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INFINITE-SERIES REPRESENTATIONS OF LAPLACE TRANSFORMS OF PROBABILITY DENSITY FUNCTIONS FOR NUMERICAL INVERSION

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Abstract In order to numerically invert Laplace transforms to calculate probability density functions (pdf's) and cumulative distribution functions (cdf's) in queueing and related models, we need to be able to calculate the Laplace transform values. In many cases the desired Laplace transform values (e.g., of a waiting-time cdf) can be computed when the Laplace transform values of component pdf's (e.g., of a service-time pdf) can be computed. However, there are few explicit expressions for Laplace transforms of component pdf's available when the pdf does not have a pure exponential tail. In order to remedy this problem, we propose the construction of infinite-series representations for Laplace transforms of pdf's and show how they can be used to calculate transform values. We use the Laplace transforms of exponential pdf's, Laguerre functions and Erlang pdf's as basis elements in the series representations. We develop several specific parametric families of pdf's in this infinite-series framework. We show how to determine the asymptotic form of the pdf from the series representation and how to truncate so as to preserve the asymptotic form for a time of interest.

1. Introduction

This paper is concerned with probability density functions (pdf's) f on $\mathbf{R}^+$ (i.e., nonnegative functions f on the positive half line for which $\int_0^\infty f(t)dt = 1$) and their Laplace transforms

$$\hat{f}(s) = \int_0^\infty e^{-st} f(t)dt \ . \tag{1.1}$$

We introduce and investigate pdf's whose Laplace transforms can be represented as infinite series.

Our motivation comes from numerical transform inversion. We have found that many descriptive quantities of interest in queueing models and other probability models in operations research can be effectively computed by numerically inverting Laplace transforms; e.g., by using the Fourier-series method or the Laguerre-series method; see Hosono [13] and [3], [4]. The most difficult step in performing the numerical inversion, if there is any difficulty at all, is usually computing the Laplace transform values; e.g., see the application to polling models in [10].

In many cases, numerical inversion is straightforward provided that Laplace transforms are available for component pdf's. A familiar example is the steady-state waiting-time in the M/G/1 queue. Numerical inversion can be applied directly to the classical Pollaczek-Khintchine transform provided that the Laplace transform of the service-time pdf is available. More generally, the steady-state waiting-time distribution in the GI/G/1 queue can be computed by numerical transform inversion provided that the Laplace transforms of both the interarrival-time pdf and the service-time pdf are available [2]. In the GI/G/1 case, an extra numerical integration is required to calculate the required waiting-time transform values. For these inversion algorithms to be effective, the pdf's also need to be suitably smooth; otherwise preliminary smoothing may need to be performed. In this paper, all pdf's will be continuous.

Of course, there are numerous pdf's on $\mathbf{R}^+$ with convenient Laplace transforms, but almost all of these pdf's have a (pure) *exponential tail*, i.e., they have the asymptotic form

$$f(t) \sim Ae^{-\alpha t} \quad \text{as} \quad t \to \infty , \tag{1.2}$$

where A and α are constants and $f(t) \sim g(t)$ means that $f(t)/g(t) \to 1$ as $t \to \infty$. For example, all phase-type pdf's satisfy (1.2). Thus we are interested in obtaining infinite-series representations of Laplace transforms of pdf's that do *not* satisfy (1.2). For example, these alternative pdf's may have a *semi-exponential tail*, i.e.,

$$f(t) \sim At^{-\beta} e^{-\alpha t} \quad \text{as} \quad t \to \infty \tag{1.3}$$

or a *power tail*, i.e.,

$$f(t) \sim At^{-\alpha} \quad \text{as} \quad t \to \infty . \tag{1.4}$$

We are interested in pdf's of class II and III in the terminology of [2], [6]. A pdf is of class I if its Laplace transforms $\hat{f}$ has rightmost singularity $-s^* < 0$ and $\hat{f}(-s^*) = \infty$. A pdf is of class II if again $-s^* < 0$ but $\hat{f}(-s^*) < \infty$. A pdf is of class III if 0 is the rightmost singularity of $\hat{f}$. Examples of class II and III, respectively, are (1.3) and (1.4). Class III pdf's are interesting because they have long (or heavy) tails. Class II pdf's are less common, but they do occur; e.g., they play a prominent role in priority queues [6]. A typical application of the series representations would be to see how the performance of a queueing system depends on a class II or III service-time distribution, as in [2]. (The effect can be dramatic, but we do not discuss queueing models here.)

We suggest a general approach for constructing class II and III pdf's for which the Laplace transform values can be computed. We suggest representing the Laplace transform as an infinite series and then numerically calculating the sum, using acceleration methods if necessary. For the inversion, we do not actually need a convenient closed-form expression for the transform. It suffices to have an algorithm to compute the transform value $\hat{f}(s)$ for the required arguments s. Thus an infinite series can be a satisfactory representation. We also show how to work with these infinite-series representations. We discuss three infinite-series representations: exponential series (Section 2), Laguerre series (Sections 4–5) and Erlang series (Sections 6–7).

2. Exponential-Series Representations

A natural way to obtain a long-tail (class III) pdf, where 0 is the rightmost singularity of its Laplace transform $\hat{f}$, is to consider an infinite-series of exponential pdf's, where the means of the exponential pdf's can be arbitrarily large. An exponential-series representation is just a countably infinite mixture of exponential pdf's, i.e.,

$$f(t) = \sum_{k=1}^{\infty} p_k \frac{e^{-t/a_k}}{a_k}, \quad t \geq 0 \quad \text{and} \quad \hat{f}(s) = \sum_{k=1}^{\infty} p_k (1 + s a_k)^{-1} , \tag{2.1}$$

where $\{p_k : k \geq 1\}$ is a probability mass function (pmf) and $\{a_k : k \geq 1\}$ is the sequence of means of the component exponential pdf's. The standard case is $a_k < a_{k+1}$ and $a_k \to \infty$ as $k \to \infty$, so that $\hat{f}(s)$ has poles at $-1/a_k$ for all k, implying that $\hat{f}(s)$ has a singularity at 0.

If we truncate the infinite series in (2.1), we obtain a finite mixture of exponential distributions, i.e., a hyperexponential (H_k) distribution. We consider how to truncate in the next section with the objective of capturing the tail asymptotics at times of interest. Thus this section together with the next constitutes an alternative approach to the method for approximating long-tail pdf's by H_k pdf's in [11].

It is significant that the n^{th} moment of f in (2.1) can be simply expressed as $m_n = n! \sum_{k=1}^{\infty} p_k a_k^n$. By choosing appropriate sequences $\{p_k\}$ and $\{a_k\}$, we can produce desired moment sequences. In many cases we can describe the asymptotic behavior of the pdf and/or its associated complementary cumulative distribution function (ccdf) $F^c(t) \equiv 1 - F(t)$ from the asymptotic behavior of the moments [1].

Example 2.1. Let $p_k = e^{-1}/k!$ and $a_k = k$. Then

$$\frac{m_n}{n!} = e^{-1} \sum_{k=1}^{\infty} \frac{k^n}{k!} = b(n) , \tag{2.2}$$

where $b(n) = \{1, 2, 5, 15, 52, 203, \ldots\}$ are the Bell numbers [21, p. 20]. The Bell number $b(n)$ is the number of ways a set of n elements can be partitioned. The Bell numbers themselves have the relatively simple exponential generating function

$$B(z) \equiv \sum_{n=0}^{\infty} \frac{b(n)}{n!} z^n = e^{e^z - 1} , \tag{2.3}$$

from which we can deduce the recurrence relation

$$b(n) = \sum_{k=0}^{n-1} \binom{n-1}{k} b(k) \tag{2.4}$$

with $b(0) = 1$ [21, p. 23]. The discrete distribution with mass $p_k = e^{-1}/k!$ at k has Laplace transform $\hat{b}(s) = B(-s)$ [5, (3.9) on p. 86].

We can easily generalize to obtain a two-parameter family; e.g., keep $p_k = e^{-1}/k!$ but now let $a_k = c(k - 1 + a)$. The parameter c is a scale parameter; $m_1 = 1$ if $c = 1/(1+a)$. In general, for this "generalized Bell pdf", the n^{th} moment is

$$m_n = \frac{n! c^n}{e} \sum_{k=1}^{\infty} \frac{(k + 1 - a)^n}{k!} . \tag{2.5}$$

Example 2.2. Let $p^k = 2^{-(k+1)}$ and $a_k = k$, $k \geq 0$. Then the n^{th} moment is

$$m_n = n! \sum_{k=0}^{\infty} \frac{k^n}{2^{k+1}} = n! \tilde{b}(n) , \tag{2.6}$$

where $\{\tilde{b}(n)\}$ is the sequence of ordered Bell numbers [21, 5.27 on p. 175]. The ordered Bell number $\tilde{b}(n)$ is the number of ordered partitions of a set of n elements, where the order of the classes is counted but not the order of the elements in the classes.

3. Truncating Infinite Series

We will want to truncate the infinite-series transform representations in applications. For the exponential series in Section 2, it is evident that truncation converts a class III pdf into a class I pdf. Hence, truncation cannot yield a good approximation for all t at once, but it can yield a good approximation for any given t (or any interval $[0,t]$). In this section we consider how to truncate.

Appropriate truncations depend on the asymptotic form of the pdf or ccdf and the times (arguments) of interest. We now show how the asymptotic form and the appropriate truncation point (as a function of the time of interest) can be identified by applying the Euler-Maclaurin summation formula to approximate the sum by an integral and then Laplace's method to determine the asymptotic behavior of the integral; see pp. 80 and 285 of Olver (1974).

We start with the ccdf

$$F^c(t) = \sum_{n=1}^{\infty} p_n F_n^c(t), \quad t \geq 0 , \tag{3.1}$$

where $F_n^c(t)$ is a ccdf for each n, which we rewrite as

$$F^c(t) = \sum_{n=1}^{\infty} e^{-\phi(n,t)} , \quad \text{where} \quad \phi(n,t) = -\log p_n - \log F_n^c(t) . \tag{3.2}$$

We now regard $\phi(x,t)$ as a continuous function of x and assume that we can approximate the sum by an integral (invoking the Euler-Maclaurin formula to quantify the error), obtaining

$$F^c(t) \approx \int_0^{\infty} e^{-\phi(x,t)} dx . \tag{3.3}$$

We now use Laplace's method on the integral in (3.3). We assume that there is a unique $x^*(t)$ that minimizes $\phi(x,t)$, i.e., for which $\phi'(x^*(t),t) = 0$ and $\phi''(x^*(t),t) > 0$. The idea is that, for suitably large t, the integral will be dominated by the contribution in the neighborhood of $x^*(t)$. Then Laplace's method yields

$$\int_0^{\infty} e^{-\phi(x,t)} dx \sim \sqrt{\frac{2\pi}{\phi''(x^*(t),t)}} e^{-\phi(x^*(t),t)} \quad \text{as} \quad t \to \infty . \tag{3.4}$$

In summary, the asymptotic form is given by (3.4) and the truncation point as a function of t is $x^*(t)$. In particular, if we are interested in time t_0, then the truncation point should be at least $x^*(t_0)$.

Example 3.1. Suppose that the weights in (3.1) are $p_n = (1-\rho)\rho^n$ (a geometric pmf with an atom $(1-\rho)$ at 0) and F_n is exponential with mean $a_n = bn^c$. For the special case $\rho = 1/2$, $b = 1$ and $c = 1$, we obtain the second Bell pdf in Example 2.2. In general, by (3.2),

$$\phi(n,t) = -\log(1-\rho) - n \log \rho + \frac{t}{bn^c} . \tag{3.5}$$

As indicated above, $x^*(t)$ is the root of

$$\phi'(x,t) = -\log \rho - \frac{ct}{bx^{c+1}} = 0 , \tag{3.6}$$

so that

$$x^*(t) = \left(\frac{ct}{b(-\log \rho)}\right)^{1/(c+1)} , \qquad \phi''(x,t) = \frac{c(c+1)t}{bx^{(c+2)}} , \tag{3.7}$$

$$\phi''(x^*(t),t) = (1+c) \left(\frac{b(-\log \rho)^{2+c}}{ct}\right)^{1/(1+c)} , \tag{3.8}$$

and

$$F^c(t) \sim At^{1/2(1+c)}e^{-Bt^{1/(1+c)}} \quad \text{as} \quad t \to \infty , \tag{3.9}$$

where

$$A = (1-\rho)\left(\frac{2\pi c^{1/(1+c)}}{(1+c)b^{1/(1+c)}(-\log\rho)^{(2+c)/(1+c)}}\right)^{1/2} \tag{3.10}$$

and

$$B = \left(1+\frac{1}{c}\right)\left(\frac{(-\log\rho)^c c}{b}\right)^{1/(1+c)} . \tag{3.11}$$

In the special case $\rho = 1/2$, $b = c = 1$ corresponding to Example 2.2,

$$F^c(t) \sim At^{1/4}e^{-Bt^{1/2}} \quad \text{as} \quad t \to \infty , \tag{3.12}$$

where

$$A = \frac{\sqrt{\pi}}{2(\log 2)^{3/4}} \quad \text{and} \quad B = 2\sqrt{\log 2} . \tag{3.13}$$

For the case of Example 2.2 we can verify the asymptotic by applying [1, Section 6], but we must correct errors in formulas (6.8) and (6.9) there. Assuming that $F^c(t) \sim \alpha t^\beta e^{-\eta t^\delta}$ as $t \to \infty$, we obtain estimators for the parameters α, β, η and δ from the moments. First, by [21, p.176], the ordered Bell numbers satisfy $\tilde{b}(n) \sim n!/2a^{n+1}$ as $n \to \infty$ for $a = \log 2$. Therefore, by (2.6),

$$m_n \sim \frac{1}{2\log 2}\frac{(n!)^2}{(\log 2)^n} \quad \text{and} \quad r_n \equiv \frac{m_n}{m_{n-1}} \sim \frac{n^2}{a} \quad \text{as} \quad n \to \infty , \tag{3.14}$$

$$\delta_n \equiv \frac{r_n}{n(r_{n+1}-r_n)} \to \frac{1}{2} \quad \text{and} \quad \eta_n \equiv \frac{n}{\delta r_n^\delta} \to 2\sqrt{a} \quad \text{as} \quad n \to \infty , \tag{3.15}$$

$$\beta_n \equiv \delta\left((\eta\delta/n)^{1/\delta} n r_n - n + (\delta^{-1}-1)/2\right) \to 1/4 \quad \text{as} \quad n \to \infty , \tag{3.16}$$

$$\alpha_n \equiv \frac{m_n \eta^{(n+\beta)/\delta}}{\Gamma((n+\beta)/\delta + 1)} \to \frac{\sqrt{\pi}}{2a^{3/4}} \quad \text{as} \quad n \to \infty . \tag{3.17}$$

Note that (3.15)–(3.17) agree with (3.13). Formulas (3.16) and (3.17) here correct (6.8) and (6.9) in [1].

Example 3.2. We now produce a pdf with a power tail. Suppose that $a_n = n^\gamma$ for $\gamma > 0$ and $p_n = \alpha n^{-\beta}$ for $\alpha > 0$ and $\beta > 1$. Then

$$\phi(x,t) = -\log\alpha + \beta\log x + \frac{t}{x^\gamma} , \tag{3.18}$$

so that

$$x^*(t) = (\gamma t/\beta)^{1/\gamma} , \qquad \phi''(x,t) = \frac{-\beta}{x^2} + \frac{\gamma(\gamma+1)t}{x^{\gamma+2}} , \tag{3.19}$$

$$\phi''(x^*(t),t) = \left(\frac{\beta}{\gamma t}\right)^{2/\gamma}(\beta\gamma) \tag{3.20}$$

and

$$F^c(t) \sim At^{-(\beta-1)/\gamma} \quad \text{as} \quad t \to \infty \tag{3.21}$$

for

$$A = \alpha\sqrt{2\pi/\beta\gamma}\, e^{-\beta/\gamma}(\gamma/\beta)^{-(\beta-1)/\gamma} . \tag{3.22}$$

If $\gamma = 1$, then

$$F^c(t) \sim \alpha e^{-\beta} \left(\frac{\beta}{t}\right)^{\beta-1} \quad \text{as} \quad t \to \infty \tag{3.23}$$

and

$$m_n = \alpha n! \sum_{k=1}^{\infty} \beta^{n-\beta} = \alpha n! \zeta(\beta - n) , \tag{3.24}$$

where $\zeta(z)$ is the Riemann zeta function [17, p.19]. The moment m_n is finite for $n < \beta - 1$ and otherwise infinite.

4. Laguerre-Series Representations

The second infinite-series representation we consider is the Laguerre-series representation. In [3] we studied the Laguerre-series representation as a tool to numerically invert Laplace transforms, given that the Laplace transform values are available. Our purpose here is different. Here we want to compute the values of a Laplace transform. For more on Laguerre series, see Keilson and Nunn [15], Sumita and Kijima [20] and references therein.

The *Laguerre-series representation* of a function f (e.g., the pdf) is

$$f(t) = \sum_{n=0}^{\infty} q_n L_n(t), \quad t \geq 0 , \tag{4.1}$$

where q_n are real numbers and

$$L_n(t) \equiv \sum_{k=0}^{n} \binom{n}{k} \frac{(-t)^k}{k!}, \quad t \geq 0 , \tag{4.2}$$

are the *Laguerre polynomials* [9, 22.3.9]. The Laguerre polynomials form a complete orthonormal basis for the space $L_2[0, \infty)$ of real-valued functions on the nonnegative real line $[0, \infty)$ that are square integrable with respect to the weight function e^{-t}, using the inner product

$$\langle f_1, f_2 \rangle = \int_0^{\infty} e^{-t} f_1(t) f_2(t) dt \tag{4.3}$$

and norm $\|f\| = \langle f, f \rangle^{1/2}$. Then (4.1) is valid in the sense of convergence in $L_2[0, \infty)$ with

$$q_n = \langle f, L_n \rangle \equiv \int_0^{\infty} e^{-t} f(t) L_n(t) dt \tag{4.4}$$

being the *Laguerre coefficient* in (4.1). Assuming that f is continuous, the series (4.1) converges pointwise for all t as well. The Laguerre polynomial $L_n(t)$ can be calculated in a numerically stable way via the recursion

$$L_n(t) = \left(\frac{2n - 1 - t}{n}\right) L_{n-1}(t) - \left(\frac{n-1}{n}\right) L_{n-2}(t) . \tag{4.5}$$

Our interest, however, is in the associated series representation for the Laplace transform $\hat{f}$. Since the Laplace transform of $L_n(t)$ has the simple form

$$\hat{L}_n(s) = \int_0^{\infty} e^{-st} L_n(t) dt = \frac{(s-1)^n}{s^{n+1}} , \tag{4.6}$$

the associated Laguerre-series representation for $\hat{f}$ is

$$\hat{f}(s) = \sum_{n=0}^{\infty} q_n \int_0^{\infty} e^{-st} L_n(t)dt = \sum_{n=0}^{\infty} q_n \frac{(s-1)^n}{s^{n+1}} , \tag{4.7}$$

where again q_n is the Laguerre coefficient in (4.1) and (4.4).

Above we have used the Laguerre polynomials and the weight e^{-t}. An essentially equivalent approach is to use the associated orthonormal Laguerre functions $e^{-t/2}L_n(t)$ without a weight. Although that approach is essentially the same, the resulting coefficients q_n are different.

In order to exploit the Laguerre-series representation for computing the Laplace transform, we primarily work with the Laguerre series representation of $e^t f(t)$. This requires that $e^t f(t)$ be square integrable with respect to the weight e^{-t} or, equivalently, that $e^{t/2} f(t)$ be square integrable directly. That, in turn, clearly requires that $f(t)$ be asymptotically dominated by the exponential $e^{-t/2}$, which means that its Laplace transform $\hat{f}(s)$ should have its rightmost singularity $-s^*$ satisfy $-s^* < -1/2$. This method thus applies to short-tail pdf's, but not long-tail pdf's, i.e., to pdf's in class I and II. We are especially interested in class II, because fewer alternatives are available.

Remark 4.1. Some pdf's f such as the gamma pdf with shape parameter 1/2, i.e., $\gamma(1/2;t) \equiv e^{-t}/\sqrt{\pi t}$, are not square integrable with respect to the exponential weight because $f(t) \sim t^{-p}$ as $t \to 0$ for $1/2 \leq p < 1$. As a consequence, such pdf's do not fit directly into the L_2 theory. Indeed, then $\sum_{n=1}^{\infty} q_n^2 = \infty$, as can be seen from Examples 2.1–2.4 of [3]. However, the L_2 theory can be applied by using the generalized Laguerre polynomials $L_n^{(\alpha)}(t) \equiv \sum_{k=0}^{n}(-1)^k \binom{n+\alpha}{n-k} t^k/k!$ with respect to the weight $t^\alpha e^{-t}$ for $\alpha = 1$. The expansion for $e^t f(t)$ can then be re-expressed in terms of the standard Laguerre polynomials and coefficients, i.e.,

$$e^t f(t) = \sum_{n=1}^{\infty} q_n^{(1)} L_n^{(1)}(t) = \sum_{n=1}^{\infty} q_n L_n(t) , \tag{4.8}$$

using relationships among the Laguerre coefficients and polynomials, i.e., $q_n^{(1)} = q_n - q_{n+1}$, $L_n^{(1)} = -L'_{n+1}(t)$ and $L'_{n+1}(t) = L'_n(t) - L_n(t)$ [17, p. 241]. As a consequence, the direct Laguerre-series representations are still valid, with the understanding that the L_2 properties require the generalized Laguerre polynomials. ∎

Given that we decide to seek a Laguerre-series representation, there are two problems: (1) calculating the Laguerre coefficients, and (2) calculating the sum for the Laplace transform. It turns out that both problems can be addressed very effectively when the pdf is short-tailed. There are two ways to compute the coefficients. The first way is by numerical integration using the standard integral representation of the coefficients, assuming that the pdf is known. The second way is to compute the coefficients from the moments, assuming that the moments are known. In exploiting the moments, we follow Keilson and Nunn [15], who showed that the Laguerre coefficients can be calculated from the moments under a regularity condition. However, we obtain a more elementary connection between the Laguerre coefficients and moments by altering the way the Laguerre coefficients are defined. The resulting Laguerre-series representation for the Laplace transform is convenient because the series tends to converge geometrically fast, even when the corresponding Laguerre-series in the time domain converges slowly. Moreover, it is sometimes possible to analytically determine the Laguerre coefficients, as we illustrate later.

We find it necessary to scale appropriately so that the rightmost singularity $-s^*$ of the Laplace transform $f(s)$ satisfies $-s^* < -1/2$. Such scaling can easily be done for classes I and II, but cannot be done for class III. Class III can be transformed to satisfy this condition too, e.g., by exponential damping as discussed in [2, 5], but then the moments and Laguerre coefficients are altered, so we are unable to exploit the transformation. (This point is made [15, Section 7] too.) In contrast, the moments are simply scaled under our linear scaling of class I and II pdf's.

Assuming that we are indeed considering the Laguerre-series representation for $e^t f(t)$, then the Laguerre coefficient q_n becomes

$$q_n = \int_0^\infty f(t) L_n(t) dt \ . \tag{4.9}$$

Since $L_n(t)$ is a polynomial of degree n, we see that q_n is a linear combination of the first n moments of f. Indeed, we can combine (4.2) and (4.9) to obtain the following result.

Theorem 4.1. *If the function $e^t f(t)$ has a Laguerre-series representation with respect to the weight e^{-t} then all moments of f are finite and the n^{th} Laguerre coefficient q_n of $e^t f(t)$ can be expressed as*

$$q_n = \sum_{k=0}^{n} \binom{n}{k} (-1)^k \frac{m_k}{k!} \ , \tag{4.10}$$

where m_k is the k^{th} moment of f with $m_0 = 1$.

Since the rightmost singularity is less than $-1/2$, all moments are finite. With the Laguerre coefficients as in (4.9) and (4.10), we obtain

$$f(t) = e^{-t} \sum_{n=0}^{\infty} q_n L_n(t) \quad \text{and} \quad \hat{f}(s) = \sum_{n=0}^{\infty} q_n \frac{s^n}{(s+1)^{n+1}} \ . \tag{4.11}$$

It is noteworthy that the convergence of the transform series tends to be faster than the convergence of the basic Laguerre series, because $L_n(t)$ converges to 0 slowly as $n \to \infty$. The Laguerre series in (4.11) tends to converge geometrically for all s with $\text{Re}(s) > 0$ because the Laguerre coefficients are bounded and $|s/(s+1)| < 1$ for $\text{Re}(s) > 0$. The Laguerre coefficients are bounded because $|L_n(t)| \le e^{t/2}$ for $t \ge 0$ [9, 22.14.12 on p. 786], and

$$|q_n| \le \int_0^\infty |f(t)||L_n(t)| dt \le \int_0^\infty e^{t/2} |f(t)| dt \equiv M < \infty \ . \tag{4.12}$$

Using the Hilbert space theory of the function space $L_2[0, \infty)$, we know that, with $g(t) = e^t f(t)$,

$$\|g\|_2^2 \equiv \int_0^\infty e^t f(t)^2 dt = \sum_{n=0}^{\infty} q_n^2 < \infty \ , \tag{4.13}$$

which implies that $q_n \to 0$. Hence the transform series (4.11) indeed converges geometrically fast.

However, when we do numerical inversion, we need to compute the transform $\hat{f}(s)$ not for a single value of s but for several values of s. If we use the Fourier-series method in [4], $A = 30$, say, and $k \ge 1$. A serious difficulty occurs because $|s/(1+s)| \to 1$ as $v \to \infty$ and, thus, as $k \to \infty$. Thus, the Euler summation used in the Fourier-series method helps greatly, because it enables us to restrict attention to $k \le 40$, say. Then $u = 15/t$ and $v \approx 125/t$.

We give a bound on the truncation error, omitting the easy proof.

Theorem 4.2. *Let M be the bound on $|q_n|$ in (4.12). For $s = u + iv$ with $u > 0$ and each $N \geq 1$,*

$$|\hat{f}(s) - \hat{f}_N(s)| \leq M \left| \frac{1}{s+1} \right| \frac{\epsilon^N}{1-\epsilon} , \tag{4.14}$$

where

$$\hat{f}_N(s) = \sum_{n=0}^{N} \frac{1}{s+1} q_n \left(\frac{s}{s+1} \right)^n \tag{4.15}$$

with q_n being the Laguerre coefficients in (4.9) and (4.10), and $\epsilon = (\beta^2 + \alpha^2)/(\beta^2 + 1) < 1$ with $\beta = v/(1+u)$ and $\alpha = u/(1+u)$.

When we apply the Fourier-series method of numerical transform inversion with Euler summation, we need to consider $s = u + iv$ with $u = 15/t$ and $v \approx 125/t$. From Theorem 4.2, we see that $|s/(s+1)| \approx 1 - u/\nu^2$ when $u << v$, so that here $|s/(s+1)| \approx 1 - t10^{-3}$ in the worst case. Hence, to have truncation error $|\hat{f}_N(s) - \hat{f}(s)| \leq \delta M$, we need

$$N \approx \frac{\log(\delta t 10^{-3})}{\log(1 - t10^{-3})} \approx \frac{-\log(\delta t 10^{-3})}{t10^{-3}} . \tag{4.16}$$

For example, for $\delta = 10^{-15}$ and $M = 1$, we require $N \approx 4.4 t^{-1} \times 10^4$. Thus, for $t = 1$, we require $N = 44,000$. The resulting computation is feasible, but not easy. Hence, if alternative methods are available, then they might well be preferred.

To apply Theorem 4.1, we can first scale the transform so that it has rightmost singularity -1. This is achieved by considering the scaled Laplace transform $\hat{f}_{s^*}(s) \equiv \hat{f}(s^* s)$. In terms of a random variable X with pdf f and moments m_n, this corresponds to replacing X by X/s^*, which has pdf $s^* f(s^* t)$, $t \geq 0$, and moments $m_n/(s^*)^n$. To carry out the scaling, we need to determine the rightmost singularity $-s^*$. Fortunately, algorithms to compute s^* from the moments are contained in [1], where s^* is denoted η. Under considerable generality, $n m_{n-1}/m_n \to s^*$ as $n \to \infty$, but the proposed algorithms often do much better by exploiting Richardson extrapolation [1].

It is convenient that we can relate the Laguerre coefficients of a pdf $f(t)$ to its complementary cdf (ccdf) $F^c(t) \equiv 1 - F(t)$ directly.

Theorem 4.3. *If $e^t f(t)$ has a Laguerre-series representation for a pdf $f(t)$, then so does $e^t F^c(t)$ for the associated ccdf $F^c(t)$, and the expansions are related by*

$$f(t) = e^{-t} \sum_{n=0}^{\infty} q_n L_n(t) \quad \text{and} \quad F^c(t) = e^{-t} \sum_{n=0}^{\infty} (q_n - q_{n+1}) L_n(t) . \tag{4.17}$$

Proof. Using Laplace transforms,

$$\begin{aligned} \hat{F}^c(s) &= \frac{1 - \hat{f}(s)}{s} = \frac{1}{s} + \hat{f}(s) - \left(\frac{s+1}{s} \right) \hat{f}(s) \\ &= \frac{1}{1+s} \sum_{n=0}^{\infty} q_n \left(\frac{s}{s+1} \right)^n - \frac{1}{1+s} \sum_{n=0}^{\infty} q_{n+1} \left(\frac{s}{s+1} \right)^n \end{aligned}$$

using the fact that

$$q_0 = \int_0^\infty f(t) L_0(t) dt = \int_0^\infty f(t) dt = 1 . \quad \blacksquare$$

Example 4.1. An interesting example from [5, p. 86] and [1, p. 986] is the Cayley-Einstein-Polya (CEP) pdf with mean 1 and Laplace transform satisfying the functional equation

$$\hat{f}(s) = \exp(-s\hat{f}(s)) \ . \tag{4.18}$$

Evidently, both the associated pdf and Laplace transform are unknown, but the moments are $m_n = (n+1)^{n-1}$, $n \geq 1$. The ccdf has the asymptotic form

$$F^c(t) \sim \sqrt{\frac{e^5}{2\pi}} t^{-3/2} e^{-\eta t} \quad \text{as} \quad t \to \infty \tag{4.19}$$

for $\eta = 1/e$. Even though the mean is $m_1 = 1$, the rightmost singularity of $\hat{f}(s)$ is $-s^* = -e^{-1} > -1/2$, so that scaling is needed in this case. As indicated earlier, the final decay rate has to be at least 1/2. From numerical experience, we found that the scaling becomes more effective in this example as the final decay rate approaches 1/2. Hence, we use the scale parameter $\sigma = e/2$, making the ccdf $G^c(t) \equiv F^c(\sigma t)$ with moments $m_n(G) = (n+1)^{n+1}(2/e)^n$. Then

$$G^c(t) \sim \frac{2e}{\sqrt{\pi}} t^{-3/2} e^{-t/2} \text{ as } t \to \infty \quad \text{and} \quad m_n(G) \sim \frac{2e}{\sqrt{\pi}} (2n)^{-3/2} 2^n \text{ as } n \to \infty \ . \tag{4.20}$$

For the ccdf, we obtain the Laguerre-series expansion

$$G^c(t) = \sum_{n=0}^{\infty} (q_n - q_{n+1}) L_n(t) \tag{4.21}$$

where q_n can be obtained from the moments using (4.10). We can compute the ccdf and its transform from (4.21) and (4.6). Because of the combinatorial sum, the calculation of q_n from the moments requires about $(n^{-2} 2^{5n/2} + 15)$ digits of precision. Hence, for $n = 80$, we need about 70 digits precision. For numerical examples, we used UBASIC; see Kida [16].

5. Mixtures of Exponential Distributions

In this section we obtain some results about Laguerre-series representations for Laplace transforms of completely monotone pdf's, i.e., for pdf's that are mixtures of exponential pdf's. We first obtain some general results. Then we describe an application of the method of moments in Theorem 4.1 to obtain a Laguerre-series representation for a class of pdf's called a *beta mixture of exponential* (BME) pdf's [7]. The important point here is that the method of moments can be applied *analytically* to determine simple expressions for the Laguerre coefficients.

We first establish integral representations for the Laguerre coefficients for general mixtures of exponentials.

Theorem 5.1. *If a pdf f can be represented as*

$$f(t) = \int_a^b x e^{-xt} dG(x) \tag{5.1}$$

for a cdf G, where $1/2 < a < b \leq \infty$, then $e^t f(t)$ has a Laguerre-series representation with Laguerre coefficients

$$q_n = \int_a^b \left(\frac{x-1}{x}\right)^n dG(x) \ . \tag{5.2}$$

Proof. Because of the exponentials in (5.1), the rightmost singularity of $\hat{f}(s)$ satisfies $-s^* < -1/2$. We exploit the fact that the Laplace transform of $L_n(t)$ is $(s-1)^n/s^{n+1}$. We change the order of integration to obtain

$$q_n = \int_0^\infty e^{-t}(e^t f(t)) L_n(t) dt = \int_a^b \left[\int_0^\infty x e^{-xt} L_n(t) dt\right] dG(x) = \int_a^b \left(\frac{x-1}{x}\right)^n dG(x) . \ \blacksquare$$

We next characterize Laguerre coefficients of mixtures of exponentials where the mixing pdf has support $[0,1]$. Recall that mixtures can be represented as products of independent random variables.

Theorem 5.2. *Let X and Y be independent random variables with X exponentially distributed having mean 1 and Y having support on $[0,1]$. If f is the pdf of XY, then $e^t f(t)$ has a Laguerre-series representation with the n^{th} Laguerre coefficient q_n equal to the n^{th} moment of $(1-Y)$. Hence $1 = q_0 \geq q_n \geq q_{n+1}$ for all n and $q_n \to 0$.*

Proof. The rightmost singularity of the Laplace transform is less than or equal to -1, so that the Laguerre-series representation exists. By Theorem 4.1, the Laguerre coefficient is

$$q_n = \sum_{k=0}^{n} \binom{n}{k} (-1)^k \frac{m_k}{k!} = \sum_{k=0}^{n} \binom{n}{k} (-1)^k y_k$$

where m_k (y_k) is the k^{th} moment of XY (Y). The n^{th} moment of $1-Y$ clearly has the final expression too. $\blacksquare$

Example 5.1. Consider the pdf

$$f(t) = \int_0^1 y^{-1} e^{-t/y} w(y) dy , \tag{5.3}$$

where $w(y) = -\log(1-y)$, $0 \leq y \leq 1$. The pdf $w(1-y) = -\log y$ has n^{th} moment $(n+1)^{-2}$ by [9, 4.1.50], so that the Laguerre coefficient of $e^t f(t)$ is $q_n = (n+1)^{-2}$ by Theorem 5.2.

Corollary. *If, in addition to the assumptions of Theorem 5.2, the mixing pdf $w(y)$ of Y is symmetric on $[0,1]$, i.e., if $w(y) = w(1-y)$, $0 \leq y \leq 1$, then the n^{th} Laguerre coefficient q_n of $e^t f(t)$ equals the n^{th} moment of Y.*

We now obtain even more explicit representations for the Laguerre coefficients of $e^t f(t)$ for the class of BME pdf's. A BME pdf can be expressed as

$$v(p,q;t) = \int_0^1 y^{-1} e^{-t/y} b(p,q;y) dy, \quad t \geq 0 , \tag{5.4}$$

where $b(p,q;y)$ is the standard beta pdf, i.e.,

$$b(p,q;y) = \frac{\Gamma(p+q)}{\Gamma(p)\Gamma(q)} y^{p-1}(1-y)^{q-1}, \quad 0 \leq y \leq 1 , \tag{5.5}$$

and $\Gamma(x)$ is the gamma function. A BME pdf has three parameters $p > 0$, $q > 0$ and a third positive scale parameter, which has been omitted from (5.4). (The parameter q should not be confused with the Laguerre coefficients q_n, which have the subscript.) In general, a more explicit form for the BME pdf is not known, but special cases are described in [7]. Especially tractable are the BME pdf's where both p and q are integer multiples of $1/2$.

The Laplace transform $\hat{v}(p,q;s)$ has rightmost singularity $-s^* = -1$. The BME pdf is class II with n^{th} moment $m_n(p,q) = (p)_n n!/(p+q)_n$, where $(x)_n = x(x+1)\ldots(x+n-1)$ is the Pochammer symbol with $(x)_0 = 1$.

Theorem 4.1, the moment expression and a combinatorial identity, [12, (7.1) on p. 58], yields the Laguerre coefficients in (4.9), as shown in [7, Theorems 2.2 and 2.3]; in particular,

$$q_n = \frac{(q)_n}{(p+q)_n}, \quad n \geq 0 . \tag{5.6}$$

Since $1-Y$ has a beta pdf with parameter pair (q,p) when Y has a beta pdf with parameter pair (p,q), we could also apply Theorem 5.2 to obtain (5.6).

Given the explicit expression for the Laguerre coefficients q_n in (5.6), we can apply the series representation (4.11) to compute the BME transform values. However, in this instance it turns out to be more effective to use continued fractions [8].

We can use the Laguerre-series representation for the BME pdf's to obtain Laguerre-series representations for some related pdf's. In [7] a second class of beta mixtures of exponential (B_2ME) is defined with pdf

$$v_2(p,q;t) = \frac{\Gamma(p+q)}{\Gamma(p)\Gamma(q)} \int_0^\infty y^{-1} e^{-t/y} y^{p-1} (1+y)^{-(p+q)} dy . \tag{5.7}$$

In (5.7) the mixing pdf is a beta pdf of the second kind.

However, by [7, p. 121] the B_2ME pdf is an exponentially undamped version of a BME pdf, i.e.,

$$v_2(p,q;t) = \frac{q}{p+q} e^t v(p,q+1,t), \quad t \geq 0 , \tag{5.8}$$

so that

$$v_2(p,q;t) \sim \frac{q\Gamma(p+q)}{\Gamma(p)t^{q+1}} \quad \text{as} \quad t \to \infty . \tag{5.9}$$

From (5.9), we see that the B_2ME pdf $v_2(p,q;t)$ is a long-tail (class III) pdf. From (5.9) we can obtain Laguerre-series representations for $v_2(p,q;t)$ and its Laplace transform $\hat{v}_2(p,q;s)$; in particular, from (5.6) and (5.8), we obtain

$$v_2(p,q,t) = \sum_{n=0}^{\infty} \left(\frac{q}{p+q}\right) \frac{(q+1)_n}{(p+q+1)_n} L_n(t) \tag{5.10}$$

and

$$\hat{v}_2(p,q;s) = \sum_{n=0}^{\infty} \left(\frac{q}{p+q}\right) \frac{(q+1)_n}{(p+q+1)_n} \frac{(s-1)^n}{s^{n+1}} . \tag{5.11}$$

Unfortunately, however, as noted in Section 2, the Laguerre-series representation (5.11) is not as useful as (4.11) with (5.6). Because of the undamping, s in (4.11) has been shifted to $s-1$.

6. The Erlang-Series Representation

We now consider our third and final class of infinite-series representations, the Erlang-series representation, which was introduced by Keilson and Nunn [15] as the Erlang transform. The n^{th} basis function is an Erlang pdf of order n (E_n) or its Laplace transform $(1+s)^{-n}$. The coefficients of the Erlang transform are obtained from the coefficients of a power series representation of the pdf, assuming that it exists. The Erlang-series representation is intimately related to the Laguerre-series representation and, at the same time, is a convenient alternative to it, because it often applies when the Laguerre-series representation does not and because the Erlang series for the Laplace transform tends to converge rapidly.

The Erlang-series representation is based on a Taylor series expansion of $g(t) \equiv e^t f(t)$, i.e.,

$$g(t) \equiv e^t f(t) = \sum_{n=0}^{\infty} g^{(n)}(0)\frac{t^n}{n!}, \quad t \geq 0 , \tag{6.1}$$

where $g^{(n)}(0)$ is the n^{th} (right) derivative of g evaluated at 0 and the coefficients $f^{(n)}(0)$ and $g^{(n)}(0)$ are related by

$$g^{(n)}(0) = \sum_{k=0}^{n} \binom{n}{k} f^{(n-k)}(0) . \tag{6.2}$$

Then $f(t)$ can be rewritten as

$$f(t) = \sum_{n=1}^{\infty} g^{(n-1)}(0)e(n;t), \quad t \geq 0 , \tag{6.3}$$

where, for $n \geq 1$, $e(n,t)$ is the Erlang (E_n) pdf $e(n;t) = e^{-t}t^{n-1}/(n-1)!$, $t \geq 0$, which has mean n, variance n and Laplace transform $\hat{e}(n;s) = (1+s)^{-n}$. As n increases $e(n;t)$ approaches a normal cdf with mean and variance n. The squared coefficient of variation (SCV, variance divided by the square of the mean) of $e(n;t)$ is $1/n$, showing that the variability is asymptotically negligible compared to the mean.

From (6.3) we immediately obtain an associated Erlang-series representation for the Laplace transform, i.e.,

$$\hat{f}(s) = \sum_{n=1}^{\infty} g^{(n-1)}(0)(1+s)^{-n} . \tag{6.4}$$

(It is necessary to check that the Erlang-series representation for $\hat{f}(s)$ in (6.4) converges.)

Closely paralleling Theorem 4.3, we can easily relate the Erlang-series representation of pdf's and cdf's.

Theorem 6.1. *Let f be a pdf with ccdf F^c. Then the following are equivalent:*

(i) $F^c(t) = \sum_{n=1}^{\infty} c_n e(n;t)$

(ii) $f(t) = \sum_{n=1}^{\infty} (c_n - c_{n+1}) e(n;t)$.

Proof. Note that $\hat{f}(s) = 1 + \hat{F}^c(s) - (s+1)\hat{F}^c(s)$. Hence,

$$\hat{F}^c(s) = \sum_{n=1}^{\infty} c_n(1+s)^{-n} \quad \text{if and only if} \quad \hat{f}(s) = \sum_{n=1}^{\infty} d_n(1+s)^{-n}$$

with $\sum_{n=1}^{\infty} d_n = c_1 = 1$ for the coefficients as indicated.

Remarkably, many exponential pdf's can be represented as a mixtures of Erlangs.

Theorem 6.2. *The Erlang coefficients are $c_n = \rho(1-\rho)^{n-1}$ for $\rho > 0$, if and only if the pdf f is exponential with mean $1/\rho$.*

Proof. Starting with the exponential pdf $\rho e^{-\rho t}$ with $\rho \neq 1$ we get the expansion for the Laplace transform

$$\hat{f}(s) = \frac{\rho}{1-\rho} \sum_{n=1}^{\infty} \frac{(1-\rho)^n}{(1+s)^n} = \frac{\rho}{\rho+s} , \tag{6.5}$$

and vice versa. For $\rho = 1$, we get $c_1 = 1$ and $c_n = 0$ for $n \geq 1$. ■

For $\rho > 1$, the coefficients in (6.5) alternate in sign; otherwise they are probabilities.

7. Mixtures of Exponentials Again

In this section we obtain results about Erlang-series representations when the pdf is a mixture of exponential pdf's. We start by showing that every mixture of exponential pdf's (spectral representation) where the mixing pdf has support $[0,b]$ for $b<\infty$ has an Erlang-series representation and identify the coefficients. First, we note that it suffices to consider mixing pdf's with support $[0,1]$, because we can always rescale; i.e., if $Z=YX$ where Y has support on $[0,b]$, we consider (Y/b) with support $[0,1]$ and let $Z^*=(Y/b)X$. We then consider $Z=bZ^*=YX$.

Theorem 7.1. *Consider a completely monotone pdf $f(t)$ that has a Laguerre-series representation as in Theorem 5.1, i.e.,*

$$f(t)=\int_a^b y^{-1}e^{-t/y}dH(y)=e^{-t}\sum_{n=0}^{\infty}q_nL_n(t)\ , \tag{7.1}$$

where H is a cdf with support in $[a,b]$ with $0\le a<b<2$. If

$$G^c(t)=\int_a^b e^{-tx}dH(x),\quad t\ge 0\ , \tag{7.2}$$

Then $G^c(t)$ has the Erlang-series representation

$$G^c(t)=\sum_{k=1}^{\infty}q_{k-1}e(k;t),\quad t\ge 0\ , \tag{7.3}$$

and $G^c(t)$ has a pdf $g(t)$ with

$$g(t)=\sum_{k=1}^{\infty}(q_{k-1}-q_k)e(k;t)\ . \tag{7.4}$$

To prove Theorem 7.1, we use the following lemma.

Lemma 7.1. *If (7.2) and the first relation (7.1) hold, where $0\le a<b\le\infty$, then $\hat{G}^c(s)=s^{-1}\hat{f}(s^{-1})$.*

Proof. Note that

$$\hat{f}(s)=\int_a^b(1+sy)^{-1}dH(y)$$

and

$$G^c(s)=\int_a^b(x+s)^{-1}dH(y)=\frac{1}{s}\int_a^b(1+xs^{-1})^{-1}dH(x)\ .$$

Proof of Theorem 7.1. From Lemma 7.1,

$$\hat{G}^c(s)=s^{-1}\hat{f}(s^{-1})=\int_0^{\infty}(s^{-1}e^{-x/s})f(x)dx\ ,$$

so that

$$G^c(t)=\int_0^{\infty}\mathcal{L}^{-1}(s^{-1}e^{-x/s})f(x)dx=\int_0^{\infty}J_0(2\sqrt{xt})f(x)dx \tag{7.5}$$

where

$$J_0(2\sqrt{xt})=e^{-t}\sum_{n=0}^{\infty}\frac{t^n}{n!}L_n(t)\ ; \tag{7.6}$$

see [9, 29.3.75 and 22.9.16]. Therefore,

$$G^c(t)=e^{-t}\sum_{k=0}^{\infty}\frac{q_kt^k}{k!}=\sum_{k=1}^{\infty}q_{k-1}e(k;t)$$

for q_k in (4.9).

Remark 7.1. We use the condition $b < 2$ to apply Theorem 5.1 to have the Laguerre-series representation in (7.1). As remarked before Theorem 7.1, we can rescale to have $b < 2$ or $b = 1$ if $b < \infty$. By Theorem 5.2, for the special case in which $b \leq 1$, we have $1 = q_0 \geq q_1 \geq q_n \geq q_{n+1}$ for all n.

Remark 7.2. We need the condition that the pdf $f(t)$ be completely monotone in order for $G^c(t)$ to be a bonafide ccdf in Theorem 7.1. To see this, suppose that

$$f(t) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)t^n}{n!} \quad \text{and} \quad \hat{f}(s) = \sum_{n=0}^{\infty} f^{(n)}(0)s^{-(n+1)} \ . \tag{7.7}$$

Also suppose that $\hat{G}^c(s)$ is analytic at $s = 0$, so that

$$G^c(s) = \sum_{n=0}^{\infty} \frac{m_{n+1}}{(n+1)!}(-s)^n \ . \tag{7.8}$$

The representations (7.7) and (7.8) imply that

$$(-1)^n f^{(n)}(0) = \frac{m_{n+1}(G)}{(n+1)!} > 0 \tag{7.9}$$

so that $f(t)$ must be completely monotone at 0. ∎

Next we apply Theorem 7.1 to develop an Erlang-series representation for the third beta mixture of exponential (B_3ME) pdf

$$v_3(p,q;t) = \int_0^1 xe^{-tx}b(p,q;x)dx \ , \tag{7.10}$$

for $b(p,q;x)$ in (5.5), which is dual to the BME pdf in (5.4); i.e., it is dual in that the exponential means are averaged in (5.4) while the rates are averaged in (7.10), both with respect to the same standard beta pdf. The BME pdf (5.4) has a Laguerre-series representation, but no Erlang-series representation.

Paralleling the connection between the BME pdf and the Tricomi confluent hypergeometric function $U(a,b,z)$ established in [7, Theorem 1.7], we now establish a connection between the B_3ME pdf and the Kummer confluent hypergeometric function $M(a,b,z)$; see [9, Ch. 13]. From the integral representation 13.2.1 there, we obtain the following result.

Theorem 7.2. *For all* $p > 0$ *and* $q > 0$,

$$v_3(p,q;t) = \frac{p}{p+q}M(p+1,p+q+1,-t), \quad t \geq 0 \ . \tag{7.11}$$

Proof. Starting from the definition (7.10), we obtain

$$v_3(p,q;t) = \int_0^1 xe^{-tx}b(p,q;x)dx = \frac{p}{p+q}e^{-t}M(q,p+q+1,t) = \frac{p}{p+q}M(p+1,p+q+1,-t) \ .$$

Remark 7.3. The B_3ME pdf in the form (7.11) was introduced by Mathai and Saxena [18]; [14, (67) on p. 32].

We can apply Theorem 7.2 to describe the asymptotic form of the B_3ME tail probabilities. We see that a B_3ME pdf has a long tail.

Corollary. *For each $p > 0$ and $q > 0$, $v_3(p,q;t) \sim \Gamma(p+q)p/\Gamma(q)t^{p+1}$ as $t \to \infty$.*

As before, it is significant that we can evaluate the coefficients analytically. The transform series also tend to converge quite rapidly, often much faster than the exponential series.

Theorem 7.3. *For each $p > 0$ and $q > 0$, $v_3(p,q;t)$ has an Erlang-series representation, in particular,*

$$v_3(p,q;t) = p\sum_{n=1}^{\infty} \frac{(q)_{n-1}}{(p+q)_n} e(n;t) \tag{7.12}$$

and

$$\hat{v}_3(p,q;s) = p\sum_{n=1}^{\infty} \frac{(q)_{n-1}}{(p+q)_n}(1+s)^{-n} \ . \tag{7.13}$$

Proof. Make the change of variables $x = 1-y$ in (7.10) to obtain

$$v_3(p,q;t) = \frac{\Gamma(p+q)}{\Gamma(p)\Gamma(q)} e^{-t} \int_0^1 e^{ty} y^{q-1}(1-y)^p dy \ . \tag{7.14}$$

Now expand e^{ty} in a power series and integrate term by term. Alternatively, apply (7.4) in Theorem 7.1 with (5.6). We get the Erlang-series representation with

$$q_{n-1} - q_n = \frac{(q)_{n-1}}{(p+q)_{n-1}} - \frac{(q)_n}{(p+q)_n} = \frac{p(q)_{n-1}}{(p+q)_n} \ . \quad \blacksquare \tag{7.15}$$

Let $v_{3e}(p,q;t) \equiv m(p,q)^{-1}V_3^c(p,q;t)$ denote the stationary-excess pdf associated with $v_3(p,q;t)$. Just as for the BME pdf, we can integrate to see that the B_3ME stationary-excess pdf is again a B_3ME pdf with different parameters.

Theorem 7.4. *For each $p > 1$ and $q > 0$,*

$$v_{3e}(p,q;t) = v_3(p-1,q;t), \quad t \geq 0 \ . \tag{7.16}$$

We now observe that the special case of a B_3ME pdf with $q = 1$ corresponds to a Pareto mixture of exponentials (PME); the PME is a minor modification of PME considered in [2]; a scale factor there made the mean 1. For each $r > 0$, let the Pareto ccdf be $F^c(r;y) = y^{-r}$, $y > 1$. The associated Pareto pdf is $f(r;y) = ry^{-(r+1)}$, $y > 1$. The n^{th} moment is $m_n(F) = r/(r-n)$ for $n < r$.

Theorem 7.5. *For $q = 1$, the B_3ME pdf is the PME pdf with $p = r$, i.e.,*

$$v_3(p,1;t) = \frac{p\gamma(p+1;t)}{t^{p+1}} = \int_1^{\infty} y^{-1}e^{-t/y} f(p;y)dy \ , \quad t \geq 0 \ . \tag{7.17}$$

where $\gamma(p,t) \equiv \int_0^t \mu^{p-1}e^{-\mu}d\mu$ is the incomplete gamma function, and its moments are $m_n(p,1) = p(n!)/(p-n)$.

Proof. To prove the first relation in (7.17), compare the power series for $\gamma(p+1;t)$ with

$$v_3(p,1;t) = p\sum_{n=1}^{\infty} \frac{(n-1)!}{(p+1)_n} e(n;t) = pe^{-t}\sum_{n=1}^{\infty} \frac{t^{n-1}}{(p+1)_n} \ ; \tag{7.18}$$

see [9, 6.5.4 and 6.5.29]. For the second relation, apply [9, 6.5.2] to get

$$v_3(p,1;t) = \frac{p}{t^{p+1}} \int_0^t \mu^p e^{-\mu} d\mu = p\int_0^1 x^p e^{-tx} dx = p\int_1^{\infty} e^{-t/y} y^{-(r+2)} dy \ . \quad \blacksquare \tag{7.19}$$

We now give the full asymptotic form as $t \to \infty$. It is significant that the second term is exponentially damped.

Corollary. *For all $p > 0$,*

$$v_3(p,1;t) \sim \frac{p\Gamma(p+1)}{t^{p+1}} - \frac{pe^{-t}}{t}\left(1 + \frac{p}{t} + \frac{p(p-1)}{t^2} + \cdots\right) \quad \text{as} \quad t \to \infty \ . \tag{7.20}$$

Proof. Apply [9, 6.5.3 and 6.5.32].

We now relate the ccdf to the pdf.

Theorem 7.6. *For each $p > 1$,*

$$V_3^c(p,1;t) = \frac{p}{p-1} v_3(p-1,1;t) = \frac{t}{p} v_3(p,1;t) + e^{-t}, \quad t \geq 0 \ . \tag{7.21}$$

Proof. Start with the second relation in (7.17). Then use the first relation in (7.17) with [9, 6.5.22]. ■

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