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FUZZY APPROXIMATIONS WITH NON-SYMMETRIC FUZZY PARAMETERS IN FUZZY REGRESSION ANALYSIS

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Abstract This paper proposes fuzzy regression analysis with non-symmetric fuzzy coefficients. By assuming non-symmetric triangular fuzzy coefficients and applying the quadratic programming formulation, the center of the obtained fuzzy regression model attains more central tendency compared to the one with symmetric triangular fuzzy coefficients. For a data set composed of crisp inputs-fuzzy outputs, two approximation models called an upper approximation model and a lower approximation model are considered as regression models. Thus, we also propose an integrated quadratic programming problem by which the upper approximation model always includes the lower approximation model at any threshold level under the assumption of the same centers in the two approximation models. Since non-symmetric fuzzy coefficients are assumed, we can obtain models with more reduced spreads as well as with more central tendency, compared to the ones with symmetric triangular fuzzy coefficients. Sensitivities of weight coefficients in the proposed quadratic programming approaches are investigated through real data.

1. Introduction

In fuzzy regression analysis originated by Tanaka et al. [12], to deal with a vague and uncertain phenomenon, a fuzzy structure of the given phenomenon is represented as a fuzzy linear function whose parameters are fuzzy numbers. Therefore, a fuzzy linear function is used as a regression model to describe fuzziness in the given phenomenon. We already developed several regression analyses based on linear programming (LP) and quadratic programming (QP) [7, 8, 13-17]. Some applications can be found in ergonomics [1], economic forecasting [2], and civil engineering [6]. Sakawa and Yano [9, 10] formulated fuzzy regression models for fuzzy input-output data as multiobjective programming problems. Recent developments on fuzzy regression can be found in Inuiguchi et al. [3] and Ishibuchi and Nii [5].

Given the input-output data $(\boldsymbol{x}_j, y_j)$, $j=1,\dots,m$, where $\boldsymbol{x}_j=(1,x_{j1},\dots,x_{jn})'$ is the j-th input vector and y_j is the j-th output and m is a data size, in general, a fuzzy regression model is assumed as

$$Y(\boldsymbol{x})=A_0+A_1x_1+\cdots+A_nx_n=\boldsymbol{A}\boldsymbol{x} \tag{1}$$

where $A=(A_0,\dots,A_n)$ is a fuzzy coefficient vector and $Y(\boldsymbol{x})$ is the corresponding fuzzy output. Equation (1) can be calculated by fuzzy arithmetic [12] which is similar to interval arithmetic. In former fuzzy regression models [12-14], the coefficients A_i, $(i=0,\dots,n)$ are assumed to be symmetric triangular fuzzy numbers consisting of a center a_i and a spread c_i, denoted as $A_i=(a_i,c_i)_T$ where T represents a *triangle*. Thus, the regression model is determined by minimizing sum of spreads of the estimated fuzzy outputs $Y(\boldsymbol{x}_j)$, $j=1,\dots,m$, subject to the constraint conditions such that the h-level set of the estimated output $Y(\boldsymbol{x}_j)$ denoted as $\lfloor Y(\boldsymbol{x}_j)\rfloor_h$ should include the j-th observation y_j for any j where h is given as a threshold. This can be expressed as the following LP problem:

$$\min_{\boldsymbol{a},\boldsymbol{c}} \sum_{j=1}^{m} \boldsymbol{c}^t|\boldsymbol{x}_j|$$
$$\text{subject to } y_j \in \left[Y(\boldsymbol{x}_j)\right]_h, \quad j=1,\ldots,m, \tag{2}$$
$$c_i \geq 0, \quad i=0,\ldots,n,$$

where $\boldsymbol{c}=(c_0,\ldots,c_n)^t$ and $|\boldsymbol{x}_j|=\left(1,|x_{j1}|,\ldots,|x_{jn}|\right)^t$.

In order to explain our motivation of this research, simple numerical data are given in Table 1. Let us assume that a fuzzy linear model is taken as

$$Y(x)=A_0+A_1 x \tag{3}$$

where the coefficients $A_i=(a_i,c_i)_T$, $(i=0,1)$ are assumed to be symmetric triangular fuzzy numbers. Here we assign $h=0$ for simplicity where the h-value is usually given by an expert knowledge. Thus, the following model is obtained using the LP problem (2) with $h=0$:

$$Y(x)=(18.750,1.750)_T+(2.333,0.833)_T\, x \tag{4}$$

which is depicted in Figure 1. As noticed in Figure 1, the dotted center line is not showing a proper central tendency because of the objective function in (2).

Table 1. Numerical data

No.(j)	1	2	3	4	5	6	7	8
x	1	2	3	4	5	6	7	8
y	21	20	30	26	31	37	35	40
No.(j)	9	10	11	12	13	14	15	
x	9	10	11	12	13	14	15	
y	49	45	53	35	52	60	62	

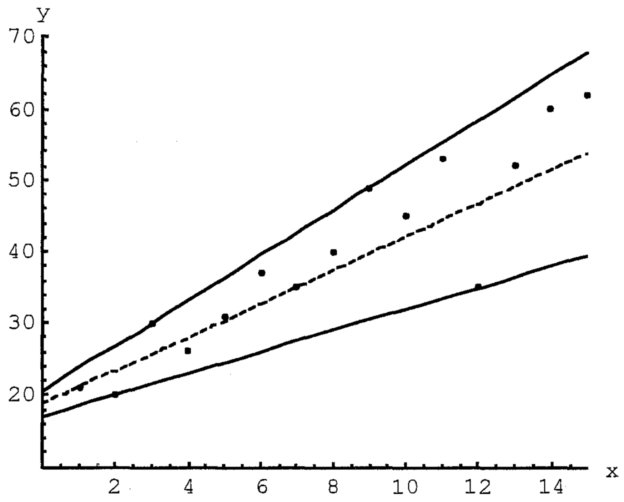

Figure 1. Regression model by LP with symmetric triangular fuzzy number coefficients

To cope with the above point, Savic and Pedrycz [11] proposed a method where center coefficients a_i $(i=0,\ldots,n)$ are obtained by least squares and then spread coefficients c_i $(i=0,\ldots,n)$ are obtained by LP problem (2). Due to symmetric coefficients, this method gives too wide spreads in spite of the reasonable central tendency.

In this paper, we propose fuzzy regression analysis with non-symmetric fuzzy coefficients by QP. The proposed QP approach can integrate the property of central tendency in least squares and the possibilistic property in fuzzy regression analysis. The characteristic of the proposed QP approach is that it allows us to obtain the center and spread coefficients simultaneously with one optimization problem while the method [11] is not. Also, the proposed QP approach gives some

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trade-off between minimum spreads and central tendency in the regression model.

For a data set composed of crisp inputs-fuzzy outputs, we can consider two approximation models, an upper approximation model and a lower approximation model which are similar to the possibility and necessity concepts. It is necessary that the upper approximation model should include the lower approximation model for any input vector. If the two approximation models are obtained by solving two separate optimization problems, there is a possibility that the upper approximation model does not include the lower approximation model for some input vector as discussed in [4]. Thus, we also propose an integrated QP problem where centers of two approximation models are assumed to be identical. The advantage of assuming a same center for two approximation models is that if the upper approximation model includes the lower approximation model at some h-level, then this inclusion relation is satisfied at any level between 0 and 1. Using the gross domestic product (GDP) data, the proposed methods are illustrated and also sensitivities of weight coefficients in the QP problems are investigated.

2. Formulation of Fuzzy Regression Model with Non-Symmetric Fuzzy Coefficients

Our former approaches [7, 8, 12-14] are based on the fuzzy linear systems whose coefficients are assumed to be symmetric fuzzy numbers. In possibilistic regression analysis, the possibilistic models are obtained by minimizing the sum of spreads subject to the constraints that the estimated models include the given outputs. Thus, if the coefficients of the fuzzy linear system are symmetric fuzzy numbers, the center of the obtained model may not show proper central tendency since the center is determined to be in the middle between the upper and lower bounds of the estimated model. To cope with that problem, if the coefficients of the fuzzy linear system are assumed to be non-symmetric fuzzy numbers, we can obtain a model with better central tendency than the fuzzy models with symmetric fuzzy numbers. Based on the motivation mentioned above, unlike our former approaches, we propose a fuzzy linear regression model with non-symmetric fuzzy coefficients. Let us assume a fuzzy regression model as

$$Y(\boldsymbol{x}) = A_0 + A_1 x_1 + \cdots + A_n x_n = \boldsymbol{A}\boldsymbol{x} \tag{5}$$

where $\boldsymbol{x} = (1, x_1, \ldots, x_n)'$ is an input vector, $\boldsymbol{A} = (A_0, \ldots, A_n)$ is a fuzzy coefficient vector, and $Y(\boldsymbol{x})$ is the estimated fuzzy output. If the coefficients A_i, $(i = 0, \ldots, n)$ are assumed to be non-symmetric triangular fuzzy numbers, A_i denoted as $A_i = (a_i, c_i, d_i)_T$ can be defined by

$$\begin{aligned} \mu_{A_i}(x) &= 1 - (a_i - x)/c_i, && \text{if } a_i - c_i \le x \le a_i, \\ &= 1 - (x - a_i)/d_i, && \text{if } a_i \le x \le a_i + d_i, \\ &= 0, && \text{otherwise,} \end{aligned} \tag{6}$$

where a_i is a center, c_i is a left-spread, and d_i is a right-spread.

Let the input-output data be given as $(\boldsymbol{x}_j, y_j) = (1, x_{j1}, \ldots, x_{jn}; y_j)$, $j = 1, \ldots, m$. Since regression coefficients $A_i = (a_i, c_i, d_i)_T$, $(i = 0, \ldots, n)$ in (5) are assumed to be non-symmetric triangular fuzzy numbers, the estimated output $Y(\boldsymbol{x}_j)$ also becomes a non-symmetric triangular fuzzy number which can be calculated by fuzzy arithmetic. Therefore, (5) can be expressed as

$$\begin{aligned} Y(\boldsymbol{x}_j) &= (a_0, c_0, d_0)_T + (a_1, c_1, d_1)_T x_{j1} + \cdots + (a_n, c_n, d_n)_T x_{jn} \\ &= \left(\theta_C(\boldsymbol{x}_j), \theta_L(\boldsymbol{x}_j), \theta_R(\boldsymbol{x}_j)\right)_T \end{aligned} \tag{7}$$

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where

$$\begin{aligned}\theta_C(x_j) &= \sum_{i=0}^{n} a_i x_{ji}, \\ \theta_L(x_j) &= \sum_{x_{ji}\geq 0} c_i x_{ji} - \sum_{x_{ji}<0} d_i x_{ji}, \\ \theta_R(x_j) &= \sum_{x_{ji}\geq 0} d_i x_{ji} - \sum_{x_{ji}<0} c_i x_{ji},\end{aligned} \tag{8}$$

represent a center, a left-spread, and a right-spread of the fuzzy output $Y(x_j)$, respectively. It should be noted that $\theta_L(x_j)$ and $\theta_R(x_j)$ in (8) are positive since c_i and d_i ($i = 0, \dots, n$) are assumed to be positive. Then, the membership function of $Y(x)$ can be defined as

$$\begin{aligned}\mu_{Y(x)}(y) &= 1 - \frac{\theta_C(x) - y}{\theta_L(x)}, \quad \text{if } \theta_C(x) - \theta_L(x) \leq y \leq \theta_C(x), \\ &= 1 - \frac{y - \theta_C(x)}{\theta_R(x)}, \quad \text{if } \theta_C(x) \leq y \leq \theta_C(x) + \theta_R(x), \\ &= 0, \quad \text{otherwise.}\end{aligned} \tag{9}$$

The h-level set of $Y(x)$ can be expressed as an interval

$$[Y(x)]_h = \left\{ y \,\middle|\, \mu_{Y(x)}(y) \geq h \right\} = [y_h^-, y_h^+] \tag{10}$$

where

$$\begin{aligned}y_h^- &= \theta_C(x) - (1-h)\,\theta_L(x), \\ y_h^+ &= \theta_C(x) + (1-h)\,\theta_R(x),\end{aligned} \tag{11}$$

represent the bounds of $[Y(x)]_h$, respectively.

To determine the optimal fuzzy coefficients $A_i = (a_i, c_i, d_i)_T$, ($i = 0, \dots, n$) of the fuzzy regression model (5), the sum of spreads of the estimated outputs can be considered as an objective function, that is, the sum of spreads of $[Y(x)]_h$ for all data is taken as an objective function:

$$(1-h)\sum_{j=1}^{m}\left(\theta_L(x_j) + \theta_R(x_j)\right) = (1-h)\sum_{j=1}^{m}\left(c^t|x_j| + d^t|x_j|\right) \tag{12}$$

where $c = (c_0, \dots, c_n)^t$ and $d = (d_0, \dots, d_n)^t$ are left and right spread coefficient vectors, and m is a data size.

Let us consider minimization of sum of squared distances between the estimated output centers and the observed outputs, denoted as

$$\sum_{j=1}^{m} (y_j - a^t x_j)^2 \tag{13}$$

which corresponds to the least squares concept and $a = (a_0, \dots, a_n)^t$ is a center vector. Thus we suggest a new objective function, by combining (12) and (13), which reflects both properties of fuzzy regression and least squares:

$$J = k_1 \sum_{j=1}^{m} (y_j - a^t x_j)^2 + k_2 (1-h) \sum_{j=1}^{m} \left(c^t|x_j| + d^t|x_j|\right) \tag{14}$$

where k_1 and k_2 are weight coefficients.

A QP problem is an optimization problem which involves minimizing a quadratic objective function subject to linear constraint conditions. To formulate fuzzy regression by QP, the followings are assumed:

(i) The input- output data are given as $(x_j, y_j) = (1, x_{j1}, \dots, x_{jn}; y_j)$, $j = 1, \dots, m$.
(ii) The given data can be represented by the fuzzy linear model (5).
(iii) Given a threshold h, the given output y_j should be included in the h-level set of the estimated fuzzy output $Y(x_j)$, that is, satisfy

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$$[Y(x_j)]_h \ni y_j \Leftrightarrow \left\{ \begin{array}{l} \theta_C(x_j) + (1-h)\,\theta_R(x_j) \ge y_j, \\ \theta_C(x_j) - (1-h)\,\theta_L(x_j) \le y_j \end{array} \right\}, \; j = 1, \dots, m, \tag{15}$$

which is regarded as one of possibilistic properties of fuzzy regression.

(iv) The objective function is defined as (14).

Based on the above assumptions, fuzzy regression by QP is to determine the optimal fuzzy coefficients $A_i = (a_i, c_i, d_i)_T$, $(i = 0, \dots, n)$ that minimize the objective function J (14) subject to the constraint conditions (15). This can be expressed as the following QP problem:

$$\begin{array}{ll} \min_{a,c,d} & J = k_1 \sum_{j=1}^{m} (y_j - a'x_j)^2 + k_2 (1-h) \sum_{j=1}^{m} \left(c'|x_j| + d'|x_j| \right) + \xi (c'c + d'd) \\ \text{subject to} & \theta_C(x_j) + (1-h)\,\theta_R(x_j) \ge y_j, \\ & \theta_C(x_j) - (1-h)\,\theta_L(x_j) \le y_j, \quad j = 1, \dots, m, \\ & c_i \ge 0, \; d_i \ge 0, \quad i = 0, \dots, n, \end{array} \tag{16}$$

where ξ is a small positive number such that $k_1, k_2 >> \xi$. The term $\xi(c'c + d'd)$ is added to (14) so that the objective function in (16) becomes a quadratic function with respect to decision variables a, c, and d. This is a well-known technique in obtaining the optimal solution by QP. By this approach, we can obtain a regression model with more central tendency comparing to the ones with symmetric triangular fuzzy coefficients by the LP problem (2).

Example: To illustrate the proposed QP approach, let us consider the numerical data shown in Table 1. By solving the QP problem (16) with $k_1 = k_2 = 1$ ($h = 0$) for the numerical data, we obtained an optimal model

$$Y(x) = (17.762, 0.762, 2.738)_T + (2.746, 1.246, 0.420)_T\, x \tag{17}$$

which is depicted in Figure 2. It can be noticed that the dotted center line in Figure 2 reflects more central tendency compared to the LP method (2) (Figure 1). The membership values of each observation by the LP method (2) and by the proposed QP method (16) are compared in Table 2 where the membership values by the proposed QP method (mean: 0.53) are slightly higher than those by the LP method (mean: 0.47). It means that the model obtained by QP method is better fitting to the given data than the one obtained by LP method is.

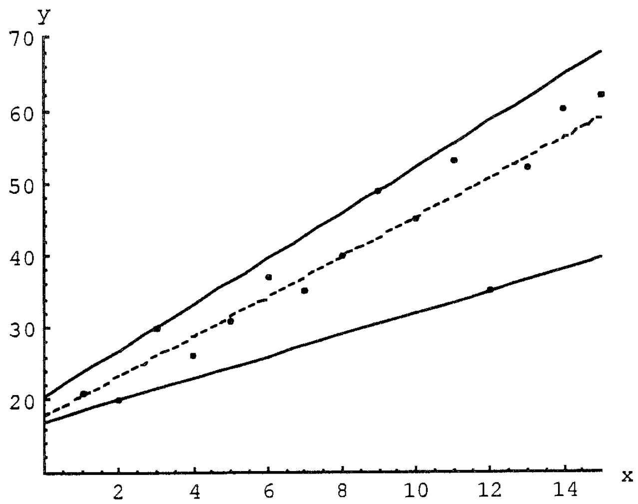

Figure 2. Fuzzy regression model with non-symmetric triangular fuzzy coefficients by QP (16)

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Table 2. Comparison of membership values between LP (2) and the proposed QP (16) for numerical data

No.(j)	1	2	3	4	5	6	7	8
μ(LP)	0.97	0	0	0.59	0.90	0.37	0.99	0.69
μ(QP)	0.84	0	0	0.52	0.93	0.47	0.80	0.96
No.(j)	9	10	11	12	13	14	15	mean
μ(LP)	0	0.71	0.21	0	0.77	0.36	0.42	0.47
μ(QP)	0	0.98	0.32	0	0.91	0.56	0.66	0.53

3. Integrated QP Approach for Upper and Lower Approximation Models

Let us consider a data set composed of crisp inputs-fuzzy outputs denoted as

$$(\boldsymbol{x}_j; Y_j) = (1, x_{j1}, \ldots, x_{jn}; Y_j),\ j = 1, \ldots, m \tag{18}$$

where the fuzzy output is defined by $Y_j = (y_j, e_j)_T$ with a center (y_j) and a spread (e_j), and m is a data size. For the data set (18), two approximation models called as a lower approximation model (LAM) and an upper approximation model (UAM) can be considered as

$$\text{LAM}: Y_*(\boldsymbol{x}_j) = A_{*0} + A_{*1}x_{j1} + \cdots + A_{*n}x_{jn} = \boldsymbol{A}_*\boldsymbol{x}_j,\ \ j = 1, \ldots, m, \tag{19}$$

$$\text{UAM}: Y^*(\boldsymbol{x}_j) = A_0^* + A_1^*x_{j1} + \cdots + A_n^*x_{jn} = \boldsymbol{A}^*\boldsymbol{x}_j,\ \ j = 1, \ldots, m, \tag{20}$$

where coefficients A_{*i} and A_i^* are non-symmetric triangular fuzzy numbers.

From the concept of lower and upper approximation models, the following inclusion relation between coefficients A_{*i} and A_i^* should be satisfied

$$A_i^* \supseteq A_{*i},\ \ i = 0, \ldots, n, \tag{21}$$

which are defined as

$$[A_i^*]_h \supseteq [A_{*i}]_h,\ \ i = 0, \ldots, n,\ \text{ for any } h. \tag{22}$$

Therefore A_{*i} of LAM and A_i^* of UAM (i = 0, ... , n) can be defined as

$$\begin{aligned} A_{*i} &= (b_i, f_i, g_i)_T, \\ A_i^* &= (b_i, f_i + p_i, g_i + q_i)_T,\ \ i = 0, \ldots, n, \end{aligned} \tag{23}$$

which are depicted in Figure 3.

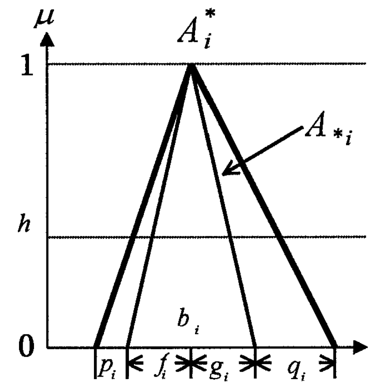

Figure 3. Fuzzy coefficients A_i^* and A_{*i} which satisfying $A_i^* \supseteq A_{*i}$

By simple fuzzy arithmetic, the inclusion relation between A_{*i} and A_i^* can be extended to the inclusion relation between LAM $Y_*(\boldsymbol{x}_j)$ and UAM $Y^*(\boldsymbol{x}_j)$, that is

$$Y^*(\boldsymbol{x}) \supseteq Y_*(\boldsymbol{x}) \text{ for any } \boldsymbol{x} = (1, x_1, \ldots, x_n)^t \text{ if } A_i^* \supseteq A_{*i}, (i = 0, \ldots, n). \tag{24}$$

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Using the coefficients $A_{*i} = (b_i, f_i, g_i)_T$, $(i = 0, \dots, n)$ in (23), LAM $Y_*(\boldsymbol{x}_j)$ (19) can be expressed as

$$\begin{aligned} \text{LAM}: Y_*(\boldsymbol{x}_j) &= (b_0, f_0, g_0)_T + (b_1, f_1, g_1)_T x_{j1} + \cdots + (b_n, f_n, g_n)_T x_{jn} \\ &= \left(\sum_{i=0}^{n} b_i x_{ji},\ \sum_{x_{ji} \ge 0} f_i x_{ji} - \sum_{x_{ji} < 0} g_i x_{ji},\ \sum_{x_{ji} \ge 0} g_i x_{ji} - \sum_{x_{ji} < 0} f_i x_{ji} \right)_T \\ &= \left(\boldsymbol{b}'\boldsymbol{x}_j,\ \theta_{*L}(\boldsymbol{x}_j),\ \theta_{*R}(\boldsymbol{x}_j) \right)_T, \qquad j = 1, \dots, m, \end{aligned} \tag{25}$$

where $\boldsymbol{b} = (b_0, \dots, b_n)'$. The spread of $\lfloor Y_*(\boldsymbol{x}_j) \rfloor_h$ can be denoted as

$$(1-h)\left(\theta_{*L}(\boldsymbol{x}_j) + \theta_{*R}(\boldsymbol{x}_j) \right) = (1-h)\left(\boldsymbol{f}'|\boldsymbol{x}_j| + \boldsymbol{g}'|\boldsymbol{x}_j| \right) \tag{26}$$

where $\boldsymbol{f} = (f_0, \dots, f_n)'$ and $\boldsymbol{g} = (g_0, \dots, g_n)'$. Using the coefficients $A_i^* = (b_i, f_i + p_i, g_i + q_i)_T$, $(i = 0, \dots, n)$ in (23), UAM $Y^*(\boldsymbol{x}_j)$ (20) can be expressed as

$$\begin{aligned} \text{UAM}: Y^*(\boldsymbol{x}_j) &= (b_0, f_0 + p_0, g_0 + q_0)_T + (b_1, f_1 + p_1, g_1 + q_1)_T x_{j1} + \cdots + (b_n, f_n + p_n, g_n + q_n)_T x_{jn} \\ &= \left(\sum_{i=0}^{n} b_i x_{ji},\ \sum_{x_{ji} \ge 0} f_i x_{ji} + \sum_{x_{ji} \ge 0} p_i x_{ji} - \sum_{x_{ji} < 0} g_i x_{ji} - \sum_{x_{ji} < 0} q_i x_{ji},\ \sum_{x_{ji} \ge 0} g_i x_{ji} + \sum_{x_{ji} \ge 0} q_i x_{ji} - \sum_{x_{ji} < 0} f_i x_{ji} - \sum_{x_{ji} < 0} p_i x_{ji} \right)_T \\ &= \left(\boldsymbol{b}'\boldsymbol{x}_j,\ \theta_L^*(\boldsymbol{x}_j),\ \theta_R^*(\boldsymbol{x}_j) \right)_T, \qquad j = 1, \dots, m. \end{aligned} \tag{27}$$

The spread of $\lfloor Y^*(\boldsymbol{x}_j) \rfloor_h$ can be denoted as

$$(1-h)\left(\theta_L^*(\boldsymbol{x}_j) + \theta_R^*(\boldsymbol{x}_j) \right) = (1-h)\left(\boldsymbol{f}'|\boldsymbol{x}_j| + \boldsymbol{g}'|\boldsymbol{x}_j| + \boldsymbol{p}'|\boldsymbol{x}_j| + \boldsymbol{q}'|\boldsymbol{x}_j| \right) \tag{28}$$

where $\boldsymbol{p} = (p_0, \dots, p_n)'$ and $\boldsymbol{q} = (q_0, \dots, q_n)'$.

The h-level set of the given output Y_j should be included in the h-level set of the estimated UAM $Y^*(\boldsymbol{x}_j)$, which can be denoted as

$$\left[Y^*(\boldsymbol{x}_j) \right]_h \supseteq \left[Y_j \right]_h \Leftrightarrow \left\{ \begin{array}{l} \boldsymbol{b}'\boldsymbol{x}_j + (1-h)\,\theta_R^*(\boldsymbol{x}_j) \ge y_j + (1-h)\,e_j, \\ \boldsymbol{b}'\boldsymbol{x}_j - (1-h)\,\theta_L^*(\boldsymbol{x}_j) \le y_j - (1-h)\,e_j \end{array} \right\}, \quad j = 1, \dots, m. \tag{29}$$

As UAM, it can be considered that $\left[Y^*(\boldsymbol{x}_j) \right]_h$ should be approached to $\left[Y_j \right]_h$ from the upper side subject to (29), i.e., $\left[Y^*(\boldsymbol{x}_j) \right]_h$ should be the least interval among all feasible solutions satisfying (29). Therefore, we should consider to minimize the sum of spreads of the estimated $\left[Y^*(\boldsymbol{x}_j) \right]_h$ for all data. Thus, the following objective function should be minimized:

$$J_U = t_1 \sum_{j=1}^{m} (y_j - \boldsymbol{b}'\boldsymbol{x}_j)^2 + t_2 (1-h) \sum_{j=1}^{m} \left(\boldsymbol{f}'|\boldsymbol{x}_j| + \boldsymbol{g}'|\boldsymbol{x}_j| + \boldsymbol{p}'|\boldsymbol{x}_j| + \boldsymbol{q}'|\boldsymbol{x}_j| \right) \tag{30}$$

where t_1 and t_2 are weight coefficients and the term $\sum_{j=1}^{m} (y_j - \boldsymbol{b}'\boldsymbol{x}_j)^2$ is inserted to obtain the property of central tendency in least squares. Thus, the optimization problem for obtaining UAM $Y^*(\boldsymbol{x}_j)$ can be described as follows:

$$\begin{aligned} [\text{UAM}]: \quad & \min_{\boldsymbol{b}, \boldsymbol{f}, \boldsymbol{g}, \boldsymbol{p}, \boldsymbol{q}} \quad (30) \\ & \text{subject to} \quad (29), \\ & \qquad f_i \ge 0,\ g_i \ge 0,\ p_i \ge 0,\ q_i \ge 0,\ i = 0, \dots, n. \end{aligned} \tag{31}$$

On the contrary to UAM, the h-level set of the estimated LAM $Y_*(\boldsymbol{x}_j)$ should be included in the h-level set of the given output Y_j, which can be denoted as

$$\left[Y_*(\boldsymbol{x}_j) \right]_h \subseteq \left[Y_j \right]_h \Leftrightarrow \left\{ \begin{array}{l} \boldsymbol{b}'\boldsymbol{x}_j + (1-h)\,\theta_{*R}(\boldsymbol{x}_j) \le y_j + (1-h)\,e_j, \\ \boldsymbol{b}'\boldsymbol{x}_j - (1-h)\,\theta_{*L}(\boldsymbol{x}_j) \ge y_j - (1-h)\,e_j \end{array} \right\}, \quad j = 1, \dots, m. \tag{32}$$

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As LAM, it can be considered that $[Y_*(x_j)]_h$ should be approached to $[Y_j]_h$ from the lower side subject to (32), i.e., $[Y_*(x_j)]_h$ should be the greatest interval among all feasible solutions satisfying (32). Therefore, we should consider to maximize the sum of spreads of the estimated $[Y_*(x_j)]_h$ for all data. Thus, the following objective function should be maximized:

$$J_L = t_2(1-h)\sum_{j=1}^{m}\left(f'|x_j| + g'|x_j|\right) + t_1\sum_{j=1}^{m}(y_j - b'x_j)^2 \tag{33}$$

where the term $\sum_{j=1}^{m}(y_j - b'x_j)^2$ is inserted to obtain the property of central tendency in least squares. Thus, the optimization problem for obtaining LAM $Y_*(x_j)$ can be described as follows:

$$\begin{aligned}
\text{[LAM]}: \quad & \max_{b,f,g} \ (33) \\
& \text{subject to} \ (32), \\
& f_i \ge 0, g_i \ge 0, \ i = 0, \dots, n.
\end{aligned} \tag{34}$$

If the coefficients A_i^* of UAM and A_{*i} of LAM (i = 0, ... , n) are obtained separately by solving two optimization problems (31) and (34), there is a possibility that the inclusion relation

$$[Y_*(x)]_h \subseteq [Y^*(x)]_h \tag{35}$$

does not be satisfied for some input vector x as discussed in [4]. Thus, to obtain two approximation models satisfying (35) for any input vector x, we need to combine the above problems (31) and (34) in a single optimization problem. In order to integrate these two (min, max) optimization problems into a single problem, letting weight coefficients $k_1 = 2t_1$ and $k_2 = t_2$, a new objective function combining (30) and (33) can be introduced as

$$\min\left(J_U - J_L\right) = \min\ k_1\sum_{i=1}^{m}(y_j - b'x_j)^2 + k_2(1-h)\sum_{i=1}^{m}\left(p'|x_j| + q'|x_j|\right). \tag{36}$$

Using the above objective function, to determine the optimal fuzzy coefficients A_{*i} of LAM and A_i^* of UAM (i = 0, ... , n) simultaneously, we propose the following Integrated QP (IQP) problem by combining two optimization problems (31) and (34):

$$\begin{aligned}
\text{[IQP]}: \quad & \min_{b,f,g,p,q} \ J = k_1\sum_{j=1}^{m}(y_j - b'x_j)^2 + k_2(1-h)\sum_{j=1}^{m}\left(p'|x_j| + q'|x_j|\right) + \xi(f'f + g'g + p'p + q'q) \\
& \text{subject to} \ b'x_j + (1-h)\,\theta_R^*(x_j) \ge y_j + (1-h)\,e_j, \\
& \qquad b'x_j - (1-h)\,\theta_L^*(x_j) \le y_j - (1-h)\,e_j, \\
& \qquad b'x_j + (1-h)\,\theta_{*R}(x_j) \le y_j + (1-h)\,e_j, \\
& \qquad b'x_j - (1-h)\,\theta_{*L}(x_j) \ge y_j - (1-h)\,e_j, \quad j = 1, \dots, m, \\
& \qquad f_i \ge 0,\ g_i \ge 0,\ p_i \ge 0,\ q_i \ge 0, \qquad i = 0, \dots, n,
\end{aligned} \tag{37}$$

where ξ is a small positive number such that $k_1, k_2 >> \xi$. The term $\xi(f'f + g'g + p'p + q'q)$ is inserted to (36) so that the objective function in (37) becomes a quadratic function with respect to decision variables $b, f, g, p,$ and q. The obtained UAM and LAM by the above IQP always satisfy the inclusion relation $Y_*(x) \subseteq Y^*(x)$ at h^*-level $(0 \le h^* \le 1)$ for any input vector $x = (1, x_1, \dots, x_n)'$.

Example: To illustrate the IQP (37), let us consider the grinding data in Table 3. In Table 3, the input is a feed speed of a grinding wheel and the output is the roughness of a work surface. The fuzzy outputs were obtained from the maximum and minimum surface roughness values from the three repeated finishing experiments for each input.

The fuzzy linear system is taken as

$$Y(x) = A_0 + A_1 x + A_2 x^2. \tag{38}$$

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Thus, we can assume two approximation models as

$$\begin{aligned} \text{LAM}: Y_*(x) &= A_{*0} + A_{*1}x + A_{*2}x^2, \\ \text{UAM}: Y^*(x) &= A_0^* + A_1^*x + A_2^*x^2, \end{aligned} \tag{39}$$

where coefficients A_{*i} and A_i^* $(i = 0, 1, 2)$ are defined as (23). Using IQP (37) with $k_1 = k_2 = 1$ (h = 0) for the grinding data, we obtained two approximation models

$$\begin{aligned} Y_*(x) &= (0.3749, 0.0288, 0)_T + (-0.1137, 0, 0)_T\,x + (0.0288, 0.0002, 0)_T\,x^2, \\ Y^*(x) &= (0.3749, 0.0965, 0.0154)_T + (-0.1137, 0.0033, 0.0695)_T\,x + (0.0288, 0.0002, 0)_T\,x^2, \end{aligned} \tag{40}$$

which are depicted in Figure 4. In Figure 4, the outer two real lines represent UAM $Y^*(x_j)$ and the inner two real lines represent LAM $Y_*(x_j)$.

Table 3. Data of feed speed and surface roughness

No.(j)	x : Feed speed (10 mm/min)	y_j: Roughness (μ) min. value	y_j: Roughness (μ) max. value	Y_j: Roughness (μ) $Y_j = (y_j, e_j)_T$
1	0.75	0.27	0.31	(0.290, 0.020)
2	1.00	0.19	0.29	(0.240, 0.050)
3	1.25	0.20	0.28	(0.240, 0.040)
4	1.50	0.24	0.32	(0.280, 0.035)
5	1.75	0.23	0.33	(0.280, 0.050)
6	2.00	0.20	0.27	(0.235, 0.035)
7	2.25	0.17	0.29	(0.230, 0.060)
8	2.50	0.20	0.46	(0.330, 0.130)
9	2.75	0.20	0.35	(0.275, 0.075)
10	3.00	0.22	0.38	(0.300, 0.080)
11	3.25	0.26	0.41	(0.335, 0.075)
12	3.50	0.22	0.33	(0.275, 0.055)
13	3.75	0.30	0.50	(0.400, 0.100)
14	4.00	0.35	0.56	(0.455, 0.105)
15	4.25	0.34	0.50	(0.420, 0.080)
16	4.50	0.37	0.60	(0.485, 0.115)
17	4.75	0.40	0.60	(0.500, 0.100)
18	5.00	0.41	0.89	(0.650, 0.240)
19	5.25	0.48	0.80	(0.640, 0.160)

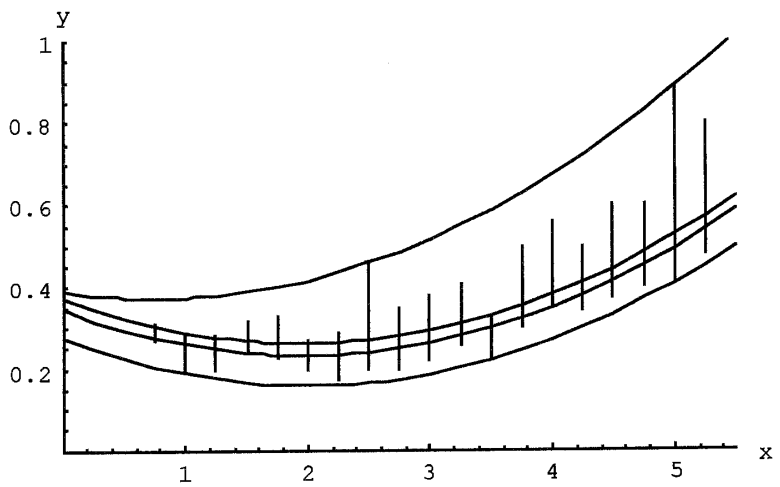

Figure 4. Estimated UAM and LAM by IQP and the given outputs for grinding data

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4. Discussion on Sensitivities of Weight Coefficients in QP (16) & IQP (37)

In this section, let us investigate sensitivities of weight coefficients in the proposed QP approaches in Sections 2 and 3. Since the proposed QP formulations have parameters of weight coefficients, it is meaningful to investigate the sensitivities of weight coefficients to the estimated approximation models. If the estimated approximation models are changing much in accordance with values of weight coefficients, an analyst may have difficulty to assign values of weight coefficients. Thus, by checking sensitivities of weight coefficients, let us find some guidance for weight coefficients in the proposed QP formulations.

The gross domestic product (GDP) data are given in Table 4 where inputs are income (x_1), working population (x_2), and output (y) is GDP of Japan during 1975 - 1992. All inputs-outputs are ratios formed by assigning the year 1970 a value of 100. Furthermore, to apply IQP to the fuzzy data, we formed fuzzy outputs $Y_j = (y_j, e_j)_T$ in the last column of Table 4 by assigning 5 % of each output y_j as the corresponding spread e_j. It should be noted that $Y_j = (y_j, e_j)_T$ is a symmetric triangular fuzzy number.

Table 4. GDP of Japan related to income and working population (1970: 100)

No. (j)	Year	Income (x_1)	Working population (x_2)	GDP (y_j)	GDP $Y_j = (y_j, e_j)_T$
1	1975	137.0	102.5	124.5	(124.5, 6.23)
2	1976	138.1	103.5	129.4	(129.4, 6.47)
3	1977	141.5	104.8	135.1	(135.1, 6.76)
4	1978	144.8	106.1	142.3	(142.3, 7.12)
5	1979	148.1	107.5	150.1	(150.1, 7.51)
6	1980	146.0	108.7	154.3	(154.3, 7.72)
7	1981	148.8	109.5	159.2	(159.2, 7.96)
8	1982	151.0	110.7	164.0	(164.0, 8.20)
9	1983	151.4	112.5	167.9	(167.9, 8.40)
10	1984	153.7	113.2	174.4	(174.4, 8.72)
11	1985	155.6	114.0	182.1	(182.1, 9.11)
12	1986	158.8	114.9	187.4	(187.4, 9.37)
13	1987	161.9	116.0	195.2	(195.2, 9.76)
14	1988	166.1	118.0	207.3	(207.3, 10.37)
15	1989	169.1	120.3	217.3	(217.3, 10.87)
16	1990	173.0	122.6	228.3	(228.3, 11.42)
17	1991	176.0	125.0	237.0	(237.0, 11.85)
18	1992	174.8	126.3	239.4	(239.4, 11.97)

Crisp inputs - outputs: First, let us analyze crisp inputs-outputs data with the following fuzzy linear system:

$$Y(\boldsymbol{x}) = A_0 + A_1 x_1 + A_2 x_2 \tag{41}$$

where the coefficients $A_i = (a_i, c_i, d_i)_T$ $(i = 0, 1, 2)$ are non-symmetric triangular fuzzy numbers. By solving QP (16) with three combinations of weight coefficients k_1 and k_2 (h = 0), optimal coefficients are obtained as shown in Table 5. It can be noticed in Table 5 that the obtained center vector $\boldsymbol{a}$ is not so sensitive to weight coefficients k_1 and k_2 in QP (16), while the spread coefficients $\boldsymbol{c}$ and $\boldsymbol{d}$ are slightly related to weight coefficients.

As the proposed QP method (16) combines the properties of least squares and fuzzy regression where $M_1 = \sum_j (y_j - \boldsymbol{a}'\boldsymbol{x}_j)^2$ represents a measure of central tendency in least squares and $M_2 = \sum_j \left(\boldsymbol{c}'|\boldsymbol{x}_j| + \boldsymbol{d}'|\boldsymbol{x}_j|\right)$ represents a measure of possibilistic property in fuzzy regression analysis, it is meaningful to check values of M_1 and M_2 as weight coefficients k_1 and k_2 in QP (16) change. Thus, using the optimal coefficients in Table 5, M_1 and M_2 are shown in Table 6. Comparing the case (a) against the case (b) in Table 6, M_1 is reduced by 0.1 % while M_2 is increased by 0.2 %. On the other hand, comparison of the case (c) against the case (b) in Table 6 shows that M_1 is increased by 0.4 % while M_2 is reduced by 0.3 %. Thus, it can be said that fluctuations of M_1 and M_2

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corresponding to change of weight coefficients are very small. Furthermore, it can be noticed that values of $M_1 + M_2$ for the cases (a), (b), and (c) in Table 6 are almost same.

Table 5. Optimal coefficients by QP (16) for GDP data with crisp inputs-outputs

Case	Weights k_1	k_2	Optimal coefficient vectors a^t	c^t	d^t
(a)	10	1	(-341.571, 1.418, 2.641)	(0.383, 0, 0.018)	(0.393, 0, 0.013)
(b)	1	1	(-341.507, 1.417, 2.643)	(0.455, 0, 0.018)	(0.541, 0, 0.012)
(c)	1	10	(-340.588, 1.430, 2.617)	(2.305, 0, 0)	(1.967, 0, 0)

Table 6. Comparison results using the optimal coefficients in Table 5

Case	k_1	k_2	$M_1 = \sum_j (y_j - a^t x_j)^2$		$M_2 = \sum_j \left(c^t\lvert x_j\rvert + d^t\lvert x_j\rvert\right)$		$M_1 + M_2$
(a)	10	1	34.704	(99.9 %)	77.293	(100.2 %)	111.997
(b)	1	1	34.724	(100.0 %)	77.175	(100.0 %)	111.899
(c)	1	10	34.872	(100.4 %)	76.909	(99.7 %)	111.781

To summarize results in Tables 5 and 6, weight coefficients k_1 and k_2 in QP (16) are not so influential in determining an optimal model. Insensitivity of weight coefficients are caused by constraint conditions in QP (16) and the assumption of non-symmetric fuzzy coefficients. Considering that value of M_1 by conventional least squares method is 34.5, the optimal models of the cases (a), (b), and (c) in Table 5 represent good central tendency.

For the case (b) in Table 5 , the optimal model can be denoted as

$$Y(x) = (-341.507, 0.455, 0.541)_T + (1.417, 0, 0)_T x_1 + (2.643, 0.018, 0.012)_T x_2 \tag{42}$$

which is depicted in Figure 5. Using (42), the estimated outputs and membership values are shown in Table 7 where four samples have zero membership values.

Table 7. Estimated outputs and membership values by QP (16) ($k_1 = k_2 = 1$) for GDP data when the given outputs (y_j) are crisp

No. (j)	GDP (y_j)	Estimated outputs $Y(x_j) = \left(\theta_C(x_j), \theta_L(x_j), \theta_R(x_j)\right)_T$	μ
1	124.5	(123.51, 2.26, 1.72)	0.42
2	129.4	(127.71, 2.28, 1.73)	0
3	135.1	(135.96, 2.30, 1.75)	0.63
4	142.3	(144.07, 2.32, 1.76)	0.24
5	150.1	(152.45, 2.35, 1.78)	0
6	154.3	(152.64, 2.37, 1.79)	0.08
7	159.2	(158.73, 2.38, 1.80)	0.74
8	164.0	(165.01, 2.40, 1.81)	0.58
9	167.9	(170.34, 2.43, 1.83)	0
10	174.4	(175.45, 2.45, 1.84)	0.57
11	182.1	(180.25, 2.46, 1.85)	0
12	187.4	(187.16, 2.48, 1.86)	0.87
13	195.2	(194.46, 2.50, 1.87)	0.61
14	207.3	(205.70, 2.53, 1.90)	0.16
15	217.3	(216.03, 2.57, 1.92)	0.34
16	228.3	(227.63, 2.61, 1.95)	0.66
17	237.0	(238.23, 2.65, 1.98)	0.54
18	239.4	(239.96, 2.68, 1.99)	0.79

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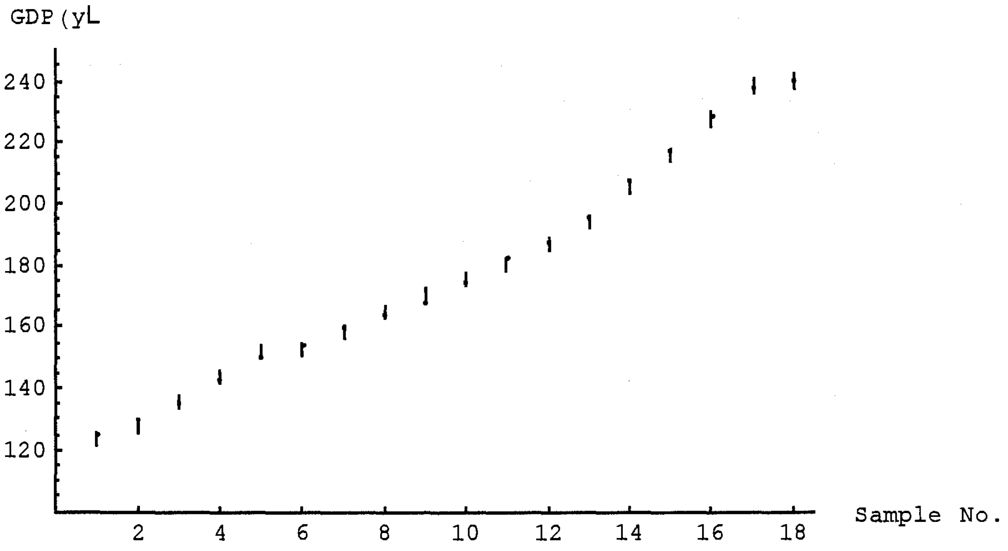

Figure 5. Estimated outputs by QP (16) ($k_1 = k_2 = 1$) for GDP data when the given outputs (y_j) are crisp

Crisp inputs - fuzzy outputs: Let us analyze crisp inputs - fuzzy outputs data where the given outputs are fuzzy numbers $Y_j = (y_j, e_j)_T$ as shown in the last column of Table 4. The two approximation models are considered as

$$\begin{aligned} &\text{LAM}: Y_*(x) = A_{*0} + A_{*1}x_1 + A_{*2}x_2, \\ &\text{UAM}: Y^*(x) = A_0^* + A_1^*x_1 + A_2^*x_2, \end{aligned} \tag{43}$$

where coefficients A_{*i} and A_i^* $(i = 0, 1, 2)$ are defined as (23). Applying IQP (37) with three combinations of weight coefficients k_1 and k_2 (h = 0), optimal coefficient vectors are obtained as shown in Table 8. In Table 8, comparison of the case (a) against the case (b) shows that weight coefficients k_1 and k_2 are not so sensitive to determine the optimal coefficient vectors, while the optimal coefficients in the case (c) are slightly changed comparing to the case (b).

Table 8. Optimal coefficients by IQP for GDP data with crisp inputs-fuzzy outputs

Case	Weights		Optimal coefficient vectors				
	k_1	k_2	b^t	f^t	g^t	p^t	q^t
(a)	10	1	(-339.392, 1.465, 2.558)	(0, 0.0342, 0)	(0, 0.0340, 0)	(0, 0.0401, 0)	(0, 0.0375, 0)
(b)	1	1	(-340.009, 1.376, 2.686)	(0, 0.0362, 0)	(0, 0.0344, 0)	(0, 0.0363, 0)	(0, 0.0388, 0)
(c)	1	10	(-331.216, 1.463, 2.489)	(0, 0.0338, 0)	(0, 0, 0.0502)	(0, 0.0433, 0)	(0, 0.0310, 0)

Using the optimal coefficients in Table 8, values of $M_3 = \sum_j (y_j - b^t x_j)^2$ and $M_4 = \sum_j \left(p^t|x_j| + q^t|x_j| \right)$ are shown in Table 9. In Table 9, we can notice that M_3 decreases as the ratio k_1/k_2 increases while M_4 increases as the ratio k_1/k_2 increases. It can be noticed that values of $M_3 + M_4$ for the cases (a), (b), and (c) in Table 9 do not indicate so much differences.

Table 9. Comparison results using the optimal coefficients in Table 8

Case	k_1	k_2	$M_3 = \sum_j (y_j - b^t x_j)^2$		$M_4 = \sum_j \left(p^t\|x_j\| + q^t\|x_j\| \right)$		$M_3 + M_4$
(a)	10	1	34.578	(96.0 %)	216.779	(103.2 %)	251.357
(b)	1	1	36.016	(100.0 %)	209.957	(100.0 %)	245.973
(c)	1	10	40.512	(112.5 %)	207.804	(99.0 %)	248.316

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Thus, from results in Tables 8 and 9, it can be said that weight coefficients k_1 and k_2 in IQP (37) are not so critical in determining an optimal model since constraint conditions in IQP (37) and assumption of non-symmetric fuzzy coefficients allow good central tendency and minimum spreads in the obtained regression model.

For the case (a) in Table 8, estimated outputs of LAM $Y_*(x_j)$ and UAM $Y^*(x_j)$ are shown in Table 10 and Figure 6 where it can be noticed that UAM $Y^*(x_j)$ includes LAM $Y_*(x_j)$ for any j.

Table 10. Estimated LAM $Y_*(x_j)$ and UAM $Y^*(x_j)$ by IQP ($k_1 = 10, k_2 = 1$) for GDP data when the given outputs are fuzzy numbers denoted as $Y_i = (y_i, e_i)_T$

No. (j)	GDP $Y_j = (y_j, e_j)_T$	LAM $Y_*(x_j) = \left(b^t x_j, \theta_{*L}(x_j), \theta_{*R}(x_j)\right)_T$	UAM $Y^*(x_j) = \left(b^t x_j, \theta^*_L(x_j), \theta^*_R(x_j)\right)_T$
1	(124.5, 6.23)	(123.52, 4.68, 4.66)	(123.52, 10.17, 9.79)
2	(129.4, 6.47)	(127.69, 4.72, 4.69)	(127.69, 10.26, 9.87)
3	(135.1, 6.76)	(136.00, 4.84, 4.81)	(136.00, 10.51, 10.11)
4	(142.3, 7.12)	(144.16, 4.95, 4.92)	(144.16, 10.75, 10.35)
5	(150.1, 7.51)	(152.58, 5.06, 5.03)	(152.58, 11.00, 10.58)
6	(154.3, 7.72)	(152.57, 4.99, 4.96)	(152.57, 10.84, 10.43)
7	(159.2, 7.96)	(158.72, 5.09, 5.06)	(158.72, 11.05, 10.63)
8	(164.0, 8.20)	(165.01, 5.16, 5.13)	(165.01, 11.21, 10.79)
9	(167.9, 8.40)	(170.20, 5.17, 5.15)	(170.20, 11.24, 10.82)
10	(174.4, 8.72)	(175.36, 5.25, 5.22)	(175.36, 11.41, 10.98)
11	(182.1, 9.11)	(180.19, 5.32, 5.29)	(180.19, 11.56, 11.12)
12	(187.4, 9.37)	(187.18, 5.43, 5.40)	(187.18, 11.79, 11.35)
13	(195.2, 9.76)	(194.54, 5.53, 5.50)	(194.54, 12.02, 11.57)
14	(207.3, 10.37)	(205.81, 5.68, 5.65)	(205.81, 12.34, 11.87)
15	(217.3, 10.87)	(216.09, 5.78, 5.75)	(216.09, 12.56, 12.08)
16	(228.3, 11.42)	(227.69, 5.91, 5.88)	(227.69, 12.85, 12.36)
17	(237.0, 11.85)	(238.22, 6.02, 5.98)	(238.22, 13.07, 12.58)
18	(239.4, 11.97)	(239.79, 5.97, 5.94)	(239.79, 12.98, 12.49)

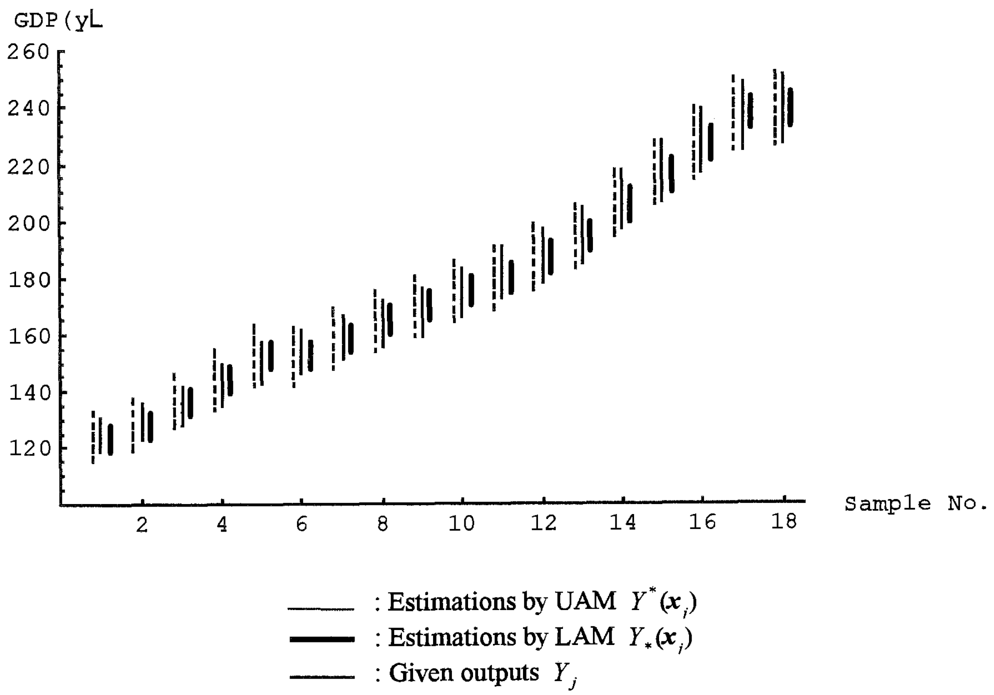

Figure 6. Estimated UAM and LAM by IQP ($k_1 = 10, k_2 = 1$) and the given outputs for GDP data when the given outputs are fuzzy numbers denoted as $Y_i = (y_i, e_i)_T$

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Simulation results showed that weight coefficients in the QP approaches are not so influential to determine fuzzy models. Especially, results of Table 6 and Table 9 do not indicate much difference. The strict constraint conditions and the assumption of non-symmetric fuzzy coefficients in (16) and (37) cause the approximation models to be insensitive, even though the models are expressed in slightly different forms, to the change of weight coefficients in the proposed QP problems.

5. Concluding Remarks

In this paper, fuzzy approximation models with non-symmetric fuzzy coefficients are proposed using the QP formulations. By assuming non-symmetric triangular fuzzy coefficients and using the QP formulations, the obtained fuzzy regression models attain more central tendency compared to the ones with symmetric triangular fuzzy coefficients. For a data set with crisp inputs-fuzzy outputs, the upper and lower approximation models can be obtained to reflect fuzziness of outputs in the analyzed phenomenon. If the two approximation models are obtained by solving two separate optimization problems, it is possible that the upper approximation model does not include the lower approximation model for some input vectors. Thus, an integrated QP formulation is proposed to obtain two approximation models satisfying the inclusion relation mentioned above.

Application results by GDP data showed that weight coefficients in the proposed QP approaches are not so critical in determining fuzzy approximation models while obtained models attain good central tendency and minimum spreads. Insensitivity of weight coefficients in the proposed QP approaches are due to the strict constraint conditions in QP problems and the assumption of non-symmetric fuzzy coefficients. Also, it is shown that the proposed integrated QP approach ensures that the upper approximation model always includes the lower approximation model at any h-level for any input vector. As future study, we can consider the extension of the proposed QP approaches for the fuzzy input-out data, which are dealt with in [9, 10].

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