

OPTIMAL SEARCH FOR A MOVING TARGET WITH NO TIME INFORMATION MAXIMIZING THE EXPECTED REWARD

Ryusuke Hohzaki Koji Iida Masayoshi Teramoto
National Defense Academy

(Received March 12, 1998; Revised June 2, 1998)

Abstract This paper investigates a search problem for a moving target on a network in which any time information of the target position is not available to a searcher. The searcher has to distribute the limited amount of search efforts on a search space to detect the target, knowing only route information of target paths but not time information about when the target passes there. On detection of the target, the searcher gains some value but expends search cost. There have been few papers which mathematically deal with such a search model without any time information of the target position so far. We formulate the search-efforts-optimizing problem under the expected reward criterion as a convex programming problem and obtain necessary and sufficient conditions for optimal solutions. Using the conditions, a new algorithm is proposed to give an optimal solution. It is shown that the algorithm has the high efficiency for computational time and the robustness for the size of problems comparing with some well-known methods for non-linear programming problems: the gradient projection method and the multiplier method, by numerical examinations. We also elucidate some properties of the optimal solution by the sensitivity analysis of system parameters.

1. Introduction

This paper investigates a search problem for a moving target, in which time information about the target position is not available to a searcher. Many papers have been published for the moving-target search problem. In the search problem, there are usually two participants, the searcher and the target. Iida[3] and Brown[1] proposed a method to obtain a searcher's optimal strategy of maximizing the detection probability given the information of the target's movement. Washburn[9] devised a useful numerical method by generalizing their idea, which is called the FAB(Forward and Backward) Algorithm. The mathematical organization and more generalization of these results mainly owe to Stone[7] and Stromquist[8]. In these earlier works, the detection probability of the target is often adopted as criteria of problems. Iida and Hohzaki[4] dealt with the problem under expected reward criterion, in which some gains occurred by the detection and the search cost were considered.

Two-sided problems for the strategies of the target as well as the searcher have been studied. In many search models, it is assumed that the searcher distributes the limited amount of search efforts on the search space and the target selects his path from some options at the beginning of the search. Iida, Hohzaki and Furui[5] formulated a two-sided search problem as a linear programming problem and found an optimal distribution of search efforts and optimal probabilities of target's selecting paths.

In the search models, several types of the target motions are considered: the Markov motion, the type of target's selecting a path from some options, the so-called conditionally deterministic motion, the diffusive motion and so on. Whichever type the target motion has, it is assumed in almost all studies that information of the target paths contains time information about where the target is at each time. However, there might be many other

cases where time information is not available to the searcher, for example, the guerrilla warfare or the search for smugglers moving in secret. In such models, the searcher could anticipate routes of the target with uncertainty but could not have any information about when the target passes them. Therefore, the searcher has to ambush the target continuously at some points on the route of the target. As one of such models, Washburn and Wood[10] study a two-person zero-sum game between the searcher and the target on a network. However, the payoff function of the game is assumed to be a linear function and their model is very simple. Shiihasi[6] deals with a one-sided problem for the searcher's strategy in the case of no time information. However, he numerically solves the problem under the criterion of the detection probability just by applying the well-known general-purpose method, that is, the gradient projection method. For the no-time-information version of the search problem for a moving target, there are few papers published so far except the above two works.

From another point of view, this problem can be regarded as a kind of resource allocation problems or an optimizing problem under a limited amount of resources[2], which will be known below in the formulation of the problem. In the field, many fruitful outcomes have been obtained, even polynomial-time algorithms for many problems. It mainly depends on a convenient property of the objective function, that is, the separability for variables. On the other hand, the problem dealt with in this paper has more complicated objective function which is never separable. Therefore, it is expected that an exact algorithm for an optimal solution is inevitable to be more complicated.

Purposes of this paper are to theoretically clarify conditions of an optimal solution of the one-sided search problem under the expected reward criterion which is more general than the detection probability or the expected cost until detection of the target and to propose a new method of solving the problem in smaller computational time than some general-purpose methods. In the next section, some assumptions of the search model are described and the problem is formulated as a convex programming problem. In Section 3, the necessary and sufficient conditions of an optimal solution are obtained and a new algorithm of giving the optimal solutions is proposed which is accompanied by the statement of validity of the algorithm. In Section 4, some numerical examples are examined in order to clarify some characteristics of the optimal solution and compare the proposed method with other general-purpose methods in terms of computational time.

2. Modeling of Search Problem and Formulation

Here, assumptions of the search problem are described and the problem is mathematically formulated as an allocation problem of searching efforts on a network.

- (1) A search space is represented by a network $G(V, A)$ with a set of nodes V and a set of arcs A . The total number of arcs is $n = |A|$ and all arcs are numbered by integers $1, 2, \dots, n$.
- (2) A target has a set of paths Ω . He selects one from Ω and goes along it during the search. Every path is assumed not to contain any loop. Path $l \in \Omega$ consists of m_l arcs, $l(1), l(2), \dots, l(m_l)$. A searcher knows that the target goes along l in this order of the arcs but does not know when the target passes each arc. Then, path l is denoted by $l = \{l(1), \dots, l(m_l)\}$. The probability that the target selects path l is anticipated to be $\pi(l) > 0$ by the searcher in advance of the search. The total of these probabilities is one, that is, $\sum_{l \in \Omega} \pi(l) = 1$.
- (3) The total amount of search efforts $M > 0$ are available to the searcher and are divisible arbitrarily. The searcher distributes the search efforts M on arcs to detect the target, but does not need to exhaust them. A search plan of distributing search efforts $\varphi_k \geq 0$

on arc k , $k = 1, \dots, n$ is denoted by $\varphi = \{\varphi_1, \dots, \varphi_n\}$.

- (4) When the searcher allocates search efforts φ_k on arc k and the target passes there, the searcher detects the target with probability $p_k = 1 - \exp(-\alpha_k \varphi_k)$ where $\alpha_k \geq 0$. By the detection of the target on the arc k , the searcher gains value $V_k > 0$ but expends search cost $C_k \geq 0$ per search effort. The value V_k is assumed to be non-increasing for every target path, that is, $V_{l(i)} \geq V_{l(i+1)}$, $i = 1, \dots, m_l - 1$, $l \in \Omega$, which means that the earlier detection of the target is desirable for the searcher.
- (5) The searcher wants to maximize the expected reward during the search, which is defined as the expected value gained by the target detection minus the expected search cost.

If there is a certain arc k which any target path never run through or has parameter $\alpha_k = 0$, the searcher should not waste any search efforts by distributing efforts there. By this reason, without loss of generality, we may additionally assume that every arc has at least one of target paths running on it and its detectability parameter is $\alpha_k > 0$. From the assumptions, we can formulate an objective function, the expected reward of the search, as follows. First, the probability $P(l, i)$ that the target is detected somewhere on arcs $l(1), l(2), \dots, l(i)$ provided that the target takes path $l \in \Omega$ is given by

$$P(l, i) = 1 - \exp\left(-\sum_{j=1}^i \alpha_{l(j)} \varphi_{l(j)}\right). \tag{1}$$

Therefore, the detection probability on the arc $l(i)$ is calculated by $P(l, i) - P(l, i - 1)$ if we define $P(l, 0) = 0$, $l \in \Omega$ for convenience. The detection brings the searcher value $V_{l(i)}$. The search cost expended by plan φ is estimated by $\sum_{k=1}^n C_k \varphi_k$. Now we obtain the expected reward $R(\varphi)$ which is the objective function of this problem as follows.

$$\begin{aligned} R(\varphi) &= \sum_{l \in \Omega} \pi(l) \sum_{i=1}^{m_l} V_{l(i)} (P(l, i) - P(l, i - 1)) - \sum_{k=1}^n C_k \varphi_k \\ &= \sum_{l \in \Omega} \pi(l) \left\{ V_{l(m_l)} P(l, m_l) + \sum_{i=1}^{m_l-1} (V_{l(i)} - V_{l(i+1)}) P(l, i) \right\} - \sum_{k=1}^n C_k \varphi_k. \end{aligned} \tag{2}$$

As known from Eqs.(1) and (2), the objective function is strictly concave for φ and the problem is formulated as the following one of maximizing a concave function over a polytope.

$$(P0) \quad \max_{\varphi} R(\varphi)$$

s.t.

$$\sum_{k=1}^n \varphi_k \leq M, \tag{3}$$

$$\varphi_k \geq 0, \quad k = 1, \dots, n. \tag{4}$$

In a special case of $V_k = 1$, $C_k = 0$, we can generate a problem with the criterion of the detection probability. The problem (P0) is one of so-called convex programming problems where a convex objective function is to be minimized over a convex region.

3. Necessary and Sufficient Conditions of Optimal Solution

Here we derive the necessary and sufficient conditions of an optimal distribution of search efforts. The conditions are used to construct an algorithm for an optimal solution. Let $\Phi_M = \{\varphi \in R^n | \sum_{k=1}^n \varphi_k \leq M, \varphi_k \geq 0\}$ be a set of feasible solutions, Ω_k be a set of target paths passing through arc k and δ_{ij} be the Kronecker's delta which means $\delta_{ij} = 1$ if $i = j$ and 0 if $i \neq j$. Because the objective function of (P0) is strictly concave and the feasible region Φ_M is a bounded closed set, there exists a unique optimal solution for the

problem (P0). We have the following theorem of the necessary and sufficient conditions of the optimal solution.

Theorem 1 *The distribution of search efforts $\varphi = \{\varphi_1, \dots, \varphi_n\} \in \Phi_M$ is optimal if and only if there is a non-negative real number μ satisfying the following conditions,*

$$\text{If } \varphi_k > 0, A_k(\varphi) \exp(-\alpha_k \varphi_k) - C_k = \mu, \quad (5)$$

$$\text{If } \varphi_k = 0, A_k(\varphi) - C_k \leq \mu \quad (6)$$

for $k = 1, \dots, n$ and

$$\text{If } \mu > 0, \sum_{k=1}^n \varphi_k = M, \quad (7)$$

where function $A_k(\varphi)$ does not include variable φ_k and is defined by

$$\begin{aligned} A_k(\varphi) = & \alpha_k \sum_{l \in \Omega_k} \pi(l) \left\{ V_{l(m_l)} \sum_{\xi=1}^{m_l} \delta_{kl(\xi)} \exp \left(- \sum_{j=1, j \neq \xi}^{m_l} \alpha_{l(j)} \varphi_{l(j)} \right) \right. \\ & \left. + \sum_{i=1}^{m_l-1} (V_{l(i)} - V_{l(i+1)}) \sum_{\xi=1}^i \delta_{kl(\xi)} \exp \left(- \sum_{j=1, j \neq \xi}^i \alpha_{l(j)} \varphi_{l(j)} \right) \right\}. \end{aligned} \quad (8)$$

Proof. Let us define a Lagrangean function $L(\varphi; \mu, \lambda)$ by introducing multipliers μ and $\lambda_k, k = 1, \dots, n$.

$$L(\varphi; \mu, \lambda) = R(\varphi) + \mu \left(M - \sum_{k=1}^n \varphi_k \right) + \sum_{k=1}^n \lambda_k \varphi_k. \quad (9)$$

The Kuhn-Tucker conditions for the problem give us the necessary and sufficient conditions of the optimal solution.

$$\frac{\partial L}{\partial \varphi_k} = \frac{\partial R}{\partial \varphi_k} - \mu + \lambda_k = 0, \quad k = 1, \dots, n, \quad (10)$$

$$\sum_{k=1}^n \varphi_k \leq M, \quad (11)$$

$$\mu \left(M - \sum_{k=1}^n \varphi_k \right) = 0, \quad (12)$$

$$\varphi_k \geq 0, \quad k = 1, \dots, n, \quad (13)$$

$$\lambda_k \varphi_k = 0, \quad k = 1, \dots, n, \quad (14)$$

$$\mu \geq 0, \quad (15)$$

$$\lambda_k \geq 0, \quad k = 1, \dots, n. \quad (16)$$

Noting $\partial P(l, i) / \partial \varphi_k = \alpha_k \sum_{\xi=1}^i \delta_{kl(\xi)} \exp \left(- \sum_{j=1}^i \alpha_{l(j)} \varphi_{l(j)} \right)$, $\partial R / \partial \varphi_k$ in Eq.(10) can be written down by Eqs.(1) and (2) as follows.

$$\frac{\partial R}{\partial \varphi_k} = A_k(\varphi) \exp(-\alpha_k \varphi_k) - C_k. \quad (17)$$

From Eq.(14), if $\varphi_k > 0$, $\lambda_k = 0$. It means that Eq.(5) holds from Eq.(10). If $\varphi_k = 0$, we have $A_k(x) - C_k = \mu - \lambda_k \leq \mu$, that is, Eq.(6). Equation (7) can be derived from the complementary slackness condition Eq.(12).

Reversibly, we can prove that the Kuhn-Tucker conditions (10)~(16) are derived from the conditions of the theorem. Using definition of $\lambda_k = \mu - \partial R / \partial \varphi_k$, Eqs.(10) and (16) hold from Eqs.(5) and (6). Equation (7) implies Eq.(12). If $\varphi_k > 0$, λ_k becomes zero just from the definition of λ_k and Eq.(5). Therefore, we have Eq.(14). Equations (11), (13) and (15) are self-explanatory from assumptions of the theorem. **Q.E.D.**

The optimal Lagrangean multiplier μ has a close relationship with the upper limit M of the total amount of search efforts. For emphasizing the value M , we use new symbols (P_M)

and R_M which denote the problem (P0) with M and the optimal function value, respectively.

$$(P_M) \quad \max_{\varphi} \{R(\varphi) \mid \varphi_k \geq 0, k = 1, \dots, n, \sum_k \varphi_k \leq M\},$$

$$R_M = \max_{\varphi \in \Phi_M} R(\varphi).$$

Concerning with the relation between μ and M , we have the following lemma.

Lemma 1 *Between a limit M and an optimal multiplier μ , the following relations hold.*

(i) *Let us define $\bar{\mu}$ by $\bar{\mu} = \max_k \{\bar{A}_k - C_k\}$ where*

$$\bar{A}_k = \alpha_k \sum_{l \in \Omega_k} \pi(l) \left\{ V_{l(m_k)} + \sum_{i=1}^{m_k-1} (V_{l(i)} - V_{l(i+1)}) \right\}.$$

For $\mu \geq \bar{\mu}$, an optimal solution becomes $\varphi = \{0, \dots, 0\}$ which indicates the corresponding limit $M = 0$.

(ii) *Let $\mu_1, \mu_2 > 0$ be optimal multipliers for two problems $(P_{M_1}), (P_{M_2})$ with limits $M_1, M_2 > 0$, respectively. If $M_1 > M_2$, then $\mu_1 < \mu_2$.*

(iii) *Let φ^* and $\mu^* = 0$ be an optimal solution and an optimal multiplier for (P_M) , respectively, and suppose that C is the total amount of φ^* , that is, $C = \sum_k \varphi_k^* \leq M$. For any M' of $C \leq M'$, an optimal solution for the problem $(P_{M'})$ remains φ^* and an optimal multiplier stays being $\mu^* = 0$. In other words, C is a minimum among the limits of the total amount corresponding to $\mu^* = 0$. In this sense, we call C the marginal amount.*

Proof: (i) From Eq.(8), $A_k(\varphi) \leq \bar{A}_k$ holds. Therefore, there is no $\varphi_k > 0$ that satisfies Eq.(5) of Theorem 1 for $\mu \geq \bar{\mu}$. In the result, we conclude that $\varphi_k = 0, k = 1, \dots, n$.

(ii) Let φ and ψ be optimal solutions for problems (P_{M_1}) and (P_{M_2}) , respectively. Since μ_1 and μ_2 are positive, $\sum_k \varphi_k = M_1$ and $\sum_k \psi_k = M_2$ hold. This means $\varphi \neq \psi$. From the strictly concavity of $R(\cdot)$, we have

$$\begin{aligned} R(\psi) &< R(\varphi) + \nabla R(\varphi) \cdot (\psi - \varphi) = R(\varphi) + \sum_{k=1}^n \frac{\partial R}{\partial \varphi_k} (\psi_k - \varphi_k) \\ &= R(\varphi) + \sum_{\{k \mid \varphi_k > 0\}} \frac{\partial R}{\partial \varphi_k} (\psi_k - \varphi_k) + \sum_{\{k \mid \varphi_k = 0\}} \frac{\partial R}{\partial \varphi_k} \psi_k. \end{aligned}$$

From the optimality of φ , $\partial R / \partial \varphi_k = \mu_1$ for $\varphi_k > 0$ and $\partial R / \partial \varphi_k \leq \mu_1$ for $\varphi_k = 0$. Using these facts, the above expression can be transformed as follows.

$$\begin{aligned} &\leq R(\varphi) + \sum_{\{k \mid \varphi_k > 0\}} \mu_1 (\psi_k - \varphi_k) + \sum_{\{k \mid \varphi_k = 0\}} \mu_1 \psi_k \\ &= R(\varphi) + \mu_1 \sum_{k=1}^n (\psi_k - \varphi_k) = R(\varphi) + \mu_1 (M_2 - M_1). \end{aligned}$$

In the result, we have $\mu_1 (M_1 - M_2) < R(\varphi) - R(\psi)$. In the same manner, we obtain another inequality $R(\varphi) - R(\psi) < \mu_2 (M_1 - M_2)$. It proves $\mu_1 < \mu_2$.

(iii) Because of $\Phi_C \subseteq \Phi_{M'}$ for $C \leq M'$, $R_C \leq R_{M'}$ follows. Let ψ be an optimal solution of problem $(P_{M'})$. If $\sum_k \psi_k \leq C$, $R_{M'} \leq R_C$ holds which indicates $R_{M'} = R_C$ and then the proof is completed. So, now let us suppose $\sum_k \psi_k > C = \sum_k \varphi_k^*$. Then, from $\psi \neq \varphi^*$ and the strictly concavity of $R(\varphi)$, we have

$$\begin{aligned} R_{M'} &= R(\psi) \\ &< R(\varphi^*) + \nabla R(\varphi^*) \cdot (\psi - \varphi^*) = R(\varphi^*) + \sum_{\{k \mid \varphi_k^* > 0\}} \frac{\partial R}{\partial \varphi_k^*} (\psi_k - \varphi_k^*) + \sum_{\{k \mid \varphi_k^* = 0\}} \frac{\partial R}{\partial \varphi_k^*} \psi_k. \end{aligned}$$

From the optimality of φ^* , it follows that $\partial R / \partial \varphi_k^* = 0$ if $\varphi_k^* > 0$ and $\partial R / \partial \varphi_k^* \leq 0$ if $\varphi_k^* = 0$. Therefore, we can transform the above expression as follows.

$$= R(\varphi^*) + \sum_{\{k|\varphi_k^*=0\}} \frac{\partial R}{\partial \varphi_k^*} \psi_k \leq R(\varphi^*) = R_C .$$

This contradicts the assumption of $R_C \leq R_{M'}$. That is, we have $R_{M'} = R_C$ and have proved $\psi = \varphi^*$ from the uniqueness of the optimal solution. The corresponding optimal multiplier is $\mu^* = 0$, of course. **Q.E.D.**

From Lemma 1, we see that there is a monotonous relation between the limit M and the optimal multiplier μ . That is, as μ becomes larger from 0 to $\bar{\mu}$, M monotonously decreases from the marginal amount C to 0. Even if M grows beyond C , the optimal solution stays unchanged and the optimal multiplier μ remains being zero too. On the other hand, any $\mu \geq \bar{\mu}$ corresponds to the limit $M = 0$.

From now, let us discuss a concrete numerical method of calculating optimal solutions. The conditions of Theorem 1 can be transformed into the following formula.

$$\varphi_k = \frac{1}{\alpha_k} \left[\log \left(\frac{A_k(\varphi)}{\mu + C_k} \right) \right]^+ , \quad (18)$$

where symbol $[]^+$ denotes $[x]^+ = \max\{0, x\}$. The validity of this calculation is verified from the following facts: $\varphi_k = 0$ if $A_k(\varphi) - C_k \leq \mu$ and $A_k(\varphi) \exp(-\alpha_k \varphi_k) - C_k = \mu$ if $A_k(\varphi) - C_k > \mu$, which come from Eqs. (5) and (6). By the representation (18), we can define the total amount of used search efforts as a function of the Lagrangean multiplier μ .

$$\rho(\mu; \varphi) = \sum_{k=1}^n \frac{1}{\alpha_k} \left[\log \left(\frac{A_k(\varphi)}{\mu + C_k} \right) \right]^+ . \quad (19)$$

The function $\rho(\mu; \varphi)$ is monotone decreasing for μ . This fact teaches us that there exists a unique optimal multiplier μ^* for the optimal solution φ^* . If $\rho(0; \varphi^*)$ is less than or equal to M , the optimal multiplier is given by $\mu^* = 0$. If $\rho(0; \varphi^*)$ is greater than M , the optimal multiplier is a $\mu^* \in (0, \max_k \{A_k(\varphi^*) - C_k\})$ satisfying $\rho(\mu^*; \varphi^*) = M$.

4. Method for Optimal Solution

Here we propose an algorithm of giving the optimal solution by using Theorem 1. The proposed algorithm is a numerical one in which a sequence of feasible solutions is generated on the process and it converges to an optimal solution step by step.

An outline of the algorithm is as follows. Suppose that we have a feasible solution φ^i now. We find a tentative multiplier μ^i corresponding to φ^i by calculating $\rho(\mu; \varphi^i)$ and generate another feasible solution ψ^i by Eq.(18). Furthermore, we produce the next feasible solution φ^{i+1} from ψ^i so as to increase the objective function value. By the iteration of the procedure, a sequence of feasible solutions $\{\varphi^i\}$ converges to an optimal one.

(Step1) Set a positive infinitesimal $\varepsilon \approx 0$ and $i = 1$. And let φ^i be $\{0\}$ as an initial feasible solution.

(Step2) If $\rho(0; \varphi^i) \leq M$, set a tentative multiplier by $\mu^i = 0$. Otherwise, find μ^i satisfying $\rho(\mu^i; \varphi^i) = M$. Using μ^i , generate a new feasible solution ψ^i by

$$\psi_k^i = \frac{1}{\alpha_k} \left[\log \left(\frac{A_k(\varphi^i)}{\mu^i + C_k} \right) \right]^+ , \quad k = 1, \dots, n . \quad (20)$$

(Step3) If $\sum_k |\psi_k^i - \varphi_k^i| < \varepsilon$, terminate. Then, the solution $\psi^i \simeq \varphi^i$ satisfies Theorem 1 and φ^i is the optimal solution.

Otherwise, execute the following line search.

$$R(\varphi^i + \theta^*(\psi^i - \varphi^i)) = \max_{0 \leq \theta \leq 1} R(\varphi^i + \theta(\psi^i - \varphi^i)) ,$$

where $\bar{\theta} = \min\{\theta_1, \theta_2\}$ and θ_1, θ_2 are defined by

$$\theta_1 = \begin{cases} \min\{-\varphi_k^i/(\psi_k^i - \varphi_k^i) \mid \psi_k^i - \varphi_k^i < 0\}, & \text{if } \psi_k^i - \varphi_k^i < 0 \text{ for some } k, \\ +\infty, & \text{otherwise,} \end{cases} \quad (21)$$

$$\theta_2 = \begin{cases} (M - \sum_k \varphi_k^i) / \sum_k (\psi_k^i - \varphi_k^i), & \text{if } \sum_k (\psi_k^i - \varphi_k^i) > 0, \\ +\infty, & \text{otherwise.} \end{cases} \quad (22)$$

Set $\varphi_k^{i+1} = \varphi_k^i + \theta^*(\psi_k^i - \varphi_k^i)$ for $k = 1, \dots, n$.

(Step4) Increase i by one, $i = i + 1$, and go to (Step2).

We name the above algorithm A0. The next theorem states the validity of Algorithm A0.

Theorem 2 *The proposed algorithm terminates and on the termination, an optimal solution of the problem is obtained within a tolerance of total error ε .*

Proof. Theorem is proved by verifying the monotonic increase of the expected reward for a sequence of solutions $\varphi^1, \varphi^2, \dots$ and the feasibility of the solutions. In order to verify $R(\varphi^{i+1}) = R(\varphi^i + \theta^*(\psi^i - \varphi^i)) > R(\varphi^i)$, it is enough to confirm that $\psi^i - \varphi^i$ is an ascent direction, that is, $\nabla R(\varphi^i) \cdot (\psi^i - \varphi^i) > 0$. We partition $k = 1, \dots, n$ into two classes, $I(\psi^i) = \{k \mid \psi_k^i > 0\}$ and $\{k \mid \psi_k^i = 0\}$. From Eq.(17),

$$\begin{aligned} \nabla R(\varphi^i) \cdot (\psi^i - \varphi^i) &= \sum_{k \in I(\psi^i)} (A_k(\varphi^i) e^{-\alpha_k \varphi_k^i} - C_k) (\psi_k^i - \varphi_k^i) \\ &\quad + \sum_{k \notin I(\psi^i)} (A_k(\varphi^i) e^{-\alpha_k \psi_k^i} - C_k) (\psi_k^i - \varphi_k^i). \end{aligned} \quad (23)$$

For $k \in I(\psi^i)$ and $k \notin I(\psi^i)$, we have $A_k(\varphi^i) e^{-\alpha_k \psi_k^i} - C_k = \mu^i$ and $A_k(\varphi^i) e^{-\alpha_k \psi_k^i} - C_k \leq \mu^i$, respectively. Therefore, the first term of Eq.(23) is transformed as follows.

$$\begin{aligned} \text{First term} &= \sum_{k \in I(\psi^i)} (A_k(\varphi^i) e^{-\alpha_k \psi_k^i} e^{+\alpha_k (\psi_k^i - \varphi_k^i)} - C_k) (\psi_k^i - \varphi_k^i) \\ &= \sum_{k \in I(\psi^i)} \{(\mu^i + C_k) e^{\alpha_k (\psi_k^i - \varphi_k^i)} - C_k\} (\psi_k^i - \varphi_k^i). \end{aligned}$$

In the same manner, the second term is transformed as follows.

$$\text{Second term} \geq \sum_{k \notin I(\psi^i)} \{(\mu^i + C_k) e^{\alpha_k (-\varphi_k^i)} - C_k\} (-\varphi_k^i).$$

In the result, we have

$$\begin{aligned} \nabla R(\varphi^i) \cdot (\psi^i - \varphi^i) &\geq \sum_{k=1}^n \{(\mu^i + C_k) e^{\alpha_k (\psi_k^i - \varphi_k^i)} - C_k\} (\psi_k^i - \varphi_k^i) \\ &= \sum_{k=1}^n \{C_k (e^{\alpha_k (\psi_k^i - \varphi_k^i)} - 1) + \mu^i (e^{\alpha_k (\psi_k^i - \varphi_k^i)} - 1) + \mu^i\} (\psi_k^i - \varphi_k^i). \end{aligned}$$

Noting that function $f(x) = (e^{\alpha x} - 1)x$ has a minimum value zero only at $x = 0$ if $\alpha > 0$, we have

$$\sum_{k=1}^n C_k (e^{\alpha_k (\psi_k^i - \varphi_k^i)} - 1) (\psi_k^i - \varphi_k^i) \geq 0, \quad \sum_{k=1}^n \mu^i (e^{\alpha_k (\psi_k^i - \varphi_k^i)} - 1) (\psi_k^i - \varphi_k^i) \geq 0.$$

In the case of $\mu^i = 0$, $\sum_k \mu^i (\psi_k^i - \varphi_k^i) = 0$, of course. In the case of $\mu^i > 0$, $\sum_k \psi_k^i = M$ from Eq.(7). Then we have

$$\sum_k \mu^i (\psi_k^i - \varphi_k^i) = \mu^i (M - \sum_k \varphi_k^i) \geq 0.$$

From the above discussion, it follows that

$$\nabla R(\varphi^i) \cdot (\psi^i - \varphi^i) \geq 0,$$

where equality holds only if $\psi^i = \varphi^i$. While the algorithm runs, ψ^i is not equal to φ^i . It means $\nabla R(\varphi^i) \cdot (\psi^i - \varphi^i) > 0$. Therefore, we conclude that $\psi^i - \varphi^i$ is the ascent direction and $R(\varphi^i) < R(\varphi^{i+1})$ holds.

From here, let us prove that if φ^i is feasible, φ^{i+1} becomes feasible too. First, we confirm that $\varphi_k^i + \theta(\psi_k^i - \varphi_k^i) \geq 0$ is valid for $0 < \theta \leq \theta_1$. If $\psi_k^i - \varphi_k^i \geq 0$, $\varphi_k^i + \theta(\psi_k^i - \varphi_k^i) \geq 0$ holds. If $\psi_k^i - \varphi_k^i < 0$, from Eq.(21),

$$\varphi_k^i + \theta(\psi_k^i - \varphi_k^i) \geq \varphi_k^i + \theta_1(\psi_k^i - \varphi_k^i) \geq \varphi_k^i - \frac{\varphi_k^i}{\psi_k^i - \varphi_k^i}(\psi_k^i - \varphi_k^i) = 0 .$$

For $0 < \theta \leq \theta_2$, we have

$$\begin{aligned} \sum_k \{ \varphi_k^i + \theta(\psi_k^i - \varphi_k^i) \} &= \sum_k \varphi_k^i + \theta \left(\sum_k \psi_k^i - \sum_k \varphi_k^i \right) \\ &\leq \begin{cases} \sum_k \varphi_k^i + \theta_2 (\sum_k \psi_k^i - \sum_k \varphi_k^i) , & \text{if } \sum_k \psi_k^i - \sum_k \varphi_k^i > 0 \\ \sum_k \varphi_k^i , & \text{if } \sum_k \psi_k^i - \sum_k \varphi_k^i \leq 0 \end{cases} . \end{aligned}$$

From Eq.(22), the above summation is less than or equal to M . From definitions (21) and (22), it is self-evident that $\theta_1 \geq 1$ and $\theta_2 \geq 1$ hold and at least one of θ_1 or θ_2 is finite. Consequently, we obtain the fact that φ^{i+1} becomes feasible, that is, $\varphi^{i+1} \in \Phi_M$. **Q.E.D.**

5. Numerical Examples

In this section, the sensitivity of system parameters to optimal solutions is investigated to clarify some characteristics of optimal solutions, and performance of the algorithm proposed in the previous section is compared with some well-known methods, the multiplier method and the gradient projection method, from the view point of computational time.

5.1 Sensitivity analysis

(1) Basic case

Suppose a search space is represented by a network illustrated in Figure 1. A target has 5 paths of options starting from a left node and entering in a right node. A searcher has total search efforts $M = 5$ which are to be distributed on 12 arcs. The routes of all paths and the estimated probabilities of target's selecting them are given by the followings.

- Path 1 : $l_1 = \{1, 2, 3\}$, $\pi(l_1) = 1/5$.
- Path 2 : $l_2 = \{1, 7, 8, 3\}$, $\pi(l_2) = 1/5$.
- Path 3 : $l_3 = \{4, 5, 6\}$, $\pi(l_3) = 1/5$.
- Path 4 : $l_4 = \{4, 9, 10, 6\}$, $\pi(l_4) = 1/5$.
- Path 5 : $l_5 = \{11, 12\}$, $\pi(l_5) = 1/5$.

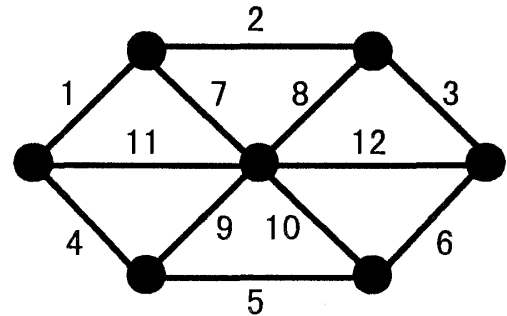


Figure 1: Search network

The parameters C_k and α_k on arc k are assumed to be constant for all arcs, $C_k = 1.0$, $\alpha_k = 0.2$, but V_k s vary as presented in Table 1. We calculate an optimal distribution of search efforts $\{\varphi_k^*\}$ by the proposed method and tabulate them in the same table. The optimal expected reward is 1.03. Total efforts 4.17 are used and are concentrated on arcs 1 and 4. Since two paths l_1 and l_2 run through arc 1 and paths l_3 and l_4 run through arc 4, it is expected that the target goes on arc 1 or 4 with high probabilities. There is another arc 3 which both of paths l_1 and l_2 go through and there is another arc 6 as for paths l_3 and l_4 . However, arcs 1 and 4 have higher values V_k s than arcs 3 and 6, respectively. This is why the search efforts are focused on not arcs 3, 6 but arcs 1, 4. Since the search efforts

$M = 5$ are not exhaustively used and the Lagrangean multiplier becomes $\mu = 0$, φ_1^* and φ_4^* are determined by $A_k(\varphi) \exp(-\alpha_k \varphi_k) - C_k = 0$. On arc 1, more efforts are allocated than arc 4 because of its higher value.

Table 1: Parameters and optimal efforts on arcs

Arc k	V_k	C_k	α_k	Optimal efforts
1	20.0	1.0	0.2	2.35
2	17.0	1.0	0.2	0
3	15.0	1.0	0.2	0
4	18.0	1.0	0.2	1.82
5	17.0	1.0	0.2	0
6	15.0	1.0	0.2	0
7	17.0	1.0	0.2	0
8	16.0	1.0	0.2	0
9	17.0	1.0	0.2	0
10	16.0	1.0	0.2	0
11	19.0	1.0	0.2	0
12	16.0	1.0	0.2	0

(2) Effect of C_k

To examine the effects of cost parameter C_k on optimal solutions, we vary cost C_1 of arc 1 between $[0, 1.6]$ in the basic case and obtain the optimal solutions. The change of optimal efforts and the total amount of used efforts are illustrated in Figure 2-a but non-distributions $\{\varphi_k^* = 0\}$ are omitted. Figure 2-b illustrates the change of maximum expected rewards.

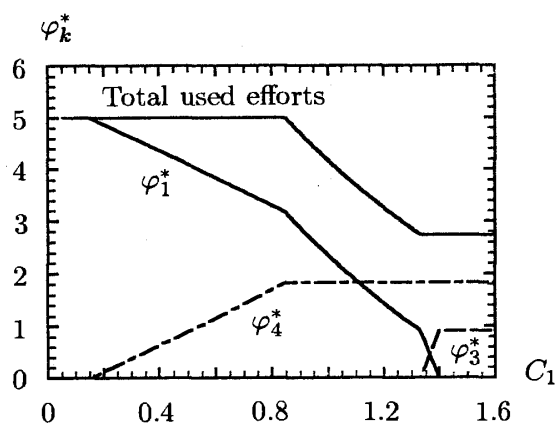


Figure 2-a: Optimal distributions for C_1

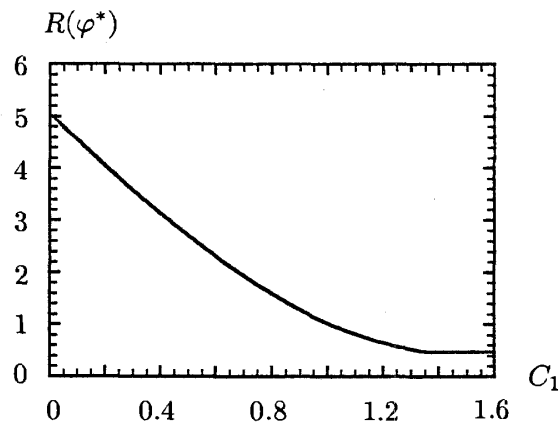


Figure 2-b: Maximum expected rewards for C_1

In the case of $C_1 \leq 0.14$, the whole search efforts are distributed only on arc 1 because of its lower cost. As the cost becomes larger beyond 0.14, search efforts begin to be distributed to arc 4 too. Search effort φ_1^* decreases and φ_4^* increases as the cost grows larger. In the case of $C_1 \leq 0.84$, the optimal distribution uses all of available efforts $M = 5$. In the case of $0.84 < C_1$, only the part of them is used and φ_1^* continues to decrease though φ_4^* stands unchanged. When the cost goes over 1.34, the optimal distribution begins to give arc 3

some efforts while decreasing φ_1^* continuously and stops the allocation of efforts on arc 1 at $C_1 = 1.40$, which means that arc 3 completely takes the place of arc 1 as an efficient arc for the search on paths 1 and 2. For larger cost than $C_1 = 1.40$, the optimal solution stays unchanged. As C_1 becomes larger during $[0, 1.40]$, the optimal expected reward decreases in the uniformly convex form and remains constant for $1.40 \leq C_1 \leq 1.60$ because the optimal solution does not change.

(3) Effect of α_k

To examine the effects of parameter α_k on optimal solutions, we vary detection rate α_1 between $[0.1, 0.5]$ in the basic case. The change of optimal distributions of search efforts is illustrated in Figure 3-a and the change of optimal expected rewards in Figure 3-b.

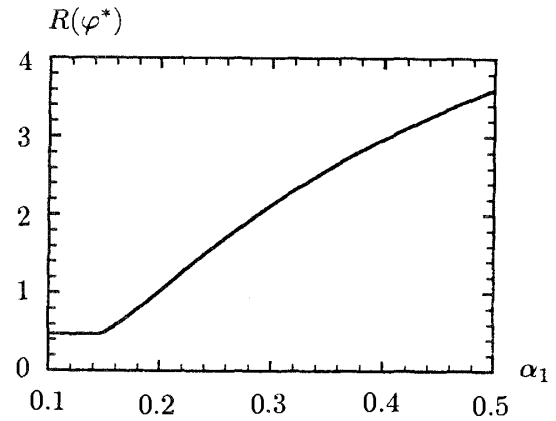
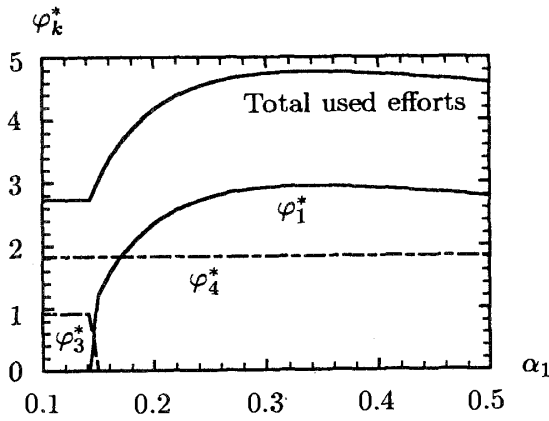


Figure 3-a: Optimal distributions for α_1

Figure 3-b: Maximum expected rewards for α_1

In the case of $\alpha_1 \leq 0.14$, the most efficient arc on paths 1 and 2 is arc 3 and that on paths 3 and 4 is arc 4. When α_1 grows over 0.14, some search efforts are distributed to arc 1, too. At $\alpha_1 = 0.15$, the most efficient arc on paths 1 and 2 becomes arc 1 instead of arc 3. During $0.15 < \alpha_1 \leq 0.5$, the change of φ_1^* on arc 1 has a unimodal form with a maximum value at $\alpha_1 = 0.34$ while φ_4^* on arc 4 stays unchanged. Total search efforts $M = 5$ are not exhausted. Because of that, φ_1^* is determined by Eq.(18) with $\mu = 0$, $\varphi_1^* = \log(8.0\alpha_1)/\alpha_1$ for $0.15 \leq \alpha_1$ in this case, which gives the change of φ_1^* the unimodal form. In the case of large α_k , the searcher would better restrain too much search efforts on enough efficient arc k considering search cost. The optimal expected reward is constant for $0.1 \leq \alpha_1 \leq 0.15$ but increases as α_1 becomes larger during $[0.15, 0.50]$.

(4) Effect of M

To examine the effect of total search efforts M on optimal solutions, we vary M between $[0, 5.0]$ in the basic case. Figure 4 illustrates the change of optimal efforts.

It is reasonable that φ_1^* and φ_4^* increase as M becomes larger, though, for $0 \leq M \leq 0.52$, the total search efforts M is used exclusively on arc 1. However, at $M = 4.16$, the increase of search efforts on arcs 1, 4 and the exhaustively use of the total efforts M stop. In the case of larger M than 4.16, optimal solutions and optimal rewards do not change. Too much available efforts are useless. We see that the marginal amount of search efforts is $C = 4.16$ in this case. More search efforts linearly increase the search cost while they increase the detection probability or the expected captured value in a concave form and at last, too much efforts saturate the detection probability. The optimal expected reward increases in a concave fashion as M becomes larger during $[0, 4.16]$ and stays constant for $4.16 < M \leq 5.0$.

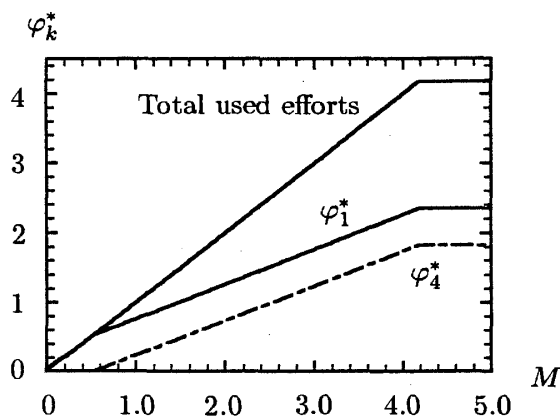


Figure 4-a: Optimal distributions for M

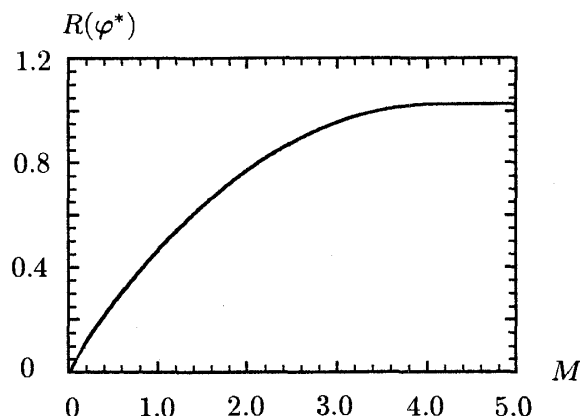


Figure 4-b: Maximum expected rewards for M

5.2 Comparison of computational time

Here, in terms of computational time, we compare the proposed method A0 with two well-known methods for the non-linear programming problem: the gradient projection method and the multiplier method.

The search problems to be examined are generated as follows. In a 2-dimensional Euclidean space, a starting node of the target is located at $s = (0, 0)$ and a terminating node at $e = (1, 1)$. On a square with the diagonal $(0, 0) - (1, 1)$, N nodes are randomly scattered. A node, say node a_1 , is randomly selected from nodes located within a certain distance r from node s in such a way that a_1 is nearer to e than s . These nodes s and a_1 are connected by an arc. In the same manner, the next node is selected within the distance r from node a_1 except already-selected nodes and an arc is made. The procedure is repeated until the terminating node is selected, and a path has been determined. This path generation is repeated $|\Omega|$ times permitting two different paths have the same arc. System parameters α_k and C_k are randomly picked up during $[\underline{\alpha}, \bar{\alpha}]$ and $[\underline{C}, \bar{C}]$, respectively. As for V_k , on the starting arc of path $l \in \Omega$, $V_{l(1)}$ is randomly selected from $[\underline{V}, \bar{V}]$ and successive values are generated by $V_{l(i+1)} = \beta V_{l(i)}$, where $0 < \beta < 1$, so as to satisfy $V_{l(i)} \geq V_{l(i+1)}$ for $i = 1, \dots, m_l - 1$.

Setting $M = 5$, $|\Omega| = 5$, $r = 0.8$, $\underline{\alpha} = 0.2$, $\bar{\alpha} = 0.4$, $\underline{C} = 1.0$, $\bar{C} = 3.0$, $\underline{V} = 19.5$, $\bar{V} = 20.5$ and $\beta = 0.95$, 1000 search problems are generated for each of $N = 30, 50, 100$ and solved by three methods. A unix machine DEC3000 and the programming language C are used for the computation. The total number of arcs generated on a network is an adequate index denoting the size of the problem. In Table 2, the number N , the mean numbers of arcs in the generated networks and the average CPU-times(sec) for three methods are presented. Figures in parentheses denotes ratios to data of $N = 30$. This is why all figures in parentheses are 1.0s for $N = 30$. We abbreviate three methods as the algorithm A0, the GP method and the M method for short.

Table 2: Comparison of CPU-times by three methods

N	# of arcs	Algorithm A0	GP method	M method
30	25.9(1.0)	0.589(1.0)	5.65(1.0)	54.1(1.0)
50	38.8(1.5)	0.806(1.4)	15.5(2.7)	186.5(3.5)
100	51.9(2.0)	1.082(1.8)	28.6(5.1)	448.1(8.3)

From Table 2, we might conclude that the proposed algorithm A0 is the most efficient for computation and the most robust for the size of problems while other methods, the GP method and the M method, show a tendency of sharply increasing CPU-time for the size of problems.

6. Conclusions

This paper investigates a search problem for a moving target on a network where any time information of the target position is not available to a searcher. The searcher has to determine an optimal distribution of the limited amount of search efforts on a search space knowing only the routes of target paths and the estimated probabilities of target's selecting paths. There have been few papers which mathematically deal with such a search model so far. We formulate the problem under the expected reward criterion as a convex programming problem and obtain necessary and sufficient conditions for the optimal distribution. A new algorithm is proposed to give an optimal solution by using the conditions.

In the algorithmic point of view, the search theory has developed many methods so far which the resource allocation problem had originated from. In many cases of the stationary target search problems, the objective functions become separable for variables by which many efficient algorithms, e.g. the polynomial-time algorithms, are devised. For the moving target search problems with time information of target paths, this convenient property of the problem is lost but clever researchers made use of the separability as computational technique in such a way that they could calculate search effort $\varphi(k, t)$ on a search point k at search time t without using efforts $\{\varphi(i, t), i \neq k\}$ on other points. For the case of no time information, even this technique is difficult to be found. We think that this is one of reasons why the proposed algorithm inevitably becomes such an iterative numerical method as seen in nonlinear programming methods. However, we showed that the proposed algorithm had the high efficiency for computational time and the robustness for the size of problems by numerical examination comparing with some well-known methods for non-linear programming problems. We also elucidated some characteristics of the optimal solution by the sensitivity analysis of system parameters.

The search problem has one constraint on the total amount of resources and can be regarded as one of optimizing problems with the limited amount of resources, which suggests that our methodology could be applied to many other optimizing models. In context of the search theory, we proposed a method to solve a one-sided problem for an optimal strategy of the searcher. Two-sided problems on both sides of the searcher and the target are thought to be located on the extension of this problem.

References

- [1] S.S. Brown: Optimal search for a moving target in discrete time and space. *Operations Research*, **28**(1980) 1275–1289.
- [2] T. Ibaraki and N. Katoh: *Resource Allocation Problems: Algorithmic Approaches* (The MIT Press, London, 1988).
- [3] K. Iida: An optimal distribution of searching efforts for a moving target (in Japanese). *Keiei Kagaku*, **16**(1972) 204–215.
- [4] K. Iida and R. Hohzaki: The optimal search plan for a moving target minimizing the expected risk. *Journal of the Operations Research Society of Japan*, **31**(1988) 294–320.
- [5] K. Iida, R. Hohzaki and S. Furui: A search game for a mobile target with the conditionally deterministic motion defined by paths. *Journal of the Operations Research*

- Society of Japan*, **39**(1996) 501–511.
- [6] N. Shiihasi: Optimal distribution of search efforts on a network (in Japanese). Bachelor Thesis, National Defense Academy of Japan (1995).
- [7] L.D. Stone: *Theory of Optimal Search* (Academic Press, New York, 1975).
- [8] W.R. Stromquist and L.D. Stone: Constrained optimization of functionals with search theory applications. *Mathematics of Operations Research*, **16**(1981) 518–529.
- [9] A.R. Washburn: Search for a moving target: The FAB algorithm. *Operations Research*, **31**(1983) 739–751.
- [10] A.R. Washburn and K. Wood: Two-person zero-sum games for network interdiction. *Operations Research*, **43**(1995) 243–251.

Ryusuke Hohzaki
Department of Applied Physics
National Defense Academy
1-10-20 Hashirimizu, Yokosuka,
239-8686, Japan
E-mail: hozaki@cc.nda.ac.jp