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A MINIMAX DISTRIBUTION FREE PROCEDURE FOR STOCHASTIC INVENTORY MODELS WITH A RANDOM BACKORDER RATE

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Abstract The stochastic inventory models analyzed in this paper involve two models that are continuous review and periodic review in which the backorder rate is a random variable. For these two models with a mixture of backorders and lost sales, we respectively assume that their mean and variance of lead time demand and review period demand are known, but their probability distribution are unknown. Instead of having a stockout cost term in the objective function, a service level constraint is added to the models. We develop a procedure to find the optimal solution for each case. Furthermore, the sensitivity analysis is performed.

1. Introduction

Among the modern production management, the Japanese successful experiences of using Just-In-Time (JIT) production show that the advantages and benefits associated with efforts to control the lead time can be clearly perceived. The goal of JIT inventory management philosophies is the focus which emphasizes high quality, keeps low inventory level and lead time to a practical minimum. In 1983, Monden [1] studied Toyota production system, and clearly declared that shortening lead time is a crux of elevating productivity.

In most of the early literature dealing with inventory problems, either in deterministic or probabilistic model, lead time is viewed as a prescribed constant or a stochastic variable, which therefore, is not subject to control (see, e.g., Naddor [2] and Silver and Peterson [3]). Recently, there have been some inventory literature which consider lead time as a decision variable. Liao and Shyu [4] first presented a continuous review inventory model in which the order quantity was predetermined and lead time was a decision unique variable. Ben-Daya and Raouf [5] extended Liao and Shyu's [4] model by considering both the lead time and order quantity to be decision variables where shortages were neglected. Ouyang *et al.* [6] generalized Ben-Daya and Raouf's [5] model by allowing shortages and the total amount of stockout is considered as a mixture of backorders and lost sales. In a recent research article, Ouyang and Wu [7] considered an inventory model with a mixture of backorders and lost sales in which a service level constraint was used instead of shortage cost in the objective function. However, in those models previously mentioned [5-7], reorder point had not been taken into account, and merely focused on the relationship between lead time and order quantity; that is, they neglected the possible impact of reorder point on the economic ordering strategy. In this article, we attempt to allow the reorder point as one of the decision variables in the modeling.

In addition, for practical inventory system, shortages are unavoidable due to various uncertainties. While a demand is unsatisfied during the lead time, generally there exists a mixture of backorders and lost sales; but existing literature mainly discussed that the backorder rate was a fixed constant. In this article, we consider that customers' patience is hard to estimate, and hence we here allow the backorder rate to be a random variable to agree with the real inventory environment.

In this study, we adopt Liao and Shyu's [4] assumption, which suppose that lead time can be decomposed into n mutually independent components each having a different crashing cost for reducing lead time. We also assume that instead of having a stockout cost term in the objective function, a service level constraint is added to the models. Two purposes of this paper are to establish a (Q, r, L) inventory model for the continuous review case and to propose a new (T, R, L) inventory model for the periodic review case. For these two models with a mixture of backorders and lost sales, we respectively consider that the form of the probability distribution of lead time demand and review period demand is unknown, and merely assume that their first and second moments are known (and hence, mean and variance are also known), and solve these inventory models by using the minimax distribution free approach. Moreover, the sensitivity analysis is included and two illustrative numerical examples are provided.

2. Notations and Assumptions

The mathematical models in this paper are developed on the basis of the following notations and assumptions.

Notations :

- D = expected demand per year
- A = ordering cost per order
- h = holding cost per unit per year
- μ = mean of the demand per unit time
- σ^2 = variance of the demand per unit time
- α = proportion of demands that are not met from stock. Hence, $1-\alpha$ is the service level, $0 < \alpha < 1/2$
- β = the fraction of the demand during the stockout period that will be backordered, $0 \leq \beta \leq 1$, a random variable
- $g(\beta)$ = the probability density function ($p.d.f.$) of β
- Q = order quantity, a decision variable
- r = reorder point for the continuous review case, a decision variable
- R = target inventory level for the periodic review case, a decision variable
- L = length of lead time, a decision variable
- T = length of a review period, a decision variable
- X = the lead time (or revrew period plus lead time) demand which has a $p.d.f.$ $f_X(x)$
- $E(\cdot)$ = mathematical expectation
- x^+ = maximum value of x and 0, i.e., $x^+ = Max\{x, 0\}$.

Assumptions :

1. The lead time L consists of n mutually independent components. The i-th component has a minimum duration a_i and normal duration b_i, and a crashing cost per unit time c_i. Further, for convenience, we rearrange c_i such that $c_1 \leq c_2 \leq \cdots \leq c_n$. Then, it is clear that the reduction of lead time should be first on component 1 because it has the minimum unit crashing cost, and then component 2, and so on.
2. For the continuous review case, the length of the lead time L does not exceed an inventory cycle time Q/D so that there is never more than a single order outstanding in any cycle.

3. For the periodic review case, the lead time L does not exceed the review period T so that no more than a single order will be outstanding at any moment.
4. If the demand per unit time, Y, has a probability distribution with mean μ and variance σ^2, then the distribution of the demand during the length of time t is a t-fold; that is, the demand Y during t follows a *p.d.f.* $f_Y(y)$ with mean μt and variance $\sigma^2 t$.

3. Continuous Review Model

In this section, we assume that inventory is continuously reviewed. An order quantity of size Q is ordered whenever the inventory level drops to the reorder point r. The reorder point r = expected demand during lead time + safety stock, and the safety stock = $k\times$ (standard deviation of the demand during L), i.e., $r = \mu L + k\sigma\sqrt{L}$, where k is the safety factor. When the lead time demand X has a *p.d.f.* $f_X(x)$ with mean μL and standard deviation $\sigma\sqrt{L}$, the expected demand shortage at the end of cycle is given by $E(X-r)^+ = \int_r^\infty (x-r)f_X(x)dx$; and thus the expected number of backorders per cycle is $\beta E(X-r)^+$ and the expected number of lost sales per cycle is $(1-\beta)E(X-r)^+$, where $0 \le \beta \le 1$.

The expected net inventory level just before the order arrives is $r-\mu L+(1-\beta)E(X-r)^+$, and the expected net inventory level at the beginning of the cycle is $Q+r-\mu L+(1-\beta)E(X-r)^+$ when an order lot Q arrives. Therefore, the expected holding cost per cycle is $h\frac{Q}{D}\left[\frac{Q}{2}+r-\mu L+(1-\beta)E(X-r)^+\right]$.

Under the Assumption 1 as in the above section, if we let L_0 denote the length of lead time with each component having a normal duration, and L_i be the length of lead time with first i components $1,2,\cdots,i$ crashed to their minimum duration and the rest components are normal duration, then L_0 and L_i can be expressed as $L_0 = \sum_{j=1}^{n} b_j$ and $L_i = \sum_{j=1}^{i} a_j + \sum_{j=i+1}^{n} b_j$, $i = 1,2,\cdots,n$, respectively; and hence the lead time crashing cost $C(L)$ per cycle for $L \in [L_i, L_{i-1}]$ can be written as $C(L) = c_i(L_{i-1}-L) + \sum_{j=1}^{i-1} c_j(b_j-a_j)$.

With the preceding statements, the expected total cost per cycle is given by

$$\begin{aligned} T(Q,r,L) &= \text{ordering cost } + \text{ holding cost } + \text{ lead time crashing cost} \\ &= A + h\frac{Q}{D}\left[\frac{Q}{2}+r-\mu L+(1-\beta)E(X-r)^+\right] + C(L). \end{aligned} \tag{1}$$

Notice that the stockout cost term is not included in the objective function.

Because the backorder rate β is a random variable with *p.d.f.* $g(\beta)$, the expected backorder rate is $M_\beta = \int_0^1 \beta g(\beta)d\beta$. And the expected order number per year is D/Q, therefore, the expected total annual cost is

$$\begin{aligned} EAC(Q,r,L) &= E[T(Q,r,L)]\frac{D}{Q} \\ &= \frac{D[A+C(L)]}{Q} + \frac{hQ}{2} + h[r-\mu L+(1-M_\beta)E(X-r)^+]. \end{aligned} \tag{2}$$

Our objective is to minimize the expected total annual inventory cost, subject to a constraint on service level. That is, the mathematical model of this problem can be expressed as follows:

$$Min EAC(Q,r,L) = \frac{D[A+C(L)]}{Q} + \frac{hQ}{2} + h[r-\mu L + (1-M_\beta)E(X-r)^+] \quad (3)$$
$$\text{Subject to} \quad \frac{E(X-r)^+}{Q} \le \alpha.$$

Since the form of the probability distribution of lead time demand X is unknown, we can not find the exact value of $E(X-r)^+$. Hence, we use minimax distribution free procedure to solve our problem. If we let $\mathcal{T}$ denote the class of *p.d.f.*'s with finite mean μL and standard deviation $\sigma\sqrt{L}$, then the minimax distribution free approach for this problem is to find the most unfavorable *p.d.f.* f_X in $\mathcal{T}$ for each (Q,r,L) and then minimize over (Q,r,L); that is, our problem is to solve

$$\underset{Q,r,L}{Min}\underset{f_X\in\mathcal{T}}{Max} EAC(Q,r,L) \quad (4)$$
$$\text{Subject to} \quad \frac{E(X-r)^+}{Q} \le \alpha.$$

For this purpose, we need the following proposition which was asserted by Gallego and Moon [8].

Proposition For any $f_X \in \mathcal{T}$,

$$E(X-r)^+ \le \frac{1}{2}\left[\sqrt{\sigma^2 L + (r-\mu L)^2} - (r-\mu L)\right]. \quad (5)$$

Moreover, the upper bound (5) is tight.

Given that $r = \mu L + k\sigma\sqrt{L}$, and for any probability distribution of the lead time demand X, the above inequality always holds. Then, using inequality (5) and considering the safety factor k as a decision variable instead of r, the model (4) is reduced to

$$Min\ EAC(Q,k,L) = \frac{D[A+C(L)]}{Q} + \frac{hQ}{2} + h\sigma\sqrt{L}\left[k + \frac{1}{2}(1-M_\beta\left(\sqrt{1+k^2}-k\right)\right] \quad (6)$$
$$\text{Subject to} \quad \sigma\sqrt{L}\left(\sqrt{1+k^2}-k\right) \le 2\alpha Q.$$

The inequality constraint in model (6) can be converted into equality by adding a non-negative slack variable, S^2. Thus, the Lagrangean function is given by

$$\begin{aligned} EAC(Q,k,L,\lambda,S) &= EAC(Q,k,L) + \lambda\left[\sigma\sqrt{L}\left(\sqrt{1+k^2}-k\right) + S^2 - 2\alpha Q\right] \quad (7)\\ &= \frac{D[A+C(L)]}{Q} + \frac{hQ}{2} + h\sigma\sqrt{L}\left[k + \frac{1}{2}(1-M_\beta)\left(\sqrt{1+k^2}-k\right)\right] \\ &\quad +\lambda\left[\sigma\sqrt{L}\left(\sqrt{1+k^2}-k\right) + S^2 - 2\alpha Q\right], \end{aligned}$$

where λ is a Lagrange multiplier.

Notice that, for any given (Q,k,λ,S), $EAC(Q,k,L,\lambda,S)$ is a concave function in $L \in [L_i, L_{i-1}]$, because

$$\frac{\partial^2 EAC(Q,k,L,\lambda,S)}{\partial L^2} = -\frac{1}{4}hk\sigma L^{-3/2} - \frac{1}{8}\sigma L^{-3/2}\left(\sqrt{1+k^2}-k\right)[h(1-M_\beta)+2\lambda] < 0.$$

Hence, for fixed (Q, k, λ, S), the minimum expected total annual cost will occur at the end points of the interval $[L_i, L_{i-1}]$. On the other hand, we can further prove that, for any given $L \in [L_i, L_{i-1}]$, $EAC(Q, k, L, \lambda, S)$ satisfies the Kuhn-Tucker necessary conditions for minimization problem and it gets the slack variable $S^2 = 0$ (hence, the inequality constraint in model (6) will become an equality). Therefore, for fixed $L \in [L_i, L_{i-1}]$, the minimum value of function (7) (in which the variable $S = 0$) will occur at the point (Q, k, λ) which satisfies

$$0 = \frac{\partial EAC(Q,k,L,\lambda)}{\partial Q} = -\frac{D[A+C(L)]}{Q^2} + \frac{h}{2} - 2\lambda\alpha, \tag{8}$$

$$0 = \frac{\partial EAC(Q,k,L,\lambda)}{\partial k} = h\sigma\sqrt{L}\left[1 - \frac{1}{2}(1-M_\beta)\frac{\sqrt{1+k^2}-k}{\sqrt{1+k^2}} - \frac{\lambda}{h}\frac{\sqrt{1+k^2}-k}{\sqrt{1+k^2}}\right], \tag{9}$$

and

$$0 = \frac{\partial EAC(Q,k,L,\lambda)}{\partial \lambda} = \sigma\sqrt{L}\left(\sqrt{1+k^2}-k\right) - 2\alpha Q. \tag{10}$$

Solving (8), (9) and (10) for Q, λ, and k, respectively, leads to

$$Q = \left\{\frac{2D[A+C(L)]}{h-4\lambda\alpha}\right\}^{1/2}, \tag{11}$$

$$\lambda = h\left[\frac{\sqrt{1+k^2}}{\sqrt{1+k^2}-k} - \frac{1}{2}(1-M_\beta)\right], \tag{12}$$

and

$$\sqrt{1+k^2} - k = \frac{2\alpha Q}{\sigma\sqrt{L}}. \tag{13}$$

Combining equations (11), (12), and (13), we get the order quantity

$$Q = \left\{\frac{4\alpha D[A+C(L)] + h\sigma^2 L}{2\alpha h(1-2\alpha M_\beta)}\right\}^{1/2}. \tag{14}$$

Thus, we can establish the following algorithm to find the optimal (Q, k, L).

Algorithm 1

Step 1. For each L_i, $i = 0, 1, 2, \cdots, n$, we use equation (14) to evaluate Q_i, and then use equation (13) to determine k_i.

Step 2. For each (Q_i, k_i, L_i), compute the corresponding expected total annual cost $EAC(Q_i, k_i, L_i)$, $i = 0, 1, 2, \cdots, n$.

Step 3. Find $\underset{i=0,1,2,\cdots,n}{Min} EAC(Q_i, k_i, L_i)$. If $EAC(Q^*, k^*, L^*) = \underset{i=0,1,2,\cdots,n}{Min} EAC(Q_i, k_i, L_i)$, then (Q^*, k^*, L^*) is the optimal solution. And hence, the optimal reorder point r^* is $r^* = \mu L^* + k^*\sigma\sqrt{L^*}$.

4. Periodic Review Model

In this section, we assume that the inventory level is reviewed every T units of time and a sufficient ordering quantity is ordered up to the target inventory level R, and the ordering quantity is arrived after L units of time. The target inventory level R = expected demand during the period of length $(T + L)$ + safety stock, and the safety stock = $\delta \times$ (standard deviation of the demand during $T + L$), i.e., $R = \mu(T+L) + \delta\sigma\sqrt{T+L}$, where δ is a safety factor. When the demand X during $(T + L)$ has a *p.d.f.* $f_X(x)$ with mean $\mu(T + L)$ and standard deviation $\sigma\sqrt{T+L}$, the expected demand shortage at the end of cycle is given by $E(X-R)^+ = \int_R^\infty (x-R)f_X(x)dx$, and the expected number of lost sales per cycle is $(1-\beta)E(X-R)^+$.

By a similar discussion as in Montgomery *et al.* [9] to discuss the periodic review case, the review period T will be used as the time between the arrival of two successive orders rather than between the placement of two successive orders. Hence, the expected net inventory level at the beginning of the period is $R-\mu L+(1-\beta)E(X-R)^+$, and the expected net inventory level at the end of the period is $R-\mu(T+L)+(1-\beta)E(X-R)^+$. Thus, the expected holding cost per year is $h\left[R-\mu L-\frac{\mu T}{2}+(1-\beta)E(X-R)^+\right]$. Therefore, the mathematical model of the period review case can be formulated as

$$Min\ EAC(T,R,L) = \frac{A}{T} + h\left[R-\mu L-\frac{\mu L}{2}+(1-M_\beta)E(X-R)^+\right] + \frac{C(L)}{T} \quad (15)$$

$$\text{Subject to} \quad \frac{E(X-R)^+}{D(T+L)} \le \alpha.$$

Based upon the same proposition as presented in the previous continuous review case and $R=\mu(T+L)+\delta\sigma\sqrt{T+L}$, we can obtain the following inequality:

$$E(X-R)^+ \le \frac{1}{2}\sigma\sqrt{T+L}\left(\sqrt{1+\delta^2}-\delta\right), \quad (16)$$

for any $f_X \in \mathcal{T}$ (which is the class of *p.d.f.* f_X's with finite mean $\mu(T+L)$ and standard deviation $\sigma\sqrt{T+L}$). And the upper bound (16) is tight.

On applying this result and allowing the safety factor δ as a decision variable instead of R, the model (15) is reduced to minimize

$$EAC(T,\delta,L) = \frac{A+C(L)}{T} + \frac{h\mu T}{2} + h\sigma\sqrt{T+L}\left[\delta - \frac{1}{2}(1-M_\beta)\left(\sqrt{1+\delta^2}-\delta\right)\right] \quad (17)$$

$$\text{Subject to} \quad \sigma\sqrt{T+L}\left(\sqrt{1+\delta^2}-\delta\right) \le 2\alpha D(T+L).$$

Hence, the Lagrangean function can be formulated as

$$\begin{aligned} EAC(T,\delta,L,\lambda,S) = {} & \frac{A+C(L)}{T} + \frac{h\mu T}{2} + h\sigma\sqrt{T+L}\left[\delta + \frac{1}{2}(1-M_\beta)\left(\sqrt{1+\delta^2}-\delta\right)\right] \quad (18) \\ & +\lambda\left[\sigma\sqrt{T+L}\left(\sqrt{1+\delta^2}-\delta\right) + S^2 - 2\alpha D(T+L)\right], \end{aligned}$$

where λ is a Lagrange multiplier and S^2 is a nonnegative slack variable.

By analogous arguments in the continuous review case, it can be verified that $EAC(T,\delta, L,\lambda,S)$ is a concave function of $L \in [L_i, L_{i-1}]$ for fixed (T,δ,λ,S). Thus, for fixed (T,δ,λ,S), the minimum value of $EAC(T,\delta,L,\lambda,S)$ will occur at the end points of the interval $[L_i, L_{i-1}]$. Furthermore, for fixed $L \in [L_i, L_{i-1}]$, we can show that $EAC(T,\delta,L,\lambda,S)$ satisfies the Kuhn-Tucker necessary conditions for minimization problem and it obtains the slack variable $S^2=0$. Consequently, for fixed $L \in [L_i, L_{i-1}]$, the minimum value of function (18) (in which the variable $S=0$) will occur at the point (T,δ,λ) which satisfies

$$\frac{\partial EAC(T,\delta,L,\lambda)}{\partial T} = 0, \quad (19)$$

$$\frac{\partial EAC(T,\delta,L,\lambda)}{\partial \delta} = 0, \quad (20)$$

and

$$\frac{\partial EAC(T,\delta,L,\lambda)}{\partial \lambda} = 0. \tag{21}$$

Solving equations (19), (20), and (21) for T, λ, and δ, respectively, results in

$$\left[\frac{A+C(L)}{T^2} - \frac{h\mu}{2} + 2\lambda\alpha D\right](T+L)^{1/2} = \frac{h\sigma\delta}{2} + \frac{\sigma}{4}\left(\sqrt{1+\delta^2}-\delta\right)[h(1-M_\beta)+2\lambda], \tag{22}$$

$$\lambda = h\left[\frac{\sqrt{1+\delta^2}}{\sqrt{1+\delta^2}-\delta} - \frac{1}{2}(1-M_\beta)\right], \tag{23}$$

and

$$\sqrt{1+\delta^2}-\delta = \frac{2\alpha D}{\sigma}\sqrt{T+L}. \tag{24}$$

By substituting (23) and (24) into (22), the review period

$$T = \left\{\frac{2[A+C(L)]}{h(\mu - 2\alpha D M_\beta)}\right\}^{1/2}. \tag{25}$$

The following algorithm can be utilized to find the optimal (T, δ, L).

Algorithm 2

Step 1. For each $L_i,\ i=0,1,2,\cdots,n$, we use equation (25) to evaluate T_i, and then use equation (24) to compute δ_i.

Step 2. For each (T_i, δ_i, L_i), compute the corresponding expected total annual cost $EAC(T_i, \delta_i, L_i),\ i=0,1,2,\cdots,n$.

Step 3. Find $\underset{i=0,1,2,\cdots,n}{Min} EAC(T_i,\delta_i,L_i)$. If $EAC(T_*,\delta_*,L_*) = \underset{i=0,1,2,\cdots,n}{Min} EAC(T_i,\delta_i,L_i)$, then (T_*,δ_*,L_*) is the optimal solution. And hence, the optimal target inventory level R_* is $R_* = \mu(T_*+L_*) + \delta_*\sigma\sqrt{T_*+L_*}$.

5. Sensitivity Analysis

1. For the continuous review case, observe that the effect of expected backorder rate M_β on the expected total annual cost may be examined. From (6), for fixed Q, k, and L, we have

$$\frac{dEAC(Q,k,L)}{dM_\beta} = -\frac{h\sigma\sqrt{L}}{2}\left(\sqrt{1+k^2}-k\right) < 0.$$

This implies when increasing the expected backorder rate, the expected total annual cost will decrease. The periodic review case has the same property.

2. In the continuous review case, from equation (14), we take the derivatives of Q with respected to M_β, and obtain

$$\frac{dQ}{dM_\beta} = \frac{\alpha Q}{1-2\alpha M_\beta} > 0 \quad (\text{because } \alpha < 1/2).$$

This means that increasing the expected backorder rate M_β increase the order quantity Q.

3. For fixed $L \in [L_i, L_{i-1}]$, from equation (13), we get

$$\frac{dk}{d\alpha} = \frac{-2Q}{\sigma\sqrt{L}}\frac{\sqrt{1+k^2}}{\sqrt{1+k^2}-k} < 0,$$

or equivalently,

$$\frac{dk}{d(1-\alpha)} = \frac{2Q}{\sigma\sqrt{L}} \frac{\sqrt{1+k^2}}{\sqrt{1+k^2}-k} > 0.$$

Hence, for the continuous review model, increasing the service level $1-\alpha$ will result in an increase in the safety factor k (or, equivalently increases the reorder point r, because $r = \mu L + k\sigma\sqrt{L}$). On the other hand, the periodic review model has the same property, i.e., increasing the service level $1-\alpha$ will result in an increase in the target inventory level R.

4. Taking the derivatives of (25) with respect to α, we obtain

$$\frac{dT}{d\alpha} = \frac{hDM_\beta T^3}{2[A+C(L)]} > 0,$$

or equivalently,

$$\frac{dT}{d(1-\alpha)} = \frac{-hDM_\beta T^3}{2[A+C(L)]} < 0.$$

It implies that increasing the service level $1-\alpha$ decreases the review period T.

5. Clearly, for the continuous review case, if k is sufficiently large, we get $\sqrt{1+k^2}-k \to 0$, then the model (6) is reduced to the model of Ben-Daya and Raouf [5].

6. Numerical Examples

Case 1. *Continuous review model*

In order to illustrate the above solution procedure, let us consider an inventory system with the data used in Ouyang and Wu [7]: $D = 600$ units/year, $A = \$200$ per order, $h = \$20$ /unit/year, $\mu = 11$ units/week, $\sigma = 7$ units/week, the service level $1-\alpha = 0.985$, i.e., the proportion of demands that are not met from stock is $\alpha = 0.015$. The lead time has three components with data as shown in Table 1. Moreover, suppose that the backorder rate β during the stockout period has a uniform distribution, i.e., the *p.d.f.* of β is $g(\beta) = 1$, $0 \leq \beta \leq 1$; $= 0$, otherwise. Hence, the mean of β is $M_\beta = 0.5$.

Table 1: Lead time data

Lead time component i	Normal duration b_i (days)	Minimum duration a_i (days)	Unit crashing cost c_i (\$ /day)
1	20	6	0.4
2	20	6	1.2
3	16	9	5.0

For this case, applying the Algorithm 1 procedure yields the results as shown in Table 2. From Table 2, the optimal inventory policy with a service level constraint can be easily found by comparing $EAC(Q_i, R_i, L_i)$, $i = 0, 1, 2, 3$, under the stockout level bound $\alpha = 0.015$. Therefore, we obtain the optimal order quantity $Q^* = 142$ units, the optimal reorder point $r^* = 65$ units, and the optimal lead time $L^* = 4$ weeks. The minimum expected total annual cost $EAC(Q^*, r^*, L^*) = \$2,798.23$.

Remark: In Ouyang and Wu's [7] model, which considered a fixed reorder point r (i.e., they let $P(X > r) = 0.2$), and took $\beta = 1$ and $\alpha = 0.015$. They obtained the optimal $(Q^*, L^*) = (116, 4)$ and the minimum expected total annual cost $EAC(116, 4) = \$2,839.06$.

Table 2 : Solution procedure of Algorithm 1 (L_i in week)

i	L_i	$C(L_i)$	Q_i	$r_i(k_i)$	$EAC(Q_i, r_i, L_i)$
0	8	0	160	126(1.94)	$ 3,142.21
1	6	5.6	150	96(1.77)	$ 2,951.93
2	4	22.4	142	65(1.49)	$ 2,798.23
3	3	57.4	144	48(1.23)	$ 2,832.29

In our model, for the same case, i.e., $\beta = 1$ and $\alpha = 0.015$ but r to be considered as a decision variable, we reevaluate and obtain the optimal $(Q^*, r^*, L^*) = (124, 54, 4)$ and $EAC(124, 54, 4) = \$2,524.05$. Thus, we find that our model savings $EAC(Q^*, L^*) - EAC(Q^*, r^*, L^*) = EAC(116, 4) - EAC(125, 54, 4) = \$2,546.94 - \$2,524.05 = \$22.89$, which can be viewed as the rewards due to controlling the reorder point to be a decision variable.

Case 2. Periodic review model

The data is as in Case 1. Applying the Algorithm 2, we tabulate the results in Table 3. From Table 3, under the stockout level bound $\alpha = 0.015$, the optimal value $(T_*, R_*, L_*) = (9.8, 263, 8)$, and the minimum expected total annual cost $EAC(T_*, R_*, L_*) = \$3,522.67$.

Table 3 : Solution procedure of Algorithm 2 (T_i, L_i in week)

i	L_i	$C(L_i)$	T_i	$R_i(\delta_i)$	$EAC(T_i, R_i, L_i)$
0	8	0	9.80	263(2.29)	$ 3,522.67
1	6	5.6	9.94	243(2.43)	$ 3,554.85
2	4	22.4	10.34	226(2.58)	$ 3,648.44
3	3	57.4	11.12	224(2.60)	$ 3,819.08

Comparing the optimal results in Table 2 and Table 3, we observe that the periodic review model is more expensive than the continuous review model and the cost difference $EAC(T_*, R_*, L_*) - EAC(Q^*, r^*, L^*) = \$3,522.67 - \$2,798.23 = \$724.44$.

7. Concluding Remarks

In this study, we present two inventory models with a mixture of backorders and lost sales, where the backorder rate be a random variable and the stockout cost term in the objective function is replaced by a service level constraint. One is the (Q, r, L) inventory model for the continuous review case and the other is (T, R, L) inventory model for the periodic review case. For these two models, we respectively consider that the distributional form of lead time demand and review period demand is unknown, and apply the minimax decision criterion to solve the distribution free model.

In future research on this problem, it would be interesting to deal with an arrival order lot involving some defective items.

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