

THE RELATION BETWEEN INVESTOR'S GREEDINESS AND THE ASSET PRICE IN THE MEAN-VARIANCE MARKET

Hiroshi Konno
Tokyo Institute of Technology

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Abstract We will discuss the role of investor's "greediness", i.e., the investor's expected rate of return out of the investment, on the determination of the asset price in the multi-period portfolios owned by investors in the mean-variance capital market. We will derive the closed form of the equilibrium price vector and show that the average greediness of investors must be less than the expected rate of return of the market portfolio to guarantee the existence of a non-negative equilibrium price system. These results will be applied to the analysis of the "bubble" of the capital market and to the pricing of a new stock to be listed in the capital market.

1. Introduction

The purpose of this article is to analyze the relation between the equilibrium asset price and the greediness of investors, i.e., the expected rate of return of the portfolios owned by investors in the mean-variance capital market. Numerous papers concerning the equilibrium capital asset price have been published since the pioneering works of Sharpe [18], Lintner [9] and Mossin [11]. For a survey of these results, the readers are referred to [2,3,16,17]. Also readers can find more recent results in a series of papers by Nielsen [12~15] and Werner [19].

In a recent series of articles ([6], [7], [8]), the author derived explicit formulae of the equilibrium price vector of stocks in the capital market under several alternative assumptions on the behavior of investors.

The starting point of the research was Konno and Shirakawa[7], in which the authors assumed that the capital market satisfies the standard assumptions imposed in the CAPM type equilibrium analysis and derived a closed form of the equilibrium price vector in the mean-variance capital market where all risk averse investors choose their portfolios in view of the mean and variance of the rate of return of investment. In addition, we derived a necessary and sufficient condition for the existence of a unique non-negative equilibrium vector.

Also, Konno and Shirakawa [6] showed that a similar result holds in the mean-absolute deviation capital market where investors choose the absolute deviation instead of the variance of the rate of return of investment as a measure of risk. Further, Konno and Suzuki [8] extended these results to a more general capital market in which the assumption about the homogeneity of investors is relaxed to allow different types of investors in the market.

The fundamental idea which enabled us to derive a "closed" form of the non-negative equilibrium vector is to interpret the rate of return of each asset as exogenously determined random variables as in Mossin[11]. To be more precise, we consider that the distribution of the random variable R_j , representing the rate of return of unit investment into stock S_j is

generated prior to the transaction as a function of projected performance of the enterprise, interest rate and current stock price, etc..

This interpretation is different from the standard equilibrium approach in which the rate of return of assets is endogenously determined in the capital market as a function of the price of stocks. The analysis of the equilibrium price in this framework leads one to a class of fixed point problems, for which it is very difficult, if not impossible to derive a closed form of the equilibrium price vector.

Another important assumption which is different from the traditional approach is that short sale is not allowed in our model. This leads one to an apparently more difficult optimization problem under inequality conditions. However, this assumption plays an essential role in deriving a necessary and sufficient condition for the existence of a unique non-negative equilibrium price vector, using a standard nonlinear programming methodology for handling non-negativity conditions.

The purpose of this article is two-folds. First, we will clarify the logical structure of our model in the framework of multi-stage decision problem and point out the importance of the "greediness", namely the expected rate of return of individual investors in the market. Second, we apply the results of equilibrium analysis to the pricing of a new stock to be listed in the market.

In Section 2, we state the set of assumptions imposed in the subsequent analysis. In Section 3, we extend the results of equilibrium analysis obtained in [7] to a multi-stage decision making environment. Section 4 will be devoted to the quantitative analysis of the effect of investor's expectation on the asset prices. We show that bubble can emerge when the investor's expectation grows beyond some bound.

In Section 5, we will present another potential application to the pricing of a new stock to be listed in the market.

2. Assumptions

Let us assume that there exists n risky assets $S_j (j = 1, \dots, n)$ and one riskless asset S_0 in the capital market. Also, let us assume that there are m investors $i = 1, \dots, m$ with initial endowments $\mathbf{x}_i^0 \equiv (x_{i0}^0, x_{i1}^0, \dots, x_{in}^0)$ where x_{ij}^0 represents the units of asset S_j owned by i -th investor at the beginning of the initial period. We will assume that all investors hold a non-negative amount of asset $S_j (j = 0, 1, \dots, n)$:

$$x_{ij}^0 \geq 0, \quad j = 0, 1, \dots, n; \quad i = 1, \dots, m. \quad (2.1)$$

At the beginning of each period $t (t = 1, 2, \dots)$, all investors sell their endowment $\mathbf{x}_i^{t-1} = (x_{i0}^{t-1}, x_{i1}^{t-1}, \dots, x_{in}^{t-1})$ in the market and purchase a new asset mix $\mathbf{x}_i^t = (x_{i0}^t, x_{i1}^t, \dots, x_{in}^t)$ to maximize his/her utility U_i^t , where U_i^t is the function of the mean r and the variance v of the rate of return per period of the portfolio. Further we assume that the utility functions $U_i^t(r, v)$ satisfy the following conditions:

$$\partial U_i^t(r, v) / \partial r > 0, \quad \partial U_i^t(r, v) / \partial v < 0, \quad (2.2)$$

for all i and t .

Let R_j^t be the random variable representing the rate of return of stock S_j during period t . We assume as usual that all investors share the same planning horizon and the complete knowledge of the the first two moments of the vector of random variable $R^t = (R_1^t, \dots, R_n^t)$ at the beginning of period t .

Next we will assume that the capital market satisfies the following assumptions:

Assumption 1. There is no friction (transaction costs and tax) associated with transaction.

Assumption 2. All assets are infinitely divisible.

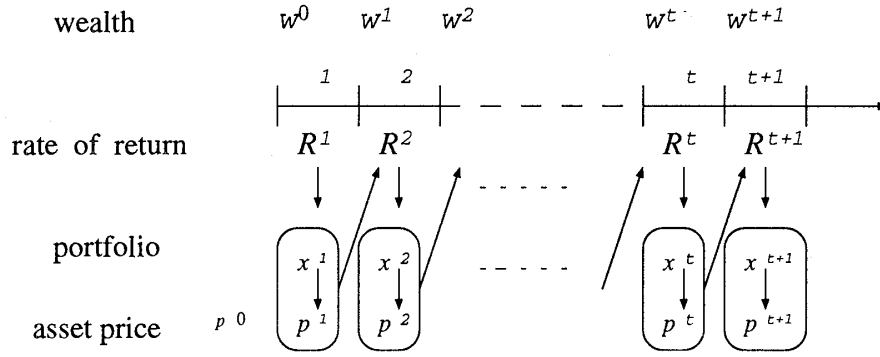


Figure 3.1: Dynamic Structure of the Portfolio Selection

Assumption 3. Investors can borrow and/or lend riskless asset with the riskfree rate r_0 without bound.

Assumption 4. Investors are not allowed to sell risky assets short.

Assumptions 1 ~ 3 are standard in the equilibrium analysis. The reason why we do not allow short sale is two-folds. First, as discussed in [7], this condition plays a crucial role in deriving a necessary and sufficient condition for the existence of a nonnegative equilibrium price vector. Second, the proportion of short sale is usually less than 10 percent in the Tokyo Stock Exchange. Also investors has to clear the short sale within a limited length of time. Therefore, we can assume no short sale as a proxy of the capital market.

3. Price of Stocks in the Equilibrium Capital Market

Let $p_j^t (j = 0, 1, \dots, n; t = 1, 2, \dots,)$ be the unit price of S_j during period t to be determined in the market at the time of transaction. We will assume without loss of generality that p_0^t , the unit price of riskless asset S_0 is unity for all t . Let x_{ij}^t be the units of $S_j (j = 0, 1, \dots, n)$ to be owned by investor $i (i = 1, \dots, m)$ during period t as a result of the transaction to be exercised at the beginning of period t .

Then the total value w_i^t of the endowment $\mathbf{x}_i^{t-1} = (x_{i0}^{t-1}, x_{i1}^{t-1}, \dots, x_{in}^{t-1})$ evaluated at the beginning of period t is given by

$$w_i^t = \sum_{j=0}^n p_j^t x_{ij}^{t-1}, \quad i = 1, \dots, m. \tag{3.1}$$

Also the rate of return $R(\mathbf{x}_i^t)$ of the portfolio $\mathbf{x}_i^t = (x_{i0}^t, x_{i1}^t, \dots, x_{in}^t)$ during period t is given by

$$R(\mathbf{x}_i^t) = \sum_{j=0}^n R_j^t p_j^t x_{ij}^t / w_i^t, \tag{3.2}$$

where R_j^t is a random variable representing the rate of return of stock S_j during period t . We assume that the probability distribution of the random variables $\mathbf{R}^t = (R_1^t, R_2^t, \dots, R_n^t)$ is generated before the transaction via the past data such as $R^s (s < t)$ and p^{t-1} as well as through various economic projections (See Figure 3.1).

Then

$$E [R(\mathbf{x}_i^t)] = \sum_{j=0}^n r_j^t p_j^t x_{ij}^t / w_i^t, \tag{3.3}$$

$$V [R(\mathbf{x}_i^t)] = \sum_{j=1}^n \sum_{k=1}^n \sigma_{jk}^t p_j^t x_{ij}^t p_k^t x_{ik}^t / (w_i^t)^2. \tag{3.4}$$

where r_j^t is the expected value of $R_j^t (j = 0, 1, \dots, n)$ and σ_{jk}^t is the covariance of R_j^t and R_k^t (Note that the rate of return R_0^t of the riskless asset S_0 is a constant r_0^t for all t so that $\sigma_{j0}^t = 0$ and $\sigma_{0k}^t = 0$ for all $j, k = 1, \dots, n$).

The minimal variance of the rate of return of investment for a given value of the expected rate of return ρ can be calculated by solving the following convex quadratic programming problem:

$$\begin{cases} \text{minimize} & \sum_{j=1}^n \sum_{k=1}^n \sigma_{jk}^t p_j^t p_k^t x_{ij}^t x_{ik}^t / (w_i^t)^2 \\ \text{subject to} & r_0^t x_{i0}^t + \sum_{j=1}^n r_j^t p_j^t x_{ij}^t = \rho w_i^t \\ & x_{i0}^t + \sum_{j=1}^n p_j^t x_{ij}^t = w_i^t \\ & x_{ij}^t \geq 0, \quad j = 1, \dots, n. \end{cases} \quad (3.5)$$

Let

$$\mathbf{x}_i^t(\rho) = ((x_{i0}^t(\rho), x_{i1}^t(\rho), \dots, x_{in}^t(\rho)), \quad (3.6)$$

be an optimal solution of (3.5) and let $v_i^t(\rho)$ be the associated minimal variance. Then the mean-variance investor i whose utility function satisfies the condition (2.2) will choose ρ in such a way as to maximize his/her utility $U_i^t(\rho, v_i^t(\rho))$. Let,

$$\rho_i^t = \operatorname{argmax}_{\rho} U_i^t(\rho, v_i^t(\rho)). \quad (3.7)$$

Let us introduce a canonical quadratic programming problem:

$$\begin{cases} \text{minimize} & \sum_{j=1}^n \sum_{k=1}^n \sigma_{jk}^t z_j z_k \\ \text{subject to} & \sum_{j=1}^n (r_j^t - r_0^t) z_j = 1 \\ & z_j \geq 0, \quad j = 1, \dots, n, \end{cases} \quad (3.8)$$

and assume that this problem has a unique optimal solution $\mathbf{z}^t = (z_1^t, z_2^t, \dots, z_n^t)$.* It is easy to show (See[7]) that $v_i(\rho)$, the minimal variance associated with problem (3.5), is given by

$$v_i(\rho) = \sum_{j=1}^n \sigma_{jk}^t z_j^t z_k^t. \quad (3.9)$$

Therefore, ρ_i^t of (3.7) is independent of p_j^t . Also we can assume without loss of generality that

$$\rho_i^t \geq r_0^t, \quad (3.10)$$

since each investor can achieve $(r, v) = (r_0^t, 0)$ by investing his/her total wealth into riskless assets.

Theorem 3.1 *Optimal asset mix $(x_{i1}^t, \dots, x_{in}^t)$ of the i -th investor is given by the following formula:*

$$p_j^t x_{ij}^t = (\rho_i^t - r_0^t) w_i^t z_j^t, \quad j = 1, \dots, n. \quad (3.11)$$

Proof

(See [7]) □

This theorem states that all investors hold a portfolio of risky assets proportional to the vector $\mathbf{z}^t = (z_1^t, \dots, z_n^t)$. Therefore the vector

$$\mathbf{y}^t \equiv \mathbf{z}^t / \sum_{j=1}^n z_j^t, \quad (3.12)$$

*This condition is satisfied if for example the following conditions hold.

- (i) $r_j > r_0$, $j = 1, \dots, n$,
- (ii) $Q = (\sigma_{jk})$ is positive definite.

defines the market portfolio associated with period t .

To achieve an equilibrium, the following market clearance condition

$$\sum_{i=1}^m x_{ij}^t = \sum_{i=1}^m x_{ij}^{t-1}, \tag{3.13}$$

has to be satisfied. From (3.13) and (3.11), we obtain the relation

$$p_j^t \sum_{i=1}^m x_{ij}^{t-1} = p_j^t \sum_{i=1}^m x_{ij}^t = \sum_{i=1}^m (\rho_i^t - r_0^t) w_i^t z_j^t. \tag{3.14}$$

Using the equation (3.1) and the assumption $p_0^t = 1$, we obtain a system of equation to be satisfied by the equilibrium price vector, namely

$$p_j^t \sum_{i=1}^m x_{ij}^{t-1} - \sum_{i=1}^m \sum_{k=1}^n (\rho_i^t - r_0^t) x_{ik}^{t-1} p_k^t z_j^t = \sum_{i=1}^m (\rho_i^t - r_0^t) x_{i0}^{t-1} z_j^t, \tag{3.15}$$

$$j = 1, \dots, n.$$

Theorem 3.2 *A necessary and sufficient condition for the system of equation (3.15) to have a unique solution is given by*

$$m_0^t \equiv \sum_{i=1}^m \sum_{j=1}^n \left\{ (\rho_i^t - r_0^t) z_j^t x_{ij}^{t-1} / \sum_{l=1}^m x_{il}^{t-1} \right\} \neq 1, \tag{3.16}$$

in which case the equilibrium price p_j^t is given by

$$p_j^t = \sum_{i=1}^m (\rho_i^t - r_0^t) x_{i0}^{t-1} z_j^t / (1 - m_0^t) \sum_{i=1}^m x_{ij}^{t-1}, \quad j = 1, \dots, n. \tag{3.17}$$

Proof

Let

$$a_j = \sum_{i=1}^m (\rho_i^t - r_0^t) x_{ij}^{t-1}, \quad j = 0, 1, \dots, n,$$

$$b_j = z_j^t / \sum_{i=1}^m x_{ij}^{t-1}, \quad j = 1, \dots, n,$$

and let $\mathbf{a} = (a_1, \dots, a_n)^T$, $\mathbf{b} = (b_1, \dots, b_n)^T$ and $\mathbf{p}^t = (p_1^t, \dots, p_n^t)^T$ where the superscript “ T ” stands for the transposition of vectors. Then the equation (3.15) can be represented as follows:

$$(I - \mathbf{b}\mathbf{a}^T)\mathbf{p}^t = \mathbf{b}a_0.$$

This equation has a unique solution if and only if $\mathbf{a}^T\mathbf{b} \neq 1$ (See [4]). It easy to check that condition is equivalent to (3.16). Also, it is easy to check that the unique solution is given by

$$\mathbf{p}^t = \frac{a_0}{1 - \mathbf{a}^T\mathbf{b}}\mathbf{b}.$$

Hence we have (3.17). □

Let us introduce an additional assumption, which is satisfied for $t = 1$ by (2.1).

Assumption 5. $\sum_{i=1}^m (\rho_i^t - r_0^t) x_{i0}^{t-1} \geq 0$ for all t .

Note that $\rho_i^t - r_0^t \geq 0$ for all i and t . Therefore this assumption holds if $x_{i0}^{t-1} \geq 0$ for all i . If there are many “disturbing” investors with large ρ_i^t and very negative x_{i0}^{t-1} , then Assumption 5 may not be valid.[†]

[†]The sum of riskless asset in the market $M_{t-1} = \sum_{i=1}^m x_{i0}^{t-1}$ is usually significantly larger than individual x_{i0}^{t-1} 's. Also, there are relatively few disturbing investors. Therefore, Assumption 5 is usually valid.

Corollary 3.3 The price of risky assets are non-negative if and only if $m_0^t < 1$ under Assumption 5.

Proof The result follows from (3.17) since

- (i) $\rho_i^t \geq r_0^t$ for all i and t by (3.10).
- (ii) $z_j^t \geq 0$ for all t since z^t is an optimal solution of (3.8).
- (iii) $\sum_{i=1}^m x_{ij}^{t-1} = \sum_{i=1}^m x_{ij}^{t-2} = \dots = \sum_{i=1}^m x_{ij}^0 > 0$ by noting (2.1). □

We see from this equation that the equilibrium price p_j^t is

- (i) an increasing function of the amount x_{i0}^{t-1} of the riskless asset owned by investors,
- (ii) a decreasing function of the rate of return r_0^t of riskless asset,
- (iii) an increasing function of ρ_i^t , the expected rate of return of i -th investor.

Let us note that similar results can be obtained under alternative assumptions on the behavior of investors. In fact, we obtained explicit formula of the equilibrium price vector under alternative measures of risk, from which we can derive the same conclusions as above. (See [7, 8] for details).

4. Role of Investor’s Expectation

It has been proved in Theorem 3.1 that $x_{ij}^t (i = 1, \dots, m; j = 1, \dots, n)$ satisfies the relation

$$x_{ij}^t = (\rho_i^t - r_0^t)w_i^t z_j^t / p_j^t. \tag{4.1}$$

Let α_{ij}^t be the fraction of security j held by investor i after the transaction at the beginning of period t . Then by (4.1), we have

$$\alpha_{ij}^t = p_j^t x_{ij}^t / \sum_{l=1}^m p_l^t x_{lj}^t = (\rho_i^t - r_0^t)w_i^t / \sum_{l=1}^m (\rho_l^t - r_0^t)w_l^t. \tag{4.2}$$

This means that the α_{ij}^t is independent of j , which in turn implies that

$$\alpha_{ij}^t = \alpha_i^t, \quad j = 1, \dots, n, \tag{4.3}$$

for all $t \geq 1$. Let us define two constants:

$$r_M^t = \sum_{j=1}^n r_j^t y_j^t, \tag{4.4}$$

$$\rho_M^t = \sum_{i=1}^m \alpha_i^t \rho_i^t. \tag{4.5}$$

where y_j ’s are defined in (3.12). The constant r_M^t is the expected rate of return of the market portfolio y^t during period t . Also ρ_M^t is the average of the expected rate of return of all investors in the market where the weight is chosen as the proportion of risky assets owned by individual investors. The constant ρ_M^t may be called the “market greediness”.

Theorem 4.1 Let $t \geq 2$. Then the condition $m_0^t < 1$ holds if and only if

$$\rho_M^t < r_M^t, \tag{4.6}$$

in which case

$$p_j^t = \frac{\rho_M^t - r_0^t}{r_M^t - \rho_M^t} (r_M^t - r_0^t) \cdot z_j^t \sum_{i=1}^m x_{i0}^{t-1} / \sum_{i=1}^m x_{ij}^0. \tag{4.7}$$

Proof

By (3.16) we have

$$m_0 = \sum_{i=1}^m \alpha_i^t (\rho_i^t - r_0^t) \sum_{j=1}^n z_j^t = (\rho_M^t - r_0^t) \sum_{j=1}^n z_j^t.$$

Also by (3.12), we have

$$\sum_{j=1}^n r_j^t z_j^t = r_M^t \cdot \sum_{j=1}^n z_j^t.$$

Further z_j^t satisfies the relation

$$\sum_{j=1}^n r_j^t z_j^t - r_0 \sum_{j=1}^n z_j^t = 1.$$

Hence

$$\sum_{j=1}^n z_j^t = \frac{1}{r_M^t - r_0^t}.$$

Therefore

$$m_0 = (\rho_M^t - r_0^t)/(r_M^t - r_0^t), \tag{4.8}$$

so that $m_0^t < 1$ if and only if $\rho_M^t < r_M^t$, in which case $p_j^t \geq 0$ for all j since $\sum_{i=1}^m x_{i0}^{t-1}$ is always nonnegative. Substituting (4.8) into (3.17) yields (4.7) by noting that $\sum_{i=1}^m x_{ij}^t = \sum_{i=1}^m x_{ij}^0$ for all $t \geq 1$. □

It is now easy to compute the total value of risky assets V_t at the beginning of period $t (\geq 2)$. **Corollary 4.2** Let M_t be the total amount of riskless asset in the capital market during period t . Then

$$V_t = \frac{\rho_M^t - r_0^t}{r_M^t - \rho_M^t} M_t, \quad \forall t \geq 2. \tag{4.9}$$

Proof

By definition

$$\begin{aligned} V_t &= \sum_{i=1}^m \sum_{j=1}^n p_j^t x_{ij}^t = \sum_{i=1}^m \sum_{j=1}^n p_j^t x_{ij}^0 \\ &= \frac{\rho_M^t - r_0^t}{r_M^t - \rho_M^t} (r_M^t - r_0^t) \sum_{j=1}^n z_j^t \cdot \sum_{i=1}^m x_{i0}^t \\ &= \frac{\rho_M^t - r_0^t}{r_M^t - \rho_M^t} M_t, \end{aligned}$$

since $\sum_{j=1}^n z_j^t = 1/(r_M^t - r_0^t)$ □

We see from Figure 4.1 that the total value V_t of risky assets is zero when $\rho_M^t = r_0^t$. It is an increasing function of the market greediness ρ_M^t and it grows to infinity as ρ_M^t approaches r_M^t . When ρ_M^t exceeds r_M^t , then the price of stock becomes negative which means to the collapse of the market. Let us define the “temperature” of the capital market

$$K_t = \frac{V_t}{V_t + M_t}, \tag{4.10}$$

which represents the proportion of the risky assets relative to total assets outstanding in the capital market. By (4.9), we have an alternative representation of K_t :

$$K_t = \frac{\rho_M^t - r_0^t}{r_M^t - r_0^t} \tag{4.11}$$

This means that K_t is an indicator of the closeness of ρ_M^t to r_M^t . If K_t is close to 1, then the market is overheated and is in danger.

The market temperature K_t is a macro-economic constant which can be recovered from the market data. Unfortunately however, it is not easy to identify M_t and V_t in the capital market. One possible surrogate variable for the total amount of the riskless asset M_t would

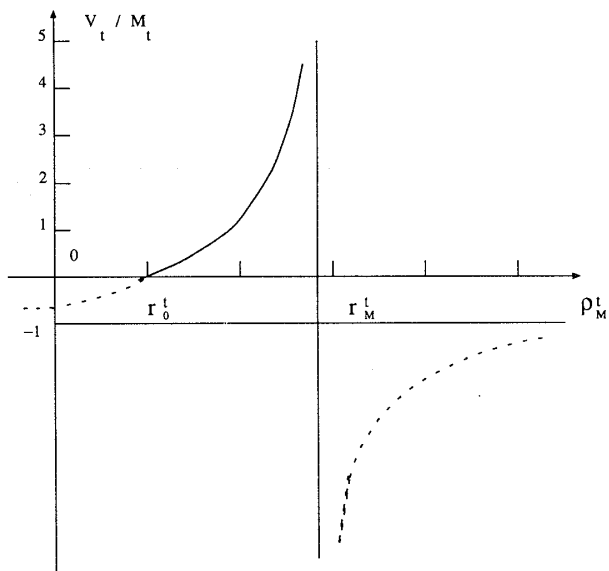


Figure 4.1: Total Value of Risky Assets as a Function to the Market Greediness

be the total amount of outstanding Government bonds. Also, the total value of risky assets V_t may be represented by the sum of the total value of stocks and bonds.

5. Pricing of a New Stock to be Listed in the Market

Let us consider the pricing of a new stock S_{n+1} to be listed in the market. Let us assume that the amount of riskless asset V_0 , the rate of return of riskless asset r_0 , and the market greediness ρ_M remain constant before and after the listing. Given the starting price p_{n+1}^0 , we may be able to calculate the distribution of the random variable $R_{n+1}(p_{n+1}^0)$ representing the rate of return of S_{n+1} .

Then we can calculate the expected rate of return $r_{n+1} \equiv r_{n+1}(p_{n+1}^0)$ and the covariance coefficient $\sigma_{jn+1} \equiv \sigma_{jn+1}(p_{n+1}^0) (j = 1, \dots, n + 1)$. A new market portfolio $(z_1^*, \dots, z_n^*, z_{n+1}^*)$ will be calculated by solving an $n + 1$ -dimensional quadratic programming problem:

$$\begin{cases}
 \text{minimize} & \sum_{j=1}^{n+1} \sum_{k=1}^{n+1} \sigma_{jk} z_j z_k \\
 \text{subject to} & \sum_{j=1}^{n+1} (r_j - r_0) z_j = 1 \\
 & z_j \geq 0, \quad j = 1, \dots, n, n + 1.
 \end{cases} \tag{5.1}$$

Thus we can calculate the price p_{n+1}^* of the stock S_{n+1} after the listing by using the formula (4.7):

$$p_{n+1}^* = \frac{\rho_M - r_0}{r_M^* - \rho_M} (r_M^* - r_0) M_0 z_{n+1}^* / s_{n+1}, \quad j = 1, \dots, n + 1, \tag{5.2}$$

where s_{n+1} is the unit of stock $n + 1$ to be released, $z_j^* (j = 1, \dots, n, n + 1)$ is an optimal solution of (5.1). Also the expected rate of return r_M^* of the new market portfolio is given by

$$r_M^* = \sum_{j=1}^n r_j z_j^* + r_{n+1} z_{n+1}^*. \tag{5.3}$$

The calculated p_{n+1}^* may not be equal to p_{n+1}^0 . The adequate price of S_{n+1} is therefore the value of p_{n+1} for which $p_{n+1}^* = p_{n+1}^0$.

Let us look into this procedure in more detail. Let

$$S = \begin{bmatrix} \sigma_{11} & \cdots & \sigma_{1n} \\ \sigma_{21} & \cdots & \sigma_{2n} \\ \vdots & \vdots & \vdots \\ \sigma_{n1} & \cdots & \sigma_{nn} \end{bmatrix}, \tilde{S} = \begin{bmatrix} \sigma_{11} & \cdots & \sigma_{1n} & \sigma_{1n+1} \\ \sigma_{21} & \cdots & \sigma_{2n} & \sigma_{2n+1} \\ \vdots & \vdots & \vdots & \vdots \\ \sigma_{n1} & \cdots & \sigma_{nn} & \sigma_{nn+1} \\ \sigma_{n+11} & \cdots & \sigma_{n+1n} & \sigma_{n+1n+1} \end{bmatrix} \tag{5.4}$$

We will assume that both S and \tilde{S} are positive definite. Let

$$\left. \begin{aligned} \mathbf{z} &= (z_1, \dots, z_n)^T \\ \mathbf{w} &= (z_1, \dots, z_n, z_{n+1})^T \\ \boldsymbol{\mu} &= (r_1 - r_0, r_2 - r_0, \dots, r_n - r_0)^T \\ \tilde{\boldsymbol{\mu}} &= (r_1 - r_0, r_2 - r_0, \dots, r_n - r_0, r_{n+1} - r_0)^T \end{aligned} \right\} \tag{5.5}$$

The canonical quadratic program corresponding to n assets and $n + 1$ assets are given by

$$\left| \begin{array}{l} \text{minimize } \mathbf{z}^t S \mathbf{z} \\ \text{subject to } \boldsymbol{\mu}^T \mathbf{z} = 1 \\ \mathbf{z} \geq \mathbf{0} \end{array} \right. \tag{5.6}$$

$$\left| \begin{array}{l} \text{minimize } \mathbf{w} \tilde{S} \mathbf{w} \\ \text{subject to } \tilde{\boldsymbol{\mu}}^T \mathbf{w} = 1 \\ \mathbf{w} \geq \mathbf{0} \end{array} \right. \tag{5.7}$$

Let us introduce quadratic programs by dropping the nonnegativity constraint from (5.6) and (5.7):

$$\left| \begin{array}{l} \text{minimize } \mathbf{z}^t S \mathbf{z} \\ \text{subject to } \boldsymbol{\mu}^T \mathbf{z} = 1 \end{array} \right. \tag{5.8}$$

$$\left| \begin{array}{l} \text{minimize } \mathbf{w} \tilde{S} \mathbf{w} \\ \text{subject to } \tilde{\boldsymbol{\mu}}^T \mathbf{w} = 1 \end{array} \right. \tag{5.9}$$

The optimal solutions of these problems are given by

$$\hat{\mathbf{z}} = S^{-1} \boldsymbol{\mu} / \boldsymbol{\mu}^T S^{-1} \boldsymbol{\mu} \tag{5.10}$$

$$\hat{\mathbf{w}} = \tilde{S}^{-1} \tilde{\boldsymbol{\mu}} / \tilde{\boldsymbol{\mu}}^T \tilde{S}^{-1} \tilde{\boldsymbol{\mu}} \tag{5.11}$$

Assumption 6. $\hat{\mathbf{z}} \geq \mathbf{0}$, $\hat{\mathbf{w}} \geq \mathbf{0}$.

Under this assumption, $\hat{\mathbf{z}}$ and $\hat{\mathbf{w}}$ of (5.10) and (5.11) give optimal solutions of (5.6) and (5.7), respectively.

Theorem 5.1 *Let (p_1, \dots, p_n) be the equilibrium price of assets S_j ($j = 1, \dots, n$) in the market. Also let $(p_1^*, \dots, p_{n+1}^*)$ be the new equilibrium price after the introduction of S_{n+1} . Then*

$$p_{n+1}^* = \frac{\tilde{\alpha}(\tilde{\boldsymbol{\mu}}_{n+1} - \sum_{j=1}^n \sigma_{jn+1} p_j s_j / \alpha)}{(\sigma_{n+1,n+1} - \boldsymbol{\sigma}^T S^{-1} \boldsymbol{\sigma})_{s_{n+1}}} \tag{5.12}$$

where

$$\alpha = (\rho_M - r_0)(r_M - r_0)M_0 / (r_M - \rho_M) \boldsymbol{\mu}^T S^{-1} \boldsymbol{\mu} \tag{5.13}$$

$$\tilde{\alpha} = (\rho_M - r_0)(r_M^* - r_0)M_0 / (r_M^* - \rho_M) \tilde{\boldsymbol{\mu}}^T \tilde{S}^{-1} \tilde{\boldsymbol{\mu}} \tag{5.14}$$

$$\boldsymbol{\sigma} = (\sigma_{1n+1}, \dots, \sigma_{nn+1})^T \tag{5.15}$$

Proof Let $\mathbf{u} = (p_1 s_1, \dots, p_n s_n)^T$ and $\mathbf{v}^* = (p_1^* s_1, \dots, p_n^* s_n, p_{n+1}^* s_{n+1})^T$. By noting the relation (5.10), (5.11) and (4.17), we have

$$\mathbf{u} = \alpha S^{-1} \boldsymbol{\mu}$$

$$\mathbf{v}^* = \tilde{\alpha} \tilde{S}^{-1} \tilde{\boldsymbol{\mu}}$$

or equivalently

$$S \mathbf{u} = \alpha \boldsymbol{\mu}$$

$$\tilde{S}v^* = \tilde{\alpha}\tilde{\mu}$$

Let $\hat{w} = (v_1^*, \dots, v_n^*)^T$. Then

$$S\hat{w} + \sigma v_{n+1}^* = \tilde{\alpha}\mu$$

$$\sigma^t \hat{w} + \sigma_{n+1,n+1} v_{n+1}^* = \tilde{\alpha}\tilde{\mu}_{n+1}$$

from which we obtain

$$\begin{aligned} v_{n+1}^* &= p_{n+1}^* s_{n+1} = \frac{\tilde{\alpha}(\tilde{\mu}_{n+1} - \sigma^T S^{-1} \mu)}{\sigma_{n+1,n+1} - \sigma^T S^{-1} \sigma} \\ &= \frac{\tilde{\alpha}(\tilde{\mu}_{n+1} - \sigma^T u / \alpha)}{\sigma_{n+1,n+1} - \sigma^T S^{-1} \sigma} \quad \square \end{aligned}$$

Let us consider a special case in which $r_M^* = r_M$ and $\tilde{\mu}^T \tilde{S}^{-1} \tilde{\mu} = \mu^T S^{-1} \mu$. Then $\tilde{\alpha} = \alpha$ so that we have

$$p_{n+1}^* s_{n+1} = \frac{\alpha \tilde{\mu}_{n+1} - \sum_{j=1}^n \sigma_{jn+1} p_j s_j}{\sigma_{n+1,n+1} - \sigma^T S^{-1} \sigma}$$

Note that R_{n+1} is a function of p_{n+1}^0 . Therefore μ_{n+1} , σ and $\sigma_{n+1,n+1}$ are functions of p_{n+1}^0 . The equilibrium price p_{n+1}^* may be obtained by solving the equation

$$p_{n+1} \cdot s_{n+1} = \frac{\tilde{\alpha} \mu_{n+1}(p_{n+1}) - \sum_{j=1}^n \sigma_{jn+1}(p_{n+1}) p_j s_j}{\sigma_{n+1,n+1}(p_{n+1}) - \sigma^T S^{-1} \sigma} \quad (5.16)$$

Therefore, we will be able to calculate the appropriate value of p_{n+1} if the explicit relation between p_{n+1} and μ_{n+1} , σ_{jn+1} ($j = 1, \dots, n+1$) can be estimated.

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Hiroshi KONNO

Department of Industrial Engineering and Management
Tokyo Institute of Technology
Meguro-ku, Tokyo, 152, Japan
(E-mail: konno@me.titech.ac.jp)