_o = v'Y_o, \\ \text{subject to} \quad u'X_o = 1, \\ \quad \quad \quad v'Y - u'X \leq 0, \\ \quad \quad \quad u, v \geq \epsilon, \end{array} \right. \quad (1.1)$$

which is a multiplier form of input oriented CCR model to evaluate  $DMU_o$ . This problem is solved  $n$  times with  $(X_o, Y_o) = (X_j, Y_j)$  for  $j = 1, \dots, n$  and the objective function value  $w_o^*$  partition the set of DMUs into two subsets; DMUs for which  $w_o^* = 1$  are efficient, while DMUs for which  $w_o^* < 1$  are inefficient.

## 2. Stochastic Data in DEA

The most representative stochastic data are noisy data with additive observational or measurement error, which is often assumed to be normally distributed. The repeated observation would improve the accuracy of estimates of stochastic error. Data given as sums of a certain period, such as a week, a month or a year, are also viewed as repeated observations for a sake of reducing the random effects.

Now, we note estimation method of the variances from the repeated observations. Let  $DMU_j$  have  $N_j$  observations  $x_j^k$ ,  $k = 1, \dots, N_j$  for a certain input or output, where  $x_j^k$  is assumed to be a sample independently drawn from a normal population with mean  $\mu_j$  and variance  $\sigma_j^2$ . The estimation of variances  $\sigma_j^2$  is categorized into the followings according to the assumption of populations.

No assumption :  $\sigma_j^2, j = 1, \dots, n$

Each variance should be separately estimated. By using a usual statistical inference, we have the estimate of variance  $\hat{\sigma}_j^2$  as follows.

$$\hat{\sigma}_j^2 = \frac{1}{N_j - 1} \sum_{k=1}^{N_j} (x_j^k - \bar{x}_j)^2, \quad (2.1)$$

where  $\bar{x}_j$  is a sample mean. It is required a sufficiently large number of observations for each DMU.

Equivalent variance assumption :  $\sigma_j^2 = \sigma^2$  for all  $j$

Each input and output has the same variance  $\sigma^2$  for all DMUs. The variance is inherent in each input and output, and does not depend on DMU. Then we have the estimates  $\hat{\sigma}^2$  by pooling (2.1) as follows.

$$\hat{\sigma}^2 = \frac{\sum_{j=1}^n (N_j - 1) \hat{\sigma}_j^2}{\sum_{j=1}^n N_j - n}, \quad (2.2)$$

which is the same to a one-factor analysis of variance (ANOVA) with repetition.

Proportional variance assumption :  $\sigma_j = K\mu_j$  for all  $j$

Each input and output has the same coefficient of variance  $K$  for all DMUs. The variance is proportional to the absolute value of input or output. The logarithmic

transformed variables  $\tilde{x}_j^k = \log(x_j^k)$  have the approximately same variance  $K^2$ .<sup>1</sup> Then we can estimate the coefficient of variance by applying (2.2) with  $\tilde{x}_j^k$ . This method can be applied for other variance assumptions.<sup>2</sup> The maximum likelihood estimation is also a useful method to estimate the variances.

### 3. Stochastic Variations in DEA

In the presence of stochastic variations, there are two situations to utilize stochastic information of DMUs. One situation takes the stochasticity into account before evaluating the efficiency, and the other situation does after. The former one evaluates the efficiency with the stochastic data directly using a distribution problem in stochastic programming, where several stochastic efficiency measures are derived. The later one evaluates the reliability and the robustness of efficient DMUs, where the efficiency is decided from deterministic data.

#### 3.1. Stochastic efficiency analysis

When we have stochastic data, i.e.,  $X(\omega)$  and  $Y(\omega)$  are random variables, problem (1.1) turns to the following stochastic programming problem.

$$\text{CCR}(\omega)\text{-D} \left\{ \begin{array}{l} \max_{u,v} \quad w_o = v'Y_o(\omega), \\ \text{subject to} \quad u'X_o(\omega) = 1, \\ \quad \quad \quad v'Y(\omega) - u'X(\omega) \leq 0, \\ \quad \quad \quad u, v \geq \epsilon. \end{array} \right. \quad (3.1)$$

The objective function value, which is an efficiency score, is stochastic and the partition of efficient or inefficient should be stochastic too. We are interested in some measures of stochastic efficiency such as an expected efficiency score, a probability being efficient, an  $\alpha$ -percentile of efficiency score and so on.

Since the stochastic problem (3.1) is very difficult to be solved theoretically, a Monte Carlo simulation method shown below is a useful technique to obtain stochastic efficiency.

1. Generate artificial data by random numbers which are drawn from the population estimated by the data set.
2. Calculate the efficiency score by solving problem (1.1) for fixed  $\omega$  given by step 1.
3. Repeat the above two steps for a sufficiently large number of runs.

Let  $\theta_1, \dots, \theta_N$  be efficiency scores for  $N$  runs. Then the estimates of some measures for stochastic efficiency are given as follows.

expectation and variance of efficiency score :  $\bar{\theta} = \frac{1}{N} \sum_{t=1}^N \theta_t$  and  $s_{\theta}^2 = \frac{1}{N-1} \sum_{t=1}^N (\theta_t - \bar{\theta})^2$

$\alpha$ -percentile of efficiency score :  $\theta^\alpha = \theta_{(N\alpha)}$ , where  $\theta_{(i)}$  is the  $i$ -th largest efficiency score.

probability being efficient :  $p = \#(\theta_t = 1)/N$ . Because the probability being efficiency is estimated based on a binomial distribution, a huge number of runs is required for accurate estimation.

<sup>1</sup> Let  $V(x_j) = \sigma_j^2$  and  $\sigma_j = K\mu_j$ . The Taylor expansion of  $f(x_j)$  at  $x_j = \mu_j$  is given by  $f(x_j) = f(\mu_j) + f'(\mu_j)(x_j - \mu_j) + \dots$ . Then we have the expectation  $E[f(x_j)] = f(\mu_j)$  and the variance  $V[f(x_j)] = \{f'(\mu_j)\}^2 \sigma_j^2$  approximately. If we select the function  $f$  such that  $f'(\mu_j)\sigma_j$  is constant,  $V[f(x_j)]$  is independent of  $j$ . By solving the differential equation  $f'(x)x = 1$ , we have  $f(x) = \log(x)$  and  $V[\log(x_j)] = K^2$ .

<sup>2</sup> Let  $\sigma_j^2 = g(\mu_j)$ . To make the variance of transformed variable being constant, we select the function  $f$  such that  $\{f'(x)\}^2 g(x) = 1$ . Then we have  $f(x) = \int \frac{1}{\sqrt{g(x)}} dx$ . For example, if  $\sigma_j^2 = K\mu_j$ , we have  $f(x) = 2\sqrt{x}$  and  $V[f(x)] = K$ .

### 3.2. Reliability analysis

Stochastic information can be also used to evaluate sensitivity and reliability of efficient DMUs. The conventional DEA model partitions the set of DMUs into two subsets that are efficient DMUs and inefficient DMUs, however, no information on stochastic variations is used for an efficiency evaluation. When there are stochastic variations, it is an inevitable consequence that the partition of efficiency is also stochastic. Then we are interested in the following points: how much reliable is the efficiency result against for the stochastic variations? how much probability of stochastic variations does ensure the efficiency? which input or output is most sensitive to the stochastic variations? and so on.

If the efficient DMU turns into inefficient by adding small change in data, the result that the DMU is efficient is not reliable for the stochastic variation, since small variation on the inputs or outputs data may bring a wrong efficiency result. The reliability of efficient DMU should be given by the amount of data variations that overturns the efficiency result. The efficient DMU that remains being efficient for large amount of variations should have a high reliability, and on the other hand, the efficient DMU such that small variation affects the efficiency result should have a low reliability. The main point of our idea is that the reliability of  $DMU_o$  corresponds to the distance from  $DMU_o$  to the efficient frontier of production possibility set excluding  $DMU_o$ .

The D-efficient<sup>3</sup> DMUs with positive slacks are evaluated to be inefficient for any small changes in inputs or outputs without slacks. Therefore, the reliability of D-efficient DMUs with positive slackness should be zero, since even a small variation may bring it inefficient. Notice that, same to the D-efficient DMU with positive slacks, the efficient DMU that is on the efficient frontier of production possibility set has the reliability of zero.

We propose the reliability measure as the amount of variation that keeps the efficiency evaluation. Measuring the amount of variation by the occurrence probability, we propose a reliability measure of D-efficient DMU as the maximal probability of stochastic variations that ensure it being efficient. In other words, any stochastic variations occurred at this probability level remain the DMU being efficient.

The stochastic evaluation of DMU as a measure of reliability and robustness provides an additional information on efficiency results. The sensitivity of inputs and outputs and the estimation of lower bound of probability being efficient can be also discussed in the framework of the following reliability analysis.

## 4. Reliability of Efficient DMUs

We denote the stochastic variations as  $\delta = (\delta_x, \delta_y)$ , where  $\delta_x = (\delta_{x1}, \dots, \delta_{xm})$  is for inputs and  $\delta_y = (\delta_{y1}, \dots, \delta_{ys})$  is for outputs, and assume that  $\delta$  has a multi-normal distribution with mean zero and finite variances. The inputs and outputs are observed including the additive stochastic variations  $\delta$ , that is, the observations of  $DMU_o$  are expressed as  $(X_o + \delta_x, Y_o - \delta_y)^4$ . We consider the lowest efficiency score under the stochastic variations occurred at the specified probability level. The lowest efficiency score is monotonically decreasing as the stochastic variations enlarge. As we make the stochastic variation large, the efficient DMU turn into inefficient in the end. Therefore, we define the reliability of efficient DMU as the amount of stochastic variations that force the efficient DMU to being inefficient. Actually, if the DMU remains being efficient in spite of large variations, the result of efficiency is robust

<sup>3</sup> The DMU with efficiency score of 1 is called D-efficient.

<sup>4</sup> Since  $\delta$  has a symmetric distribution, we are interested in the size of  $\delta$ . We use the sign of  $\delta$  such as  $X_o + \delta_x$  and  $Y_o - \delta_y$  so as to express that the positive variations lower the efficiency score

for the stochastic variations and so the reliability is high.

Let  $\Sigma_o$  be the variance-covariance matrix of stochastic variation of DMU<sub>o</sub>. Since  $\delta$  has a normal distribution  $N(0, \Sigma_o)$ , we have that  $\delta' \Sigma_o^{-1} \delta$  has a  $\chi^2$ -distribution with degrees of freedom  $m+s$ . Then the confidence region of stochastic variations  $\delta$  in DMU<sub>o</sub> at probability level  $\alpha$  is given by

$$S_\alpha = \left\{ \delta \mid \delta' \Sigma_o^{-1} \delta \leq \chi_{m+s}^2(\alpha) \right\}, \tag{4.1}$$

where  $\chi_{m+s}^2(\alpha)$  is an  $\alpha$  percentile of the  $\chi^2$ -distribution.

Assuming that the inputs and outputs include the stochastic variations  $\delta$  that realize the value in (4.1) with probability  $\alpha$ , the efficiency score of DMU<sub>o</sub> is expressed as a function of  $\delta$ . When DMU<sub>o</sub> has additional stochastic variations,  $(X_o, Y_o)$  in problem (1) is replaced by  $(X_o + \delta_x, Y_o - \delta_y)$  and we have the efficiency score by the following problem.

$$\left| \begin{array}{l} \max_{u,v} \quad \tilde{w}_o(\delta) = v'(Y_o - \delta_y) \\ \text{subject to} \quad u'(X_o + \delta_x) = 1, \\ \quad \quad \quad v'Y_o - u'X_o \leq 0, \\ \quad \quad \quad u, v \geq \epsilon. \end{array} \right. \tag{4.2}$$

where

$$X_j^o = \begin{cases} X_j & j \neq o \\ X_o + \delta_x & j = o \end{cases}, \quad Y_j^o = \begin{cases} Y_j & j \neq o \\ Y_o - \delta_y & j = o \end{cases} \tag{4.3}$$

and let  $\tilde{w}_o^*(\delta)$  be an efficiency score for a given  $\delta$ .

Now we consider the stochastic variations occurred with probability  $\alpha$ , and the minimum efficiency score when the stochastic variations are in (4.1). The following problem gives the lowest efficiency at a specific probability level  $\alpha$ :

$$\left| \begin{array}{l} \min_\delta \quad w_o(\alpha) = \tilde{w}_o^*(\delta), \\ \text{subject to} \quad \delta' \Sigma_o^{-1} \delta \leq \chi_{m+s}^2(\alpha), \end{array} \right. \tag{4.4}$$

or

$$\text{MIN}(\alpha)\text{-D} \left| \begin{array}{l} \min_\delta \max_{u,v} \quad w_o(\alpha) = v'(Y_o - \delta_y), \\ \text{subject to} \quad u'(X_o + \delta_x) = 1, \\ \quad \quad \quad v'Y_o - u'X_o \leq 0, \\ \quad \quad \quad u, v \geq \epsilon, \\ \quad \quad \quad \delta' \Sigma_o^{-1} \delta \leq \chi_{m+s}^2(\alpha). \end{array} \right. \tag{4.5}$$

Let the optimal value of problem (4.5) denote  $w_o^*(\alpha)$ . Unfortunately, it is difficult to obtain a global optimal solution as it is, since problem (4.5) has a mini-max type objective function and non-convex constraints. Problem (4.5) is rewritten to the following envelopment form, which is a dual form of problem (4.5) for fixed  $\delta$ .

$$\text{MIN}(\alpha)\text{-P} \left| \begin{array}{l} \min_{\delta, \theta, \lambda, s_x, s_y} \quad z_o(\alpha) = \theta - \epsilon' s_x - \epsilon' s_y, \\ \text{subject to} \quad Y^o \lambda - s_y = Y_o - \delta_y, \\ \quad \quad \quad X^o \lambda + s_x = \theta(X_o + \delta_x), \\ \quad \quad \quad \lambda, s_x, s_y \geq 0, \\ \quad \quad \quad \delta' \Sigma_o^{-1} \delta \leq \chi_{m+s}^2(\alpha). \end{array} \right. \tag{4.6}$$

This is not a mini-max problem, but still has a non-convex constraint.

Note that again  $w_o^*(\alpha)$  is the minimum efficiency score at a specific probability level  $\alpha$ , that is, any stochastic variations in the confidence region at probability level  $\alpha$  cannot lower

the efficiency score  $w_o^*(\alpha)$ . Therefore when  $w_o^*(\alpha) = 1$ ,  $DMU_o$  is D-efficient for any  $\delta$  in (4.1). We denote the maximum of  $\alpha$  as  $\alpha_{\max}$  when  $w_o^*(\alpha) = 1$ :

$$\alpha_{\max} = \max\{\alpha | w_o^*(\alpha) = 1\}. \tag{4.7}$$

The efficient  $DMU_o$  is evaluated to be inefficient for some stochastic variations in the confidence region at probability level more than  $\alpha_{\max}$ . When we consider the point  $(X_o + \delta_x^*, Y_o - \delta_y^*)$  that implies  $w_o^*(\alpha_{\max}) = 1$ , there is a possibility that  $(X_o + \delta_x^*, Y_o - \delta_y^*)$  is inefficient because of the positive slacks. However,  $\delta^*$  must be uniquely determined from problem (4.4) and  $DMU_o$  is efficient for any  $\delta \in S_{\alpha_{\max}}$  except for  $\delta^*$ .

From the normality assumption on  $\delta$ , we have the probability  $\Pr(\delta = \delta^*) = 0$ . Therefore,  $DMU_o$  is efficient with probability one for all  $\delta \in S_{\alpha_{\max}}$ . We define the *reliability* of  $DMU_o$  as the probability level  $\alpha_{\max}$ .

To obtain the reliability  $\alpha_{\max}$ , we consider the production possibility set excluding  $DMU_o$ , which is given by the super-efficiency (or extended efficiency) model of Andersen and Petersen [1]:

$$SE \left\{ \begin{array}{l} \min_{\theta, \lambda, s_x, s_y} \quad \hat{z}_o = \theta - \epsilon' s_x - \epsilon' s_y, \\ \text{subject to} \quad Y_{-o} \lambda - s_y = Y_o, \\ \quad \quad \quad X_{-o} \lambda + s_x = \theta X_o, \\ \quad \quad \quad \lambda, s_x, s_y \geq 0, \end{array} \right. \tag{4.8}$$

where  $X_{-o} = (X_1, \dots, X_{o-1}, X_{o+1}, \dots, X_n)$  and  $Y_{-o} = (Y_1, \dots, Y_{o-1}, Y_{o+1}, \dots, Y_n)$ . The efficient frontier of the production possibility set is constructed by excluding  $DMU_o$ , and so the super efficiency score  $\hat{z}_o^*$  measured by the efficient frontier is not bounded by one from above. Note that the efficient DMUs have the super efficiency score of more than or equal to one, while the inefficient DMUs have the exactly same efficiency score to the CCR model.

Now, we consider the super-efficiency model with stochastic variations derived from (4.6).

$$SE(\alpha) \left\{ \begin{array}{l} \min_{\delta, \theta, \lambda, s_x, s_y} \quad \hat{z}_o(\alpha) = \theta - \epsilon' s_x - \epsilon' s_y, \\ \text{subject to} \quad Y_{-o} \lambda - s_y = Y_o - \delta_y, \\ \quad \quad \quad X_{-o} \lambda + s_x = \theta(X_o + \delta_x), \\ \quad \quad \quad \lambda, s_x, s_y \geq 0, \\ \quad \quad \quad \delta' \Sigma_o^{-1} \delta \leq \chi_{m+s}^2(\alpha). \end{array} \right. \tag{4.9}$$

Problem (4.9) gives the minimum of super efficiency score at a specific probability level  $\alpha$ , however, it is not convex problem. Let  $\hat{z}_o^*(\alpha)$  be an optimal value of (4.9). Then we have

$$z_o^*(\alpha) = 1, \hat{z}_o^*(\alpha) > 1 \quad \text{for } \alpha < \alpha_{\max}, \tag{4.10}$$

$$z_o^*(\alpha) = \hat{z}_o^*(\alpha) = 1 \quad \text{for } \alpha = \alpha_{\max}, \tag{4.11}$$

$$z_o^*(\alpha) = \hat{z}_o^*(\alpha) < 1 \quad \text{for } \alpha > \alpha_{\max}. \tag{4.12}$$

If  $\theta = 1$  is a feasible solution of (4.9), we have  $\alpha \geq \alpha_{\max}$ . We can obtain the  $\alpha_{\max}$  as the minimum of  $\alpha$  subject to  $\theta = 1$ , which is given by the problem:

$$\left\{ \begin{array}{l} \min_{\delta, \lambda} \quad \alpha, \\ \text{subject to} \quad Y_{-o} \lambda \geq Y_o - \delta_y, \\ \quad \quad \quad X_{-o} \lambda \leq X_o + \delta_x, \\ \quad \quad \quad \lambda \geq 0, \\ \quad \quad \quad \delta' \Sigma_o^{-1} \delta \leq \chi_{m+s}^2(\alpha). \end{array} \right. \tag{4.13}$$

Since  $\chi_{m+s}^2(\alpha)$  is a monotonically decreasing function of  $\alpha$ , the minimum of  $\alpha$  is attained by the minimum of  $\chi_{m+s}^2(\alpha)$ . Therefore, the following quadratic programming problem can find the  $\alpha_{\max}$ .

$$R \left\{ \begin{array}{l} \min_{\lambda, \delta} \quad \Delta = \delta' \Sigma_o^{-1} \delta, \\ \text{subject to} \quad Y_{-o} \lambda \geq Y_o - \delta_y, \\ \quad \quad \quad X_{-o} \lambda \leq X_o + \delta_x, \\ \quad \quad \quad \lambda \geq 0. \end{array} \right. \quad (4.14)$$

Notice that we can easily obtain the reliability  $\alpha_{\max}$ , since problem (4.14) is a quadratic programming problem with linear constraints. Let an optimal value of (4.14) denote  $\Delta^*$ . The reliability  $\alpha_{\max}$  is given from

$$\chi_{m+s}^2(\alpha_{\max}) = \Delta^*. \quad (4.15)$$

We discuss the reliability for efficient DMUs. There are efficient DMUs on the efficient frontier of production possibility set in some cases. The DMUs on the efficient frontier have a reliability of 0, which is the same to the reliability of inefficient DMUs which is D-efficient but has positive slacks. Because both of them are located on the frontier of production possibility set and any stochastic variations affect the efficiency, our proposed measure provides the same reliability for both DMUs in spite that one is efficient and the other is inefficient.

The  $\sqrt{\Delta^*}$  denotes a stochastic distance from data  $(X_o + \delta_x, Y_o - \delta_y)$  to the envelopment surface. The optimal solution  $\delta^*$  gives us more information. We decompose  $\Delta^*$  into each input and output by  $\Delta^* = d'd$ , where

$$d = \Sigma_o^{-\frac{1}{2}} \delta^*, \quad (4.16)$$

The stochastic distance vector  $d$  shows the sensitivity of inputs and outputs. The point  $(X_o, Y_o)$  has the closest point measured by a stochastic distance on the efficient frontier in direction of  $d$ . If  $d_{x1} > d_{x2}$ , input-1 is more sensitive to the stochastic variations than input-2.

If we fix the efficiency score  $\theta$ , we can find the maximum probability level  $\alpha(\theta)$  that ensures the specific efficiency score  $\theta$  by solving the following problem:

$$\left\{ \begin{array}{l} \min_{\lambda, \delta} \quad \Delta(\theta) = \delta' \Sigma_o^{-1} \delta, \\ \text{subject to} \quad Y_{-o} \lambda \geq Y_o - \delta_y, \\ \quad \quad \quad X_{-o} \lambda \leq \theta(X_o + \delta_x), \\ \quad \quad \quad \lambda \geq 0. \end{array} \right. \quad (4.17)$$

The probability  $\alpha(\theta)$  is derived from (4.15) as a function of  $\theta$ . If we calculate  $\alpha(\theta)$  for several values of  $\theta$ , we can draw a curve of  $\alpha(\theta)$  on  $(\theta, \alpha)$ -space as shown in Figure 1. The minimum efficiency score  $\theta_0$  at probability level  $\alpha_0$ , say  $\alpha_0\%$ -efficiency, is given from  $\alpha(\theta_0) = \alpha_0$ .

### 5. Simple Example

Consider five DMUs with two inputs and one unity output. The input values for five DMUs are A(3,5), B(5,3), C(8,2), D(9,2), E(5,5). DMUs A, B and C are efficient, and DMUs D and E are inefficient. DMU D has an efficiency score of 1, however, it has a positive slack in input-1. Assume that each input has a stochastic error with zero mean and variance  $\sigma_{x1}^2 = 0.8^2$  and  $\sigma_{x2}^2 = 0.5^2$ , respectively, where they are normally distributed and independent each other.

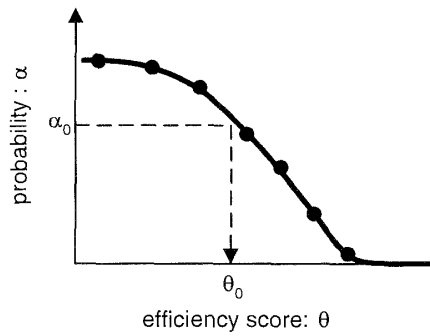


Figure 1: Curve of  $\alpha(\theta)$

First, we evaluate DMU B, whose super-efficiency score is  $\hat{z}_B = 1.133$ . The reliability of DMU B is given by solving the problem:

$$P_B \begin{cases} \min_{\lambda, \delta} & \left(\frac{\delta_1}{0.8}\right)^2 + \left(\frac{\delta_2}{0.5}\right)^2 \\ \text{subject to} & 3\lambda_A + 8\lambda_C + 9\lambda_D + 5\lambda_E \leq 5 + \delta_1 \\ & 5\lambda_A + 2\lambda_C + 2\lambda_D + 5\lambda_E \leq 3 + \delta_2 \\ & \lambda_A + \lambda_C + \lambda_D + \lambda_E = 1 \\ & \lambda_A, \lambda_C, \lambda_D, \lambda_E \geq 0 \end{cases} \tag{5.1}$$

The optimal solution is  $\Delta^* = 1.3322$ ,  $\delta_1^* = 0.639$ ,  $\delta_2^* = 0.416$ ,  $\lambda_A^* = 0.472$ ,  $\lambda_C^* = 0.528$  and  $\lambda_D^* = \lambda_E^* = 0$ . From  $\chi_2^2(.4863) = 1.3322$ , we have  $\alpha_{\max} = 48.6\%$ . The stochastic distances are  $d_1 = 0.80$  for input-1 and  $d_2 = 0.83$  for input-2. The sensitivity for each input is almost same although  $\delta_1^*$  is greater than  $\delta_2^*$ , since the variance of  $\delta_1$  is larger than that of  $\delta_2$ .

Next, we evaluate DMU C, whose super-efficiency score is  $\hat{z}_C = 1.0625$ . The reliability of DMU C is given by solving the problem:

$$P_C \begin{cases} \min_{\lambda, \delta} & \left(\frac{\delta_1}{0.8}\right)^2 + \left(\frac{\delta_2}{0.5}\right)^2 \\ \text{subject to} & 3\lambda_A + 5\lambda_B + 9\lambda_D + 5\lambda_E \leq 8 + \delta_1 \\ & 5\lambda_A + 3\lambda_B + 2\lambda_D + 5\lambda_E \leq 2 + \delta_2 \\ & \lambda_A + \lambda_B + \lambda_D + \lambda_E = 1 \\ & \lambda_A, \lambda_B, \lambda_D, \lambda_E \geq 0 \end{cases} \tag{5.2}$$

Table 1: Reliability of efficient DMUs

measures	DMU				
	A	B	C	D	E
super efficiency score : $\hat{z}$	1.667	1.133	1.063	1.000	0.800
stochastic distance : $\sqrt{\Delta^*}$	2.500	1.154	0.464	0.000	—
$\delta_1^*$	2.000	0.639	0.138	0.000	—
$\delta_2^*$	0.000	0.416	0.216	0.000	—
$d_1$	2.500	0.799	0.173	0.000	—
$d_2$	0.000	0.832	0.432	0.000	—
reliability : $\alpha_{\max}$	95.6%	48.6%	10.2%	0.0%	—
90% efficiency	1	.908	.824	.751	.665
95% efficiency	1	.884	.799	.731	.650



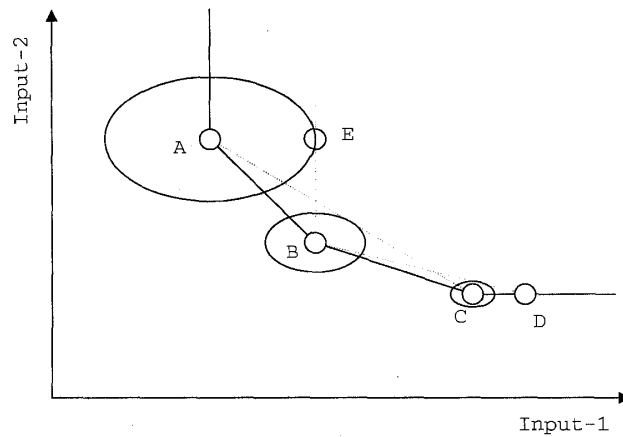


Figure 2: Ellipsoids of Reliability

The optimal solution is  $\Delta^* = 0.2155$ ,  $\delta_1^* = 0.138$ ,  $\delta_2^* = 0.216$ ,  $\lambda_B^* = 0.216$ ,  $\lambda_D^* = 0.784$  and  $\lambda_A^* = \lambda_E^* = 0$ . From  $\chi_2^2(.1022) = 0.2155$ , we have  $\alpha_{\max} = 10.2\%$ . The stochastic distances are  $d_1 = 0.17$  for input-1 and  $d_2 = 0.43$  for input-2. Input-2 is more sensitive to stochastic variations than input-1.

Figure 2 shows the ellipsoids of reliability for DMUs A, B and C. The results for other DMUs are shown in Table 1. DMU A is an extremely reliable efficiency unit, and input-2 is quite insensitive. DMU D is unreliable, because it has a slack in input-1 and any variation in input-2 brings it inefficient. DMU C has small reliability, especially it is sensitive in input-2. DMU B has medium reliability and the sensitivity of inputs is almost same.

### 6. Probability Being Efficient

There are two kinds of probability being efficient. One is the probability being efficient (type I probability  $P_I$ ) when all DMUs are assumed to have stochastic data, which may be a true probability being efficient under the stochastic variations. The simulation method shown in section 3.1 would be the most useful method, however, a huge number of iterations is required for a good estimation of the probability being efficient. The other is the probability being efficient (type II probability  $P_{II}$ ) when the DMUs out of investigation are assumed to have deterministic data. The stochasticity exists only in the  $DMU_o$  under investigation.

Seiford and Zhu [6] show the region that the efficient DMU remains being efficient. The size of the region indicates the measure of sensitivity, however, the region is not finite in many cases, so it is impossible to compare the size of region quantitatively. By introducing the stochastic information for a quantitative evaluation of sensitivity, we can evaluate the probability that the DMU remains in that region.

In any case, since it is difficult to obtain the probability being efficient itself, we show in this section a lower bound for two types of probabilities being efficient using the results of reliability analysis.

#### 6.1. Probability being efficient as an efficiency measure (Type I)

When all DMUs are assumed to have stochastic variations, the difficulty to get the probability being efficient is an indefinite envelope surface. For efficient  $DMU_o$ , there exists a plane in the data space that separates  $DMU_o$  from other DMUs. If  $DMU_o$  is separated from other DMUs by the plane,  $DMU_o$  is certainly efficient.

We select plane  $L$  tangent to the region  $S_{\alpha_{\max}} + (X_o, Y_o)$  at  $(X_o + \delta_x^*, Y_o - \delta_y^*)$ , since

the tangent plane has several good statistical properties and is easily obtained from the reliability analysis. Let  $L$  denote as  $a_x x - a_y y = 0$ , where  $a = (a_x, a_y)$  is a tangent vector at  $(X_o + \delta_x^*, Y_o - \delta_y^*)$ . Then  $L$  is expressed as

$$a = c \cdot \Sigma_o^{-1} \delta^*, \quad (6.1)$$

where  $c$  is a positive constant, and we have

$$\Delta^* = \frac{\{a'(X_o, -Y_o)\}^2}{a' \Sigma_o a}. \quad (6.2)$$

The following theorem shows the distribution of the distance from  $DMU_o$  to plane  $L$ .

**Theorem 1.** The distance  $D_o$  from point  $(X_o + \delta_x, Y_o - \delta_y)$  to plane  $L$  is normally distributed with mean  $\bar{D}_o$  and variance 1, where

$$\bar{D}_o = \frac{a'(X_o, -Y_o)}{\sqrt{a' \Sigma_o a}} \quad (6.3)$$

**Proof:** The distance  $D_o$  from point  $(X_o + \delta_x, Y_o - \delta_y)$  to plane  $L$  is given by

$$D_o = \frac{a'(X_o + \delta_x, -Y_o + \delta_y)}{\sqrt{a' \Sigma_o a}}. \quad (6.4)$$

From  $\delta \sim N(0, \Sigma_o)$ , it holds  $V[a'\delta] = a' \Sigma_o a$ . Then we have  $E[D_o] = \bar{D}_o$  and  $V[D_o] = \frac{V[a'\delta]}{a' \Sigma_o a} = 1$ . (q.e.d.)

Next, we consider the distance from  $DMU_j$  to plane  $L$ .  $DMU_j$  has stochastic variations  $(\delta_x, \delta_y)$ , which are normally distributed with mean zero and variance-covariance matrix  $\Sigma_j$ .

**Corollary 1.** The distance  $D_j$  from point  $(X_j + \delta_x, Y_j - \delta_y)$  to plane  $L$  is normally distributed with mean  $\bar{D}_j$  and variance 1, where

$$\bar{D}_j = \frac{a'(X_j, Y_j)}{\sqrt{a' \Sigma_j a}}. \quad (6.5)$$

**Proof:** The distance  $D_j$  from point  $(X_j + \delta_x, Y_j - \delta_y)$  to plane  $L$  is given by

$$D_j = \frac{a'(X_j + \delta_x, -Y_j + \delta_y)}{\sqrt{a' \Sigma_j a}} \quad (6.6)$$

From  $\delta \sim N(0, \Sigma_j)$ , it holds  $V[a'\delta] = a' \Sigma_j a$ . Then we have  $E[D_j] = \bar{D}_j$  and  $V[D_j] = 1$ . (q.e.d.)

When  $D_o = l$ , let us consider a plane parallel to plane  $L$  whose distance to plane  $L$  is  $l$ , and denote it by plane  $(L - l)$ . To remain  $DMU_o$  being efficient, it is sufficient that other  $DMU_s$  are in the opposite side of  $DMU_o$  with respect to plane  $(L - l)$ . Note that the distances  $D_o$  and  $D_j$  are stochastic distances, which are transformed by the variance-covariance matrices. When we denote  $D_o = l$ , the distance  $l$  is measured using variance-covariance matrix  $\Sigma_o$ . The same distance  $l$  should be measured as  $C_j \times l$  for  $DMU_j$ , where

$$C_j = \sqrt{\frac{a' \Sigma_o a}{a' \Sigma_j a}}. \quad (6.7)$$

For fixed  $l$ , we have a conditional probability that other DMUs are in opposite side of  $DMU_o$  as follows.

$$P(l) = \prod_{j \neq o} \Pr(D_j + C_j l \geq 0) = \prod_{j \neq o} u^{-1}(\bar{D}_j + C_j l), \tag{6.8}$$

where  $u^{-1}$  is a cumulative function of standard normal distribution. The probability  $\hat{P}_I$  that DMUs except for  $DMU_o$  are in opposite side of  $DMU_o$  is given by convoluting  $l$ .

$$\hat{P}_I = \int P(l) f(l) dl \tag{6.9}$$

where  $f(l)$  is a probability density function of normal distribution  $N(\bar{D}_o, 1)$ . The probability  $\hat{P}_I$  is certainly a lower bound of type I probability  $P_I$ , because  $DMU_o$  is separated from other DMUs by a plane.

### 6.2. Probability being efficient as a sensitivity measure (Type II)

Since we assume DMUs except for  $DMU_o$  have deterministic data, the envelopment surface of super-efficiency model is definite. The envelopment surface divides the data space into two regions. One is the region including  $DMU_o$ , say  $R_o$ , and the other is the region including  $DMU_{-o}$ , say  $\bar{R}_o$ . If  $DMU_o$  is in region  $R_o$ , it remains being efficient. The size of region  $R_o$  would be a sensitivity measure, however, region  $R_o$  is not finite in many case.

To compare the sensitivity quantitatively, the probability that  $DMU_o$  stays in the region  $R_o$  is an effective measure, which is the same to the probability being efficient. The lower bound of the probability is given by the reliability  $\Delta^*$ .  $DMU_o$  is efficient for any  $\delta$  under plane  $L$ , that is,  $a'\delta + a'(X_o, -Y_o) \geq 0$ . Since the region  $R_o$  is convex, probability  $\hat{P}_{II} = \Pr(a'\delta + a'(X_o, -Y_o) \geq 0)$  is a lower bound of the probability. From (6.2), we have  $a'\delta \sim N(0, \frac{\{a'(X_o, -Y_o)\}^2}{\Delta^*})$ , and then the lower bound is given by

$$\hat{P}_{II} = u^{-1}(\sqrt{\Delta^*}). \tag{6.10}$$

### 6.3. Numerical simulation

The numerical simulation to obtain a probability distribution of efficiency score is used to compare our proposed stochastic efficiency measures. The input data for five DMUs in section 5 are generated by random numbers according to the normal distributions, for example, input-1 of DMU A has mean 3 and variance  $0.8^2$ , and input-2 of DMU A has

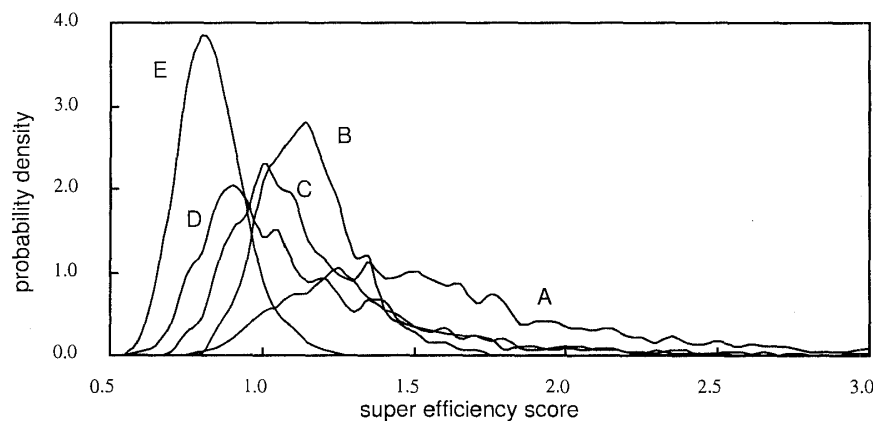


Figure 3: Distribution of super efficient score

Table 2: Stochastic efficiency

measures	DMU				
	A	B	C	D	E
expectation : $E(\hat{z})$	1.657	1.130	1.155	1.117	0.810
standard deviation : $s(\hat{z})$	0.811	0.158	0.348	0.413	0.114
90-percentile : $\hat{z}^{0.90}$	1.065	0.946	0.858	0.772	0.674
95-percentile : $\hat{z}^{0.95}$	0.977	0.900	0.812	0.727	0.646
probability being efficient : $p$	93.8%	79.0%	63.3%	50.7%	5.6%

Table 3: Probability being efficient  $\hat{P}_I$  as an efficiency measure

DMUs	lower bound	confidence interval
DMU A :	90.3%	[92.6%, 94.8%]
DMU B :	65.5%	[77.1%, 80.7%]
DMU C :	47.1%	[61.2%, 65.4%]
DMU D :	48.8%	[48.5%, 52.9%]

Table 4: Probability being efficient  $\hat{P}_{II}$  as a sensitivity measure

DMUs	lower bound	confidence interval
DMU A :	98.8%	[99.4%, 99.7%]
DMU B :	87.5%	[86.9%, 88.2%]
DMU C :	67.9%	[67.7%, 69.5%]
DMU D :	50.0%	[49.8%, 51.8%]

mean 5 and variance  $0.5^2$ . We have calculated a super efficiency score for 2000 runs. Figure 3 shows the distributions of super efficiency scores for five DMUs, and Table 2 shows the results of stochastic measures of efficiency such as expectation, standard deviation, 90%- and 95%-efficiency and the estimates of probability being efficient. The variance of efficiency score is apt to be large as the expectation of efficiency score is large. The large expectation of super efficiency score does not always have a high reliability.

Table 3 shows a lower bound of the probability being efficient (type I) as a stochastic measure. The confidence intervals of the probability are obtained by a simulation analysis with 2000 runs. The probability being efficient of DMU C is larger than that of DMU D, however, the lower bound of DMU C is smaller than that of DMU D. DMU D is apt to have an over-evaluation of efficiency, because of a positive slack in input-1. The probability  $\hat{P}_I$  is certainly a lower bound of the probability being efficient. Unfortunately, it is not a good lower bound of the probability being efficient, because we use rough approximation of envelope surface to ensure  $DMU_o$  being efficient. Table 4 shows a lower bound of probability being efficient (type II) as a sensitivity measure. The confidence intervals of the probability are obtained by a simulation analysis with 10000 runs. It seems a good lower bound of type II probability.

## 7. Application to Hospital Data

Consider the actual hospital data set in Tone [9], which is shown in Table 5 with the CCR efficiency scores. This data set consists of 14 DMUs (hospitals) with two inputs (total working time of doctors and nurses for a month) and two outputs (medical points of outpatients and inpatients for a month). Five hospitals (DMU 2, 3, 6, 8, 10) are evaluated

Table 5: Hospital Data Set

Hospital	Input-1 Doctors	Input-2 Nurses	Output-1 Outpatients	Output-2 Inpatients	CCR efficiency
DMU1	3008	20980	97775	101225	.955
DMU2	3985	25643	135871	130580	1
DMU3	4324	26978	133655	168473	1
DMU4	3534	25361	46243	100407	.702
DMU5	8836	40796	176661	215615	.827
DMU6	5376	37562	182576	217615	1
DMU7	4982	33088	98880	167278	.844
DMU8	4775	39122	136701	193393	1
DMU9	8046	42958	225138	256575	.995
DMU10	8554	48955	257370	312877	1
DMU11	6147	45514	165274	227099	.913
DMU12	8366	55140	203989	321623	.969
DMU13	13479	68037	174270	341743	.786
DMU14	21808	78302	322990	487539	.974

Table 6: Reliability of efficient hospitals

measures	Hospitals				
	2	3	6	8	10
$\hat{z}$	1.057	1.021	1.075	1.001	1.040
$\sqrt{\Delta^*}$	2.534	0.961	3.817	0.024	2.083
$\delta_{x1}^*$	17.6	9.5	104.2	0.5	0.0
$\delta_{x2}^*$	177.7	60.5	0.0	0.0	536.0
$\delta_{y1}^*$	6512.8	485.4	10589.3	12.0	6454.9
$\delta_{y2}^*$	1251.8	3058.7	9140.2	84.7	9249.2
$d_{x1}^*$	0.441	0.220	1.938	0.009	0.000
$d_{x2}^*$	0.693	0.224	0.000	0.000	1.095
$d_{y1}^*$	2.385	0.182	2.900	0.008	1.254
$d_{y2}^*$	0.236	0.908	2.100	0.022	1.478
reliability : $\alpha_{\max}$	83.00%	7.88%	99.43%	0.00%	63.81%
95% efficiency	.988	.955	1	.933	.981
99% efficiency	.975	.944	1	.920	.971
lower bound of probability being efficient					
Type I	93.2%	60.5%	99.3%	48.4%	87.9%
Type II	99.4%	83.2%	100.0%	51.0%	98.1%

Table 7: CCR, 95%- and 99%-efficiency scores

	Hospitals													
	1	2	3	4	5	6	7	8	9	10	11	12	13	14
CCR	.955	1	1	.702	.827	1	.844	1	.995	1	.913	.969	.786	.974
95%	.890	.988	.955	.654	.779	1	.789	.933	.928	.981	.851	.906	.732	.908
99%	.879	.975	.944	.646	.771	1	.779	.920	.917	.971	.839	.894	.723	.896

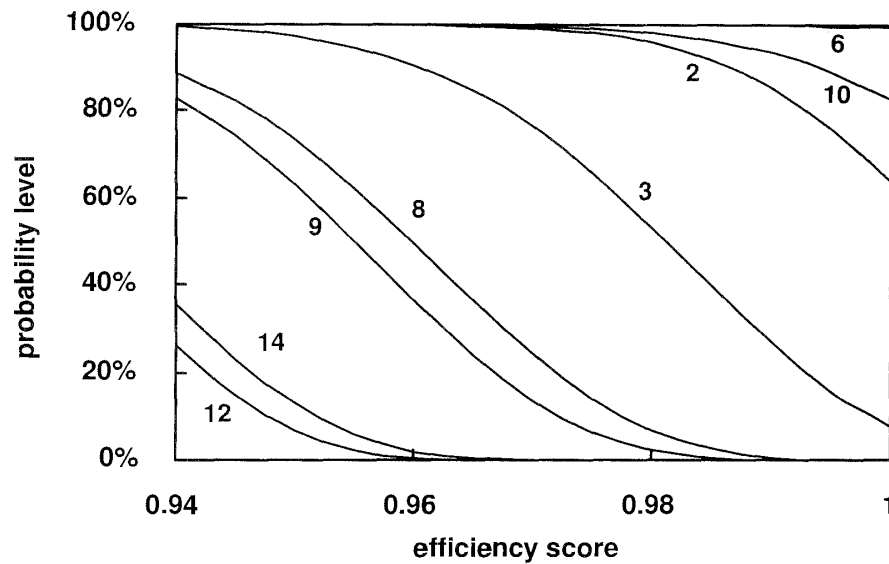


Figure 4: Probability level vs. efficiency score

to be efficient. This data set is observed for a month. Actually, each input and output has daily, weekly or monthly fluctuation. If data are observed everyday, the variance of daily data can be estimated by using daily observations, and then the variance of monthly data is given as 30 times of daily variance. Here we give some kinds of variances artificially, and compare the stochastic efficiency of hospitals.

First, we assume that the standard deviation is proportional to the amount of data with 1% for inputs and 2% for outputs, and a correlation between output-1 and 2 is  $\rho = 0.2$ , for example, the variances of DMU 1 are  $\sigma_{x_1}^2 = 30.08^2$ ,  $\sigma_{x_2}^2 = 209.80^2$ ,  $\sigma_{y_1}^2 = 1955.50^2$  and  $\sigma_{y_2}^2 = 2024.50^2$ . The reliability of efficient hospitals are shown in Table 6. The sensitivity of each input and output is given by the stochastic distance  $d$ , for example, DMU 2 is sensitive to output-1, and DMUs 6 and 8 are insensitive to input-2. DMU 6 is an extremely reliable efficient hospital, which is efficient with reliability greater than 99%. DMUs 2 and 10 are also reliable efficient hospitals, however, DMUs 3 and 8 has low reliability. DMU 3 is sensitive to output-2, so the change in output-2 may bring it an inefficient hospital. DMU 8 is very close to the efficient frontier, and may happen to be efficient by stochastic error. The lower bounds of probability being efficient discussed in section 6 are also shown, where DMUs 2, 6 and 10 have a high probability being efficient, however, DMU 8 has a fifty-fifty probability being efficient.

We can obtain the maximum probability level for a specific efficiency score by using problem (4.17). Figure 4 shows the curves of probability level vs. efficiency score for some DMUs, and Table 7 shows 95%- and 99%-efficiency for all DMUs. DMUs 6, 2 and 10 forms a group of efficient hospitals. DMU 8 is fragile to the stochastic variation and behaves like the inefficient DMU 9. The slope of curve indicates the sensitivity for the stochastic variation, that is, a steep curve is less sensitive and a gentle curve is more sensitive. Notice that the amount of reduction of efficiency score by stochastic variation closely relates to the ratio of variance to expectation, i.e., variance of coefficient. If we assume the equivalent variances through DMUs, the reduction of efficiency score for small DMU is more than that for large DMU, that is, small DMU is more sensitive than large DMU. In any way, the reduction of efficiency score from deterministic efficiency reflects the sensitivity to stochastic error.

Table 8: Reliability, 95%- and 99%-efficiency for variances (I), (II) and (III)

		Hospitals													
		1	2	3	4	5	6	7	8	9	10	11	12	13	14
CCR	Type	.955	1	1	.702	.827	1	.844	1	.995	1	.913	.969	.786	.974
reliability	I	83.00%	7.88%				99.43%		0.00%		63.81%				
	II	83.00%	7.88%				98.58%		0.00%		55.04%				
	III	99.94%	40.71%				100%		0.00%		95.01%				
95%	I	.890	.988	.955	.654	.779	1	.789	.933	.928	.981	.851	.906	.732	.908
	II	.890	.988	.955	.654	.775	1	.789	.933	.928	.976	.851	.906	.732	.908
	III	.916	1	.983	.672	.794	1	.812	.958	.953	1	.874	.932	.752	.933
99%	I	.879	.975	.944	.646	.771	1	.779	.920	.917	.971	.839	.894	.723	.896
	II	.879	.975	.944	.646	.766	.998	.779	.920	.916	.965	.839	.894	.723	.896
	III	.911	1	.976	.667	.788	1	.806	.950	.945	.933	.867	.926	.746	.925

Table 9: stochastic distance in each input and output

	DMU 3		DMU 6		DMU 10	
	I	III	I	III	I	III
$d_{x1}$	0.220	0.667	1.938	4.371	0	0
$d_{x2}$	0.224	0.679	0	0	1.095	2.395
$d_{y1}$	0.182	0.122	2.900	3.060	1.254	1.079
$d_{y2}$	0.908	1.369	2.100	2.097	1.478	1.445

Next, we give other values of variances or correlation under the proportional variance assumption.

(I) 1% for inputs, 2% for outputs and  $\rho = 0.2$

(II) 1% for inputs, 2% for outputs and  $\rho = 0.5$

(III) 1% for inputs and outputs, and  $\rho = 0.2$

(I) is used for the previous analysis. The only change of (II) is correlation coefficient. (III) has the same variance of coefficient for all inputs and outputs. Table 8 shows CCR efficiency, reliability, 95%- and 99%-efficiency. Comparing the rows (I) and (II), we find that the correlation coefficient  $\rho$  between two outputs is insensitive. The efficiency scores of (I) and (II) in Table 8 are almost same. Actually, nine DMUs (2, 3, 4, 7, 8, 11, 12, 13, 14) have the same efficiency score regardless of the value of correlation coefficient. All of them have a positive slack in either output, and so the minimized value of problem (4.17) is not depend on the correlation coefficient. The difference of reliability and efficiency score would be small even if the correlation coefficient would change.

It is obvious that a large variance lowers the reliability as well as the minimum efficiency scores at the specific probability level, and a small variance lifts them. Actually, comparing the rows (I) and (III), the reliability of DMU 3 is increased from 7.88% to 40.71%, and 95%- and 99%-efficiency for (III) are also increased than those for (I). The effect of change of variances is depend on the stochastic distance (4.16). The stochastic distances for some DMUs are shown in Table 9. Since the variances of outputs in (III) are smaller than those in (I), the stochastic distances for inputs would relatively increase. The stochastic distance for input-2 of DMU 6 is  $d_{x2} = 0$ , so the small change of variance  $\sigma_{x2}^2$  is insensitive to the stochastic efficiency of DMU 6.

## 8. Conclusion and Remarks

We have discussed on the stochastic efficiency using inputs and outputs data with stochastic noise, and proposed the reliability of efficient DMU as a sensitivity measure and the minimum efficiency score at a specific probability level. The proposed reliability measure can be obtained by solving quadratic programming problem. We have shown the reliability analysis based on CCR model, however, it is easily applied to the other DEA models such as BCC model and an additive model.

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