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SOME RESULTS ON A BAYESIAN SEQUENTIAL SCHEDULING ON TWO IDENTICAL PARALLEL PROCESSORS

Toshio Hamada
Kobe University of Commerce

Mitsushi Tamaki
Aichi University

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Abstract Suppose that there are two types J_0 of m jobs and J_1 of n jobs, which are processed by either one of two identical machines. The processing time of one job of class J_0 or J_1 is the random variable from the exponential distribution with parameter u or v, respectively. The true value of v is unknown, but it has the gamma distribution with parameters w and α as the prior distribution. Preemption is not allowed in this problem. The objective is to minimize the expected total flowtime. From the Bayesian point of view, the problem is formulated by the principle of optimality of dynamic programming and the optimal solutions are obtained for some special cases.

1. Introduction

Let us first review the classical scheduling problem with two types of jobs and two identical machines when the objective is to minimize the expected total flowtime. Suppose that there are m jobs of type 0 and n jobs of type 1, all of which are available at time 0. A job of type i $(i = 0, 1)$ is simply called J_i job. Two identical machines are available for processing these jobs. To process a job, that job must be put on one of the machines (either of the two can do) for a random duration (processing time), after which it is complete. Jobs are processed consecutively starting at time t=0, so that as soon as a job is complete, another job is put on the machine that is freed. The processing time of J_0 job or J_1 job are exponentially distributed with known parameter u and unknown parameter v, respectively. The order in which putting $m + n$ jobs on two machines determines a schedule. Given a schedule, *flowtime* is defined for each job as a time until that job has been completed. We wish to find a schedule that will minimize the expected total flowtime. In the case that u and v are known, it is well known (see, Pinedo and Weiss [7]) that processing J_1 jobs first minimizes the expected total flowtimes if u>v (order among the jobs of the same type is of course immaterial). Bruno, Downey and Frederickson [1] and Kämpke [6] generalized this to allow more than two machines.

The problem we consider here is a Bayesian version of the classical scheduling problem, in which u is known in advance but v is unknown and has a gamma distribution as its prior. We call this problem (m, n)-problem when there are m J_0-jobs and n J_1-jobs. To our knowledge, although the incomplete information cases have been studied for a single machine in several papers, no Bayesian problem with two machines has been studied so far. Gittins and Glazebrook [4] discussed a Bayesian single machine scheduling problem, in which the processing time of each job is a random variable with unknown parameter. Burnetas and Katehakis [2] considered the model of sequencing two types of jobs on a single machine. Hamada and Glazebrook [5] derived the method to calculate the critical value related with the value of index which described the optimal strategy. Rieder and

Weinhaupt [9] considered the stochastic scheduling problem with incomplete information and linear waiting costs. In these papers, only the single machine problems have been considered.

In Section 2, the (m, n)-problem is formulated via dynamic programming. We analyze the $(m, 1)$-problem in Section 3 and the $(1, n)$-problem in Section 4.

2. Formulation of the (m,n)-problem.

Let X denote the processing time for J_0 job and Y denote that for J_1 job. X and Y are assumed to be independent and exponentially distributed random variables with parameters u and v respectively, that is, if we denote by $f(x)$ and $g(y)$ their densiy functions, respectively, then they are given by

$$f(x) = ue^{-ux}, \quad x \geq 0$$

and

$$g(y) = ve^{-vy}, \quad y \geq 0.$$

Each of the processing times of the same type is also assumed to be independent. One of the remarkable properties of exponential distribution is that $min(X, Y)$ is exponentially distributed with parameter $u + v$. Thus, in particular

$$E[\min(X, \ Y)] = \frac{1}{u+v}. \tag{2.1}$$

Another property used later is

$$\Pr\{X < Y\} = \frac{u}{u+v}. \tag{2.2}$$

By scale transformation, u is assumed to be unity without loss of generality. Since v is assumed to be a random variable, we use V instead of v and a gamma prior with parameters $w > 0$ and $\alpha > 1$, denoted by $p(v|w, \alpha)$,

$$p(v|w, \alpha) = \frac{w^\alpha}{\Gamma(\alpha)} v^{\alpha-1} e^{-wv},$$

is assumed on V. Thus the density functions of X and Y are now respectively given by

$$f(x) = e^{-x} \tag{2.3}$$

and

$$\begin{aligned} g(y|w, \alpha) &= \int_0^\infty ve^{-vy} p(v|w, \alpha) dv \\ &= \int_0^\infty ve^{-vy} \frac{w^\alpha}{\Gamma(\alpha)} v^{\alpha-1} e^{-wv} dv \\ &= \frac{\alpha w^\alpha}{(w+y)^{\alpha+1}}. \end{aligned} \tag{2.4}$$

For later use, define the followings:

$$\bar{F}(x) = \int_x^\infty f(s) ds = e^{-x},$$

$$\bar{G}(y|w,\alpha) = \int_y^\infty g(s|w,\alpha)ds = (\frac{w}{w+y})^\alpha,$$

$$A_k(w,\alpha) = \int_0^\infty \left(\frac{1}{1+v}\right)^k p(v|w,\alpha)dv \tag{2.5}$$

and

$$B_k(w,\alpha) = \int_0^\infty \left(\frac{v}{1+v}\right)^k p(v|w,\alpha)dv \tag{2.6}$$

for $k = 0, 1, 2, 3, \cdots$. Then, some properties of these functions are given in the following lemma.

Lemma 1 *For $k \geq 1$ and $\alpha > 1$,*
(i) *$A_k(w,\alpha)$ is strictly increasing in w and $B_k(w,\alpha)$ is strictly decreasing in w.*
(ii) *$A_k(w,\alpha) \leq 1$, and $B_k(w,\alpha) \leq 1$,*
(iii) *$A_k(w,\alpha) \geq (\frac{w}{w+\alpha})^k$,*
(iv) *$A_1(w,\alpha) + B_1(w,\alpha) = 1$,*
(v) *$\int_0^\infty g(x|w,\alpha)e^{-x}dx = 1 - A_1(w,\alpha)$,*
(vi) *$\int_0^\infty \frac{w+x}{\alpha-1}\bar{G}(x|w,\alpha)e^{-x}dx = \frac{w}{\alpha-1} - A_1(w,\alpha)$,*
(vii) *$\int_0^\infty e^{-x}\bar{G}(x|w,\alpha)dx = A_1(w,\alpha)$,*
(viii) *$\int_0^\infty \frac{w+x}{\alpha}g(x|w,\alpha)e^{-x}dx = A_1(w,\alpha)$,*
(ix) *$\int_0^\infty A_k(w+x,\alpha+1)g(x|w,\alpha)e^{-x}dx = A_k(w,\alpha) - A_{k+1}(w,\alpha)$,*
(x) *$\int_0^\infty A_k(w+x,\alpha)e^{-x}\bar{G}(x|w,\alpha)dx = A_{k+1}(w,\alpha)$,*
(xi) *$B_n(w,\alpha) = \sum_{k=0}^{n} {}_nC_k(-1)^k A_k(w,\alpha)$.*

Proof. (i) is easily derived by rewriting (2.5) and (2.6) as

$$A_k(w,\alpha) = \int_0^\infty \left(\frac{w}{w+y}\right)^k p(y|1,\alpha)dy$$

and

$$B_k(w,\alpha) = \int_0^\infty \left(\frac{y}{w+y}\right)^k p(y|1,\alpha)dy,$$

respectively. (ii) is trivial from (2.5) and (2.6). Since the function $h(z) = (1+z)^{-k}$ is convex in z, (iii) is derived from Jensen's inequality (see, for example, Ross [8])

$$\mathrm{E}[h(Z)] \geq h(\mathrm{E}[Z]),$$

where $\mathrm{E}[Z]=\alpha/w$. Equations (iv) is immediate from (2.5) and (2.6). Since

$$\int_0^\infty g(x|w,\alpha)e^{-x}dx = \int_0^\infty g(x|w,\alpha)e^{-x}\left(\int_0^\infty p(u|w+x,\alpha+1)du\right)dx$$

$$= \int_0^\infty \frac{u}{u+1} p(u|w,\alpha)du,$$

(v) is derived from (2.6) and (iv). Also, since

$$\begin{aligned}\int_0^\infty \frac{w+x}{\alpha-1}\overline{G}(x|w,\alpha)e^{-x}dx &= \int_0^\infty \frac{w+x}{\alpha-1}\overline{G}(x|w,\alpha)e^{-x}\left(\int_0^\infty p(u|w+x,\alpha-1)du\right)dx \\ &= \int_0^\infty \left(\frac{1}{u}-\frac{1}{u+1}\right)p(u|w,\alpha)du,\end{aligned}$$

(vi) is derived from (2.5). Since

$$\int_0^\infty \frac{(w+x)^\alpha}{\Gamma(\alpha)}u^{\alpha-1}e^{-(w+x)u}du = 1,$$

(vii) is derived as follows:

$$\begin{aligned}\int_0^\infty e^{-x}\overline{G}(x|w,\alpha)dx &= \int_0^\infty e^{-x}\frac{w^\alpha}{(w+x)^\alpha}\left\{\int_0^\infty \frac{(w+x)^\alpha}{\Gamma(\alpha)}u^{\alpha-1}e^{-(w+x)u}du\right\}dx \\ &= \int_0^\infty \frac{w^\alpha}{\Gamma(\alpha)}u^{\alpha-1}e^{-wu}\left(\int_0^\infty e^{-(u+1)x}dx\right)du \\ &= A_1(w,\alpha).\end{aligned}$$

(viii) is derived from

$$\begin{aligned}\int_0^\infty \frac{w+x}{\alpha}g(x|w,\alpha)e^{-x}dx &= \int_0^\infty \frac{w+x}{\alpha}g(x|w,\alpha)e^{-x}\left(\int_0^\infty p(u|w+x,\alpha)du\right)dx \\ &= \int_0^\infty \frac{w^\alpha}{\Gamma(\alpha)}u^{\alpha-1}e^{-wu}\left(\int_0^\infty e^{-(u+1)x}dx\right)du\end{aligned}$$

and (ix) is immediate from the definitions of $g(x|w,\alpha)$ and $A_k(w,\alpha)$. (x) is derived from

$$\int_0^\infty A_k(w+x,\alpha)e^{-x}\,\bar{G}\,(x|w,\alpha)dx$$

$$= \int_0^\infty \left\{\int_0^\infty \left(\frac{1}{1+u}\right)^k \frac{(w+x)^\alpha}{\Gamma(\alpha)}u^{\alpha-1}e^{-wu}du\right\}e^{-x}\left(\frac{w}{w+x}\right)^\alpha dx$$

$$= A_{k+1}(w,\alpha)$$

Also, (xi) is derived from the equation

$$\mathrm{E}\left[\left(\frac{V}{1+V}\right)^n\right] = \sum_{k=0}^{n} {}_nC_k(-1)^k\mathrm{E}\left[\left(\frac{1}{1+V}\right)^k\right].$$

□

Preemption is not allowed, that is, every job is processed until it is completed. In this section, we formulate the (m,n)-problem via dynamic programming. The (m,n)-problem is sometimes referred to as the (m,n,w,α)-problem when the current prior is gamma with parameters w and α. Now, imagine a state where one of the machines is just freed, while the other machine is still processing J_i job. Then this state is described as (m,n,w,α,i) if there remain m J_0 jobs and n J_1 jobs yet to be processed and the distribution of V is gamma with parameters w and α. It is well known from the conjugate argument (see DeGroot [3]) that if a prior distribution of V is gamma with parameters w and α, then after having observed that the processing time of J_1 job is y, the distribution undergoes Bayesian updating and becomes gamma with parameters $w+y$ and $\alpha+1$ (Note that when we observe that job is not completed within y units of time, i.e., $Y > y$, then the posterior distribution becomes gamma with parameters $w+y$ and α).

Assume that, in state (m,n,w,α,i), J_j job is assigned to the idle machine. Then, if $i \neq j$, the state makes transition into $(m-1,n,w+X,\alpha,1)$ or $(m,n-1,w+Y,\alpha+1,0)$ in time $\min(X,Y)$ depending on whether $X \leq Y$ or $X > Y$. Similarly, depending on $i=j=0$ or $i=j=1$, the state makes transition into $(m-1,n,w,\alpha,0)$ in time $\min(X,X')$ or into $(m,n-1,w+2\min(Y,Y'),\alpha+1,1)$ in time $\min(Y,Y')$, where X and X' (Y and Y') are independent and exponentially distributed random variables with parameter 1 (with parameter V). The expectations of $\min(X,Y)$, $\min(X,X')$, and $\min(Y,Y')$ are derived as follows:

$$\begin{aligned} E[\min(X,Y)] &= A_1(w,\alpha) \\ &= \int_0^\infty \left(\frac{1}{1+v}\right) p(v|w,\alpha)dv, \\ E[\min(X,X')] &= \frac{1}{2}, \end{aligned} \tag{2.7}$$

and

$$\begin{aligned} E[\min(Y,Y')] &= \int_0^\infty \left(\frac{1}{2v}\right) p(v|w,\alpha)dv \\ &= \frac{w}{2(\alpha-1)}. \end{aligned} \tag{2.8}$$

Observe that from (2.7) the expected total flowtime when m jobs of type 0 are processed consecutively by two machines, denoted by f_m, is calculated as

$$\begin{aligned} f_m &= \sum_{k=0}^{m-2} \frac{1}{2}(m-k)+1 \\ &= \frac{m^2+m+2}{4}. \end{aligned}$$

Similarly, from (2.8) the expected total flowtime when n jobs of type 1 are processed consecutively by two machines, denoted by $g_n(w,\alpha)$, is given by

$$\begin{aligned} g_n(w,\alpha) &= \sum_{k=0}^{n-2} \frac{w}{2(\alpha-1)}(n-k)+\frac{w}{\alpha-1} \\ &= \left(\frac{w}{\alpha-1}\right)\left(\frac{n^2+n+2}{4}\right). \end{aligned}$$

Let $F(m,n,w,\alpha)$ be the minimum expected total flowtime for the (m,n,w,α)-problem. Also, let $F(m,n,w,\alpha,i)$ be the minimum expected total (remaining) flowtime under an optimal schedule starting from state (m,n,w,α,i). Then, we obviously have

$$F(m,n,w,\alpha) = \min_i\{F(m,n,w,\alpha,i)\}. \tag{2.9}$$

Let further $F^j(m,n,w,\alpha,i)$ be the minimum expected total (remaining) flowtime when we assign J_j job to the idle machine in state (m,n,w,α,i) and continue optimally thereafter. Then the principle of optimality yields for $m \geq 2-i, n \geq 1+i$ with $i=0,1$,

$$F(m,n,w,\alpha,i) = \min_j\{F^j(m,n,w,\alpha,i)\} \tag{2.10}$$

with the boundary conditions

$$F(1,n,w,\alpha,0) = F^1(1,n,w,\alpha,0) \tag{2.11}$$

for $n \geq 1$ and

$$F(m,1,w,\alpha,1) = F^0(m,1,w,\alpha,1) \tag{2.12}$$

for $m \geq 1$, where

$$F^0(m,n,w,\alpha,0) = \frac{m+n}{2} + F(m-1,n,w,\alpha,0) \tag{2.13}$$

for $m \geq 2$ and $n \geq 1$,

$$F^1(m,n,w,\alpha,1) = \left(\frac{m+n}{2}\right)\left(\frac{w}{\alpha-1}\right) + \int_0^\infty F(m,n-1,w+x,\alpha+1,1)g(x|w,\alpha)dx \tag{2.14}$$

for $m \geq 1$ and $n \geq 2$,

$$\begin{aligned} F^0(m,n,w,\alpha,1) &= F^1(m,n,w,\alpha,0) & (2.15)\\ &= (m+n)A_1(w,\alpha) + \int_0^\infty F(m-1,n,w+x,\alpha,1)e^{-x}\overline{G}(x|w,\alpha)dx \\ &\quad + \int_0^\infty F(m,n-1,w+x,\alpha+1,0)g(x|w,\alpha)e^{-x}dx & (2.16)\end{aligned}$$

for $m \geq 1$ and $n \geq 1$. Starting with the initial conditions

$$\begin{aligned} F(1,1,w,\alpha,0) = F(1,1,w,\alpha,1) = F^1(1,1,w,\alpha,0) = F^0(1,1,w,\alpha,1) \\ = 1 + \frac{w}{\alpha-1}, \end{aligned} \tag{2.17}$$

$$F(m,0,w,\alpha,0) = f_m \tag{2.18}$$

for $m \geq 1$ and

$$F(0,n,w,\alpha,1) = g_n(w,\alpha) \tag{2.19}$$

for $n \geq 1$, we can solve in principle the equations (2.9)-(2.19) recursively to yield the optimal policy and the optimal value $F(m,n,w,\alpha)$.

3. $(m,1)$-Expected Total Flowtime Problem.

In this section, the $(m,1)$-problem is considered and in this case, as to observe the processing time of a job of type 1 does not influence the decision to reduce the total flowtime for the jobs to be processed after that job. Therefore, a schedule is to determine when to start J_1 job.

Theorem 1 *For $m \geq 2$, $w > 0$ and $\alpha > 1$,*

$$F(m,1,w,\alpha,1) = \frac{m^2+m+2}{4} + \frac{w}{\alpha-1} + \frac{1}{2}\sum_{i=1}^{m-1}(m-i+1)A_i(w,\alpha) \tag{3.1}$$

and

$$F(m,1,w,\alpha,0) = \min_{1\leq k\leq m}\left[\frac{m^2+3m-2}{4} + \frac{w}{\alpha-1} + \frac{1}{2}\left\{\sum_{i=1}^{k-1}(k-i+1)A_i(w,\alpha) - (k-2)\right\}\right]. \tag{3.2}$$

Proof. From (2.12) and (2.16)

$$\begin{aligned} F(m,1,w,\alpha,1) &= F^0(m,1,w,\alpha,1) \\ &= (m+1)A_1(w,\alpha) + \int_0^\infty F(m-1,1,w+x,\alpha,1)e^{-x}\overline{G}(x|w,\alpha)dx \\ &\quad + \int_0^\infty F(m,0,w+x,\alpha+1,0)g(x|w,\alpha)e^{-x}dx. \end{aligned}$$

Since $F(m,0,w+x,\alpha+1,0) = f_m = (m^2+m+2)/4$ and (v) of Lemma 1,

$$\begin{aligned} F(m,1,w,\alpha,1) &= \frac{m^2+m+2}{4} - \frac{m^2-3m-2}{4}A_1(w,\alpha) \\ &\quad + \int_0^\infty F(m-1,1,w+x,\alpha,1)e^{-x}\overline{G}(x|w,\alpha)dx. \end{aligned}$$

By using this equation and (2.17),

$$\begin{aligned} F(2,1,w,\alpha,1) &= 2 + A_1(w,\alpha) + \int_0^\infty \left(1 + \frac{w+x}{\alpha-1}\right)e^{-x}\overline{G}(x|w,\alpha)dx \\ &= 2 + \frac{w}{\alpha-1} + A_1(w,\alpha), \end{aligned}$$

which means (3.1) is true for $n = 2$. Assume that

$$F(m-1,1,w,\alpha,1) = \frac{(m-1)^2+(m-1)+2}{4} + \frac{w}{\alpha-1} + \frac{1}{2}\sum_{i=1}^{m-2}(m-i)A_i(w,\alpha)$$

holds. Then

$$F(m,1,w,\alpha,1) = \frac{m^2+m+2}{4} - \frac{m^2-3m-2}{4}A_1(w,\alpha)$$

$$+ \int_0^\infty \left\{\frac{(m-1)^2+(m-1)+2}{4} + \frac{w+x}{\alpha-1} + \frac{1}{2}\sum_{i=1}^{m-2}(m-i)A_i(w+x,\alpha)\right\}e^{-x}\overline{G}(x|w,\alpha)dx$$

$$\begin{aligned} &= \frac{m^2+m+2}{4} - \frac{m^2-3m-2}{4}A_1(w,\alpha) + \frac{(m-1)^2+(m-1)+2}{4}A_1(w,\alpha) \\ &\quad + \frac{w}{\alpha-1} - A_1(w,\alpha) + \frac{1}{2}\sum_{i=1}^{m-2}(m-i)A_{i+1}(w+x,\alpha) \\ &= \frac{m^2+m+2}{4} + \frac{w}{\alpha-1} + \frac{1}{2}mA_1(w,\alpha) + \frac{1}{2}\sum_{i=2}^{m-1}(m-i+1)A_i(w+x,\alpha) \end{aligned}$$

$$= \frac{m^2+m+2}{4} + \frac{w}{\alpha-1} + \frac{1}{2}\sum_{i=1}^{m-1}(m-i+1)A_i(w+x,\alpha).$$

Therefore, (3.1) holds for $n \geq 2$. Also, since

$$F(m,1,w,\alpha,0) = \min\{F^0(m,1,w,\alpha,0), F^1(m,1,w,\alpha,0)\},$$

$$F^0(m,1,w,\alpha,0) = \frac{m+1}{2} + F(m-1,1,w,\alpha,0)$$

and

$$F^1(m,1,w,\alpha,0) = F(m,1,w,\alpha,1),$$

$F(m,1,w,\alpha,0)$ is rewritten as follows:

$$F(m,1,w,\alpha,0) = \min\left\{\frac{m+1}{2} + F(m-1,1,w,\alpha,0), F(m,1,w,\alpha,1)\right\},$$

For $m=2$,

$$\begin{aligned} F(2,1,w,\alpha,0) &= \min\left\{\frac{3}{2} + F(1,1,w,\alpha,0), F(2,1,w,\alpha,1)\right\} \\ &= \min\left\{\frac{3}{2} + 1 + \frac{w}{\alpha-1}, 2 + \frac{w}{\alpha-1} + A_1(w,\alpha)\right\}, \\ &= \min_{1\leq k\leq 2}\left[2 + \frac{w}{\alpha-1} + \frac{1}{2}\left\{\sum_{i=1}^{k-1}(k-i+1)A_i(w,\alpha) - (k-2)\right\}\right]. \end{aligned}$$

Therefore, (3.2) is true for $m=2$. Assume that

$$\begin{aligned} F(m-1,1,w,\alpha,0) &= \min_{1\leq k\leq m-1}\left[\frac{(m-1)^2+3(m-1)-2}{4} + \frac{w}{\alpha-1}\right. \\ &\quad \left. + \frac{1}{2}\left\{\sum_{i=1}^{k-1}(k-i+1)A_i(w,\alpha) - (k-2)\right\}\right] \end{aligned}$$

holds for $m \geq 3$. Then

$$F(m,1,w,\alpha,0) = \min\left\{\frac{m+1}{2} + F(m-1,1,w,\alpha,0), F(m,1,w,\alpha,1)\right\},$$

$$\begin{aligned} &= \min\left\{\frac{m+1}{2} + \min_{1\leq k\leq m-1}\left[\frac{(m-1)^2+3(m-1)-2}{4} + \frac{w}{\alpha-1}\right.\right. \\ &\quad \left.\left. + \frac{1}{2}\left\{\sum_{i=1}^{k-1}(k-i+1)A_i(w,\alpha) - (k-2)\right\}, F(m,1,w,\alpha,1)\right]\right\} \\ &= \min\left\{\min_{1\leq k\leq m-1}\left[\frac{m^2+3m-2}{4} + \frac{w}{\alpha-1} + \frac{1}{2}\left\{\sum_{i=1}^{k-1}(k-i+1)A_i(w,\alpha) - (k-2)\right\}\right],\right. \\ &\quad \left.\frac{m^2+m+2}{4} + \frac{w}{\alpha-1} + \frac{1}{2}\sum_{i=1}^{m-1}(m-i+1)A_i(w,\alpha)\right\} \\ &= \min_{1\leq k\leq m}\left[\frac{m^2+3m-2}{4} + \frac{w}{\alpha-1} + \frac{1}{2}\left\{\sum_{i=1}^{k-1}(k-i+1)A_i(w,\alpha) - (k-2)\right\}\right]. \end{aligned}$$

This completes the proof.□

Let

$$R_k(w,\alpha) = \sum_{i=1}^{k-1}(k-i+1)A_i(w,\alpha) - (k-2), \tag{3.3}$$

for $k \geq 1$. Then, from (3.2), the optimal schedule is one that minimizes $R_k(w,\alpha)$. Let S_k, $1 \leq k \leq m$, denote a schedule that starts processing J_1 job just after $m-k$ J_0 jobs have been completed. Note that S_m (S_1), corresponds to the schedule that starts J_1 job initially (finally).

Lemma 2 *For $k \geq 2$ and $\alpha > 1$, $R_k(w,\alpha)$ is strictly increasing in w.*

Proof. This is immediate from (3.3) and (i) of Lemma 1.□

Lemma 3 *If $A_1(w,\alpha) \geq 1/2$, then*

$$\min_{1\leq k\leq m} R_k(w,\alpha) = R_1(w,\alpha) = 1$$

for $m \geq 2$, i.e., S_1 is optimal.

Proof. Since $R_1(w,\alpha) = 1$, it suffices to show that $R_k(w,\alpha) \geq 1$ for $k \geq 1$ when $A_1(w,\alpha) \geq 1/2$. To prove this, we use Jensen's inequality which states that if φ is a convex function, then for any random variable X,

$$\mathrm{E}[\varphi(X)] \geq \varphi(\mathrm{E}[X]) \tag{3.4}$$

provided the expectations exist. Take

$$\varphi(X) = X^i$$

and

$$X = \frac{1}{1+V},$$

then the inequality (3.4) is equivalent to

$$A_i(w,\alpha) \geq \{A_1(w,\alpha)\}^i. \tag{3.5}$$

Applying (3.5) to (3.3) with $A_1(w,\alpha) \geq 1/2$ yields, for $k \geq 2$

$$\begin{aligned} R_k(w,\alpha) &\geq \sum_{i=1}^{k-1}(k-i+1)\left(\frac{1}{2}\right)^i - (k-2) \\ &= 1, \end{aligned}$$

which completes the proof. □

Lemma 4 *For $k \geq 2, w > 0$ and $\alpha > 1$,*
(i) $R_k(w,\alpha) - R_{k-1}(w,\alpha) = \sum_{i=1}^{k-1} A_i(w,\alpha) + A_{k-1}(w,\alpha) - 1$,
(ii) $R_k(w,\alpha) - R_{k-1}(w,\alpha)$ *is strictly increasing in w,*
(iii) *the following equation of w has a unique positive root $r_k(\alpha)$ and $R_k(w,\alpha) < R_{k-1}(w,\alpha)$ if and only if $w < r_k(\alpha)$:*

$$R_k(w,\alpha) - R_{k-1}(w,\alpha) = 0. \tag{3.6}$$

Proof. From the definition of $\mathrm{R}_k(w,\alpha)$,

$$R_k(w,\alpha) - R_{k-1}(w,\alpha) = \sum_{i=1}^{k-1} A_i(w,\alpha) + A_{k-1}(w,\alpha) - 1. \tag{3.7}$$

Since $A_i(w,\alpha)$ is strictly incresing in w ((i) of Lemma 1), $R_k(w,\alpha) - R_{k-1}(w,\alpha)$ is also strictly increasing in w. Since

$$\lim_{w\to 0+} A_i(w,\alpha) = \lim_{w\to 0+} \int_0^\infty \left(\frac{w}{w+y}\right)^k p(y \mid 1,\alpha)dy = 0$$

and

$$\lim_{w\to \infty} A_i(w,\alpha) = \lim_{w\to \infty} \int_0^\infty \left(\frac{w}{w+y}\right)^k p(y \mid 1,\alpha)dy = 1,$$

we can derive from (3.7) that

$$\lim_{w\to 0+} \{R_k(w,\alpha) - R_{k-1}(w,\alpha)\} = -1$$

and

$$\lim_{w\to \infty} \{R_k(w,\alpha) - R_{k-1}(w,\alpha)\} > 0.$$

This completes the proof. □

From (3.6) and (3.7), $r_k(\alpha)$ is the unique root of the following equation of w:

$$\sum_{i=1}^{k-1} A_i(w,\alpha) + A_{k-1}(w,\alpha) - 1 = 0. \tag{3.8}$$

Although the optimal strategy is difficult to obtain analitically, we can compute the values of $r_k(\alpha)$ numerically by using the equation (3.8). After the calculation of $r_k(\alpha)$ for $k = 2,3,\cdots,20$ and $\alpha = 2,3,\cdots,30$, we found that the inequality $r_{k-1}(\alpha) > r_k(\alpha)$ holds for $3 \le k \le 20$ and $2 \le \alpha \le 30$. The values of $r_k(\alpha)$ for $k = 2,3,\cdots,12$ and $\alpha = 2,3,\cdots,30$ are listed in Table 1.

Claim 1 *If $r_2(\alpha) > r_3(\alpha) > \cdots > r_n(\alpha)$ holds for some $n \ge 2$ and $\alpha > 1$, then for $2 \le m \le n$ and $w > 0$, the optimal strategy for $(m,1,w,\alpha)$-problem is described as follows:*
(i) *If $r_2(\alpha) < w$, then assign J_1 immediately after $(m-1)$ jobs of type J_0 have been completed.*
(ii) *If $r_{k+1}(\alpha) < w \le r_k(\alpha)$ $(2 \le k \le m-1)$, then assign J_1 immediately after $(m-k)$ jobs of type J_0 have been completed.*
(iii) *If $0 < w \le r_m(\alpha)$, then assign J_1 immediately.*

4. $(1,n)$-Expected Total Flowtime Problem.

Since learning mechanism is included in the case of $(1,n)$-problem if $n \ge 2$, the problem is more difficult than the single machine stochastic scheduling problem. The following theorem shows how $F(1,n,w,\alpha,0)$ is related to $\{A_i(w,\alpha)\}_{i=1}^\infty$ or $\{B_i(w,\alpha)\}_{i=1}^\infty$.

Table 1: $r_n(\alpha)$ *for* $n = 2, 3, \cdots, 12$ *and* $\alpha = 2, 3, \cdots, 30$

α \ n	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**	**11**	**12**
2	1.561	1.423	1.327	1.260	1.214	1.180	1.155	1.135	1.120	1.108	1.098
3	2.543	2.394	2.290	2.217	2.166	2.130	2.104	2.085	2.071	2.060	2.051
4	3.533	3.380	3.271	3.195	3.143	3.107	3.081	3.063	3.050	3.040	3.032
5	4.527	4.371	4.260	4.182	4.130	4.093	4.068	4.051	4.039	4.030	4.023
6	5.523	5.365	5.252	5.174	5.121	5.085	5.060	5.044	5.032	5.024	5.018
7	6.520	6.361	6.247	6.168	6.115	6.079	6.055	6.039	6.028	6.020	6.015
8	7.518	7.357	7.243	7.164	7.110	7.075	7.051	7.035	7.025	7.018	7.013
9	8.517	8.355	8.240	8.161	8.107	8.072	8.048	8.033	8.023	8.016	8.011
10	9.515	9.353	9.237	9.158	9.104	9.069	9.046	9.031	9.021	9.015	9.010
11	10.514	10.351	10.235	10.156	10.102	10.067	10.044	10.029	10.020	10.014	10.010
12	11.513	11.350	11.234	11.154	11.101	11.066	11.043	11.028	11.019	11.013	11.009
13	12.513	12.349	12.232	12.152	12.099	12.064	12.042	12.027	12.018	12.012	12.008
14	13.512	13.348	13.231	13.151	13.098	13.063	13.041	13.026	13.017	13.012	13.008
15	14.512	14.347	14.230	14.150	14.097	14.062	14.040	14.026	14.017	14.011	14.008
16	15.511	15.347	15.229	15.149	15.096	15.061	15.039	15.025	15.016	15.011	15.007
17	16.511	16.346	16.229	16.148	16.095	16.061	16.039	16.025	16.016	16.011	16.007
18	17.510	17.345	17.228	17.148	17.094	17.060	17.038	17.024	17.016	17.010	17.007
19	18.510	18.345	18.227	18.147	18.094	18.059	18.038	18.024	18.015	18.010	18.007
20	19.510	19.344	19.227	19.146	19.093	19.059	19.037	19.024	19.015	19.010	19.007
21	20.509	20.344	20.226	20.146	20.093	20.058	20.037	20.023	20.015	20.010	20.007
22	21.509	21.344	21.226	21.145	21.092	21.058	21.036	21.023	21.015	21.010	21.007
23	22.509	22.343	22.226	22.145	22.092	22.058	22.036	22.023	22.015	22.010	22.006
24	23.509	23.343	23.225	23.145	23.091	23.057	23.036	23.023	23.014	23.009	23.006
25	24.509	24.343	24.225	24.144	24.091	24.057	24.036	24.022	24.014	24.009	24.006
26	25.509	25.343	25.225	25.144	25.091	25.057	25.035	25.022	25.014	25.009	25.006
27	26.508	26.342	26.224	26.144	26.090	26.056	26.035	26.022	26.014	26.009	26.006
28	27.508	27.342	27.224	27.143	27.090	27.056	27.035	27.022	27.014	27.009	27.006
29	28.508	28.342	28.224	28.143	28.090	28.056	28.035	28.022	28.014	28.009	28.006
30	29.508	29.342	29.224	29.143	29.090	29.056	29.035	29.022	29.014	29.009	29.006

Theorem 2 *For* $n \geq 2, w > 0$ *and* $\alpha > 1$,

$$F(1,n,w,\alpha,0) = 1 + \frac{n^2+n+2}{4}\frac{w}{\alpha-1} + \sum_{i=1}^{n-1} a_{n,i}A_i(w,\alpha) \tag{4.1}$$

$$= \frac{n+2}{2} + \frac{n^2+n+2}{4}\frac{w}{\alpha-1} - \frac{1}{2}\left(\sum_{i=1}^{n-1} B_i(w,\alpha) + B_{n-1}(w,\alpha)\right) \tag{4.2}$$

where

$$a_{n,k} = -\frac{1}{2}\left({}_{n-1}C_k + \sum_{i=k}^{n-1} {}_iC_k\right)(-1)^k$$

for $n-1 \geq k \geq 1$.

Proof. Since $F(1,2,w,\alpha,0) = 1 + 2w/(\alpha-1) + A_1(w,\alpha)$ is derived from (2.11), (2.16), (2.17), (2.19) and Lemma 1, (4.1) is trivial by using $a_{2,1} = 1$ for $n = 2$. For $n \geq 3$, suppose that (4.1) is true for the $(1, n-1, w, \alpha)$-problem. That is,

$$F(1,n-1,w,\alpha,0) = 1 + \frac{(n-1)^2+(n-1)+2}{4}\frac{w}{\alpha-1} + \sum_{i=1}^{n-2} a_{n-1,i}A_i(w,\alpha).$$

For the $(1, n, w, \alpha)$-problem,

$$F(1,n,w,\alpha,0) = F^1(1,n,w,\alpha,0)$$
$$= (n+1)A_1(w,\alpha) + \int_0^\infty F(0,n,w+x,\alpha,1)\overline{G}(x|w,\alpha)e^{-x}dx$$
$$+ \int_0^\infty F(1,n-1,w+x,\alpha+1,0)g(x|w,\alpha)e^{-x}dx$$
$$= (n+1)A_1(w,\alpha) + \int_0^\infty \frac{n^2+n+2}{4}\frac{w+x}{\alpha-1}\overline{G}(x|w,\alpha)e^{-x}dx$$
$$+ \int_0^\infty \left[1 + \frac{(n-1)^2+(n-1)+2}{4}\frac{w+x}{\alpha} + \sum_{i=1}^{n-2} a_{n-1,i}A_i(w+x,\alpha+1)\right] g(x|w,\alpha)e^{-x}dx$$

Using (vi), (v), (viii) and (ix) of Lemma 1,

$$F(1,n,w,\alpha,0) = (n+1)A_1(w,\alpha) + \frac{n^2+n+2}{4}\left[\frac{w}{\alpha-1} - A_1(w,\alpha)\right] + \{1 - A_1(w,\alpha)\}$$
$$+\frac{(n-1)^2+(n-1)+2}{4}A_1(w,\alpha) + \sum_{i=1}^{n-2} a_{n-1,i}\{A_i(w,\alpha) - A_{i+1}(w,\alpha)\}$$

$$= (n+1)A_1(w,\alpha) + \frac{n^2+n+2}{4}\left[\frac{w}{\alpha-1} - A_1(w,\alpha)\right] + \{1 - A_1(w,\alpha)\}$$
$$+\frac{(n-1)^2+(n-1)+2}{4}A_1(w,\alpha) + \sum_{i=1}^{n-2} a_{n-1,i}A_i(w,\alpha) - \sum_{i=2}^{n-1} a_{n-1,i-1}A_i(w,\alpha)$$
$$= (n+1)A_1(w,\alpha) + \frac{n^2+n+2}{4}\left[\frac{w}{\alpha-1} - A_1(w,\alpha)\right] + \{1 - A_1(w,\alpha)\}$$
$$+\frac{(n-1)^2+(n-1)+2}{4}A_1(w,\alpha) + a_{n-1,1}A_1(w,\alpha)$$
$$+\sum_{i=2}^{n-2}\{a_{n-1,i} - a_{n-1,i-1}\}A_i(w,\alpha) - a_{n-1,n-2}A_{n-1}(w,\alpha)$$
$$= 1 + \frac{n^2+n+2}{4}\frac{w}{\alpha-1} + \sum_{i=2}^{n-2}\{a_{n-1,i} - a_{n-1,i-1}\}A_i(w,\alpha)$$
$$+\left\{a_{n-1,1} + \frac{n}{2}\right\}A_1(w,\alpha) - a_{n-1,n-2}A_{n-1}(w,\alpha)$$

Since

$$a_{n,1} = a_{n-1,1} + \frac{n}{2}$$

and

$$a_{n,n-1} = -a_{n-1,n-2}$$

for $n \geq 3$ and

$$a_{n,i} = a_{n-1,i} - a_{n-1,i-1}$$

for $n-2 \geq i \geq 2$ with $n \geq 4$, (4.1) is derived. From (xi) of Lemma 1,

$$\sum_{i=1}^{n-1} B_i(w,\alpha) + B_{n-1}(w,\alpha) = \sum_{i=1}^{n-1}\left[\sum_{k=0}^{i} {}_iC_k(-1)^k A_k(w,\alpha)\right] + \sum_{k=0}^{n-1} {}_{n-1}C_k(-1)^k A_k(w,\alpha)$$

$$\begin{aligned}
&= \sum_{i=1}^{n-1}\left[1+\sum_{k=1}^{i} {}_iC_k(-1)^k A_k(w,\alpha)\right] + \sum_{k=1}^{n-1} {}_{n-1}C_k(-1)^k A_k(w,\alpha)+1 \\
&= n+\sum_{i=1}^{n-1}\sum_{k=1}^{i} {}_iC_k(-1)^k A_k(w,\alpha) + \sum_{k=1}^{n-1} {}_{n-1}C_k(-1)^k A_k(w,\alpha) \\
&= +\sum_{k=1}^{n-1}\sum_{i=k}^{n-1} {}_iC_k(-1)^k A_k(w,\alpha) + \sum_{k=1}^{n-1} {}_{n-1}C_k(-1)^k A_k(w,\alpha) \\
&= n+\sum_{k=1}^{n-1}\left({}_{n-1}C_k+\sum_{i=k}^{n-1} {}_iC_k\right)(-1)^k A_k(w,\alpha) \\
&= n-2\sum_{k=1}^{n-1} a_{n,k}A_k(w,\alpha),
\end{aligned}$$

from which

$$\sum_{k=1}^{n-1} a_{n,k}A_k(w,\alpha) = \frac{n}{2}-\frac{1}{2}\left(\sum_{i=1}^{n-1} B_i(w,\alpha)+B_{n-1}(w,\alpha)\right).$$

Therefore, (4.2) is derived from (4.1), which completes the proof. □

Theorem 3 *For $w>0$ and $\alpha>1$,*

$$F(1,1,w,\alpha,1) = 1+\frac{w}{\alpha-1}, \tag{4.3}$$

$$F(1,2,w,\alpha,1) = \min\left\{1+\frac{5}{2}\frac{w}{\alpha-1},\ 2+\frac{2w}{\alpha-1}-B_1(w,\alpha)\right\}, \tag{4.4}$$

and

$$F(1,n,w,\alpha,1) = \min\left\{F^0(1,n,w,\alpha,1), F^1(1,n,w,\alpha,1)\right\} \tag{4.5}$$

for $n \geq 3$, where

$$F^0(1,n,w,\alpha,1) = \frac{n+2}{2}+\frac{n^2+n+2}{4}\frac{w}{\alpha-1}-\frac{1}{2}\left(\sum_{i=1}^{n-1} B_i(w,\alpha)+B_{n-1}(w,\alpha)\right) \tag{4.6}$$

and

$$F^1(1,n,w,\alpha,1) = \frac{n+1}{2}\frac{w}{\alpha-1}+\int_0^\infty F(1,n-1,w+x,\alpha+1,1)g(x|w,\alpha)dx. \tag{4.7}$$

Proof. (4.3) was given in Section 2. Also, (4.4) is easily derived from (2.10), (2.15), (2.14),

$$F(1,1,w+x,\alpha+1,0) = 1+\frac{w+x}{\alpha}$$

and

$$F(0,2,w+x,\alpha,1) = \frac{2(w+x)}{\alpha-1}.$$

(4.5) is the same as (2.10). (4.6) is from (2.15) and (4.2). (4.7) is the immediate consequence of (2.14). □

5. Conclusion

In this paper, we considered a Bayesian version of the sequencing problem, in which job is assigned to the machine which becomes idle. Two types, J_0 and J_1, of jobs are considered and the processing times of jobs of each type is assumed to be distributed in the exponential distribution with known parameter u and unknown parameter v, respectively. The decision of assigning jobs is made one by one dynamically after observing the processing times of jobs which have been completed up to the current time. The objective of the problem is to minimize the expected total flowtime. This problem is formulated by the principle of optimality of dynamic programming and the recursive formula are obtained. The explicit formula of the objective function is derived in both the case of m J_0 jobs and one J_1 job and the case of one J_0 job and n J_1 jobs.

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Toshio HAMADA
Department of Management Science
Kobe University of Commerce
8-2-1 Gakuen-nishimachi, Nishiku,
Kobe 651-2197, JAPAN
E-mail: hamada@kobe.kobeuc.ac.jp

Mitsushi TAMAKI
Faculty of Business Administration
Aichi University
370 Kurozasa, Miyoshi,
Aichi 470-0296, JAPAN
E-mail: tamaki@vega.aichi-u.ac.jp