

## CHARACTERIZING A VALUATED DELTA-MATROID AS A FAMILY OF DELTA-MATROIDS\*

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*Abstract* Two characterizations are given for a valuated delta-matroid. Let  $(V, \mathcal{F})$  be an even delta-matroid on a finite set  $V$  with the family  $\mathcal{F}$  of feasible sets. It is shown that a function  $\delta : \mathcal{F} \rightarrow \mathbf{R}$  is a valuation of  $(V, \mathcal{F})$  if and only if, for each linear weighting  $p : V \rightarrow \mathbf{R}$ , the maximizers of  $\delta + p$  form the family of feasible sets of a delta-matroid. It is also shown that  $\delta$  is a valuation if and only if its conjugate function is “locally bisubmodular” at each point.

### 1. Introduction

Greedy algorithms for nonlinear discrete functions are attracting renewed interest in the literature (e.g., Ando–Fujishige–Naitoh [1], Fujishige [19], Favati–Tardella [18], Hochbaum–Hong [20]). Among others, it has turned out that the valuated (delta-)matroids, introduced by Dress–Wenzel [11, 12, 14] and Wenzel [39], afford a nice combinatorial framework, in which variants of greedy algorithms work (Dress–Wenzel [11, 12], Dress–Terhalle [8, 9, 10], Murota [24, 25], Murota–Shioura [32]). These greedy-type algorithms are similar in the vein to, but not the same as, those in Korte–Lovász–Schrader [21].

In addition to these results on greedy algorithms, valuated matroids afford a nice combinatorial framework to which the duality results for matroids can be generalized. The weighted matroid intersection problem [15, 16] has been extended by Murota [26, 27] to the valuated matroid intersection problem, where the optimality criteria and algorithms for the weighted matroid intersection problem are generalized. This result has been reformulated by Murota [28] into a novel min-max duality theorem.

The concept of valuation of a matroid is defined as follows (Dress–Wenzel [11, 14]). Let  $(V, \mathcal{B})$  be a matroid on a finite set  $V$  with the basis family  $\mathcal{B}$  (see, e.g., Faigle [17], Welsh [38], White [41] for matroids), and let  $\mathbf{R}$  be the set of real numbers. A valuation of  $(V, \mathcal{B})$  is a function  $\omega : \mathcal{B} \rightarrow \mathbf{R}$  which enjoys the exchange property:

(MV) For  $B, B' \in \mathcal{B}$  and for  $u \in B - B'$ , there exists  $v \in B' - B$  such that  $B - u + v \in \mathcal{B}$ ,  $B' + u - v \in \mathcal{B}$  and

$$\omega(B) + \omega(B') \leq \omega(B - u + v) + \omega(B' + u - v),$$

where  $B - u + v = (B - \{u\}) \cup \{v\}$  and  $B' + u - v = (B' \cup \{u\}) - \{v\}$ . A matroid equipped with a valuation is called a valuated matroid.

This concept has been generalized by Murota [29, 30, 31] for a function  $\omega : B \rightarrow \mathbf{R}$  defined on the set  $B$  of integral points in the base polytope of an integral submodular system in the sense of Fujishige [19]. That is, the domain of definition  $B \subseteq \mathbf{Z}^V$  is assumed to satisfy the exchange property

**(BP)** For  $x, y \in B$  and for  $u \in \text{supp}^+(x - y)$ , there exists  $v \in \text{supp}^-(x - y)$  such that  $x - \chi_u + \chi_v \in B$  and  $y + \chi_u - \chi_v \in B$ , where  $\text{supp}^+(x - y) = \{u \in V \mid x(u) > y(u)\}$ ,  $\text{supp}^-(x - y) = \{v \in V \mid x(v) < y(v)\}$  and  $\chi_u \in \mathbf{Z}^V$  denotes the characteristic vector of  $u \in V$ , and then the exchange property (MV) is generalized to

**(EXC)** For  $x, y \in B$  and  $u \in \text{supp}^+(x - y)$  there exists  $v \in \text{supp}^-(x - y)$  such that  $x - \chi_u + \chi_v \in B$ ,  $y + \chi_u - \chi_v \in B$  and

$$\omega(x) + \omega(y) \leq \omega(x - \chi_u + \chi_v) + \omega(y + \chi_u - \chi_v).$$

It is pointed out that such functions arise naturally in the context of combinatorial optimization. Furthermore, the relationship among (EXC), submodularity and convexity is made clear by extending Lovász's observation [22] about the relationship between convexity and submodularity.

Another generalization of the concept of valuated matroid is that of valuated delta-matroid, due to Dress–Wenzel [12] and Wenzel [39]. A valuated delta-matroid is a function  $\delta : 2^V \rightarrow \mathbf{R} \cup \{-\infty\}$  such that

**(DV0)**  $\delta(I) \neq -\infty$  for some  $I \subseteq V$ ,

**(DV1)** For  $I, I' \subseteq V$  with  $\delta(I) \neq -\infty \neq \delta(I')$  and for  $u \in I \Delta I'$ , there exists  $v \in (I \Delta I') - u$  such that

$$\delta(I) + \delta(I') \leq \delta(I \Delta u \Delta v) + \delta(I' \Delta u \Delta v).$$

Here  $\Delta$  denotes the symmetric difference:  $I \Delta I' = (I - I') \cup (I' - I)$ , and  $I \Delta u \Delta v = I \Delta \{u\} \Delta \{v\}$ , etc.

It is easy to see that the underlying family

$$\mathcal{F} = \{I \subseteq V \mid \delta(I) \neq -\infty\},$$

which is nonempty by (DV0), satisfies the following property:

**(DE)** For  $I, I' \in \mathcal{F}$  and for  $u \in I \Delta I'$ , there exists  $v \in (I \Delta I') - u$  such that  $I \Delta u \Delta v \in \mathcal{F}$  and  $I' \Delta u \Delta v \in \mathcal{F}$ .

This means that  $(V, \mathcal{F})$  is an even delta-matroid. In fact, (DE) is known (Wenzel [40]) to be equivalent to the defining condition of an even delta-matroid. It is recalled (Bouchet [2], [3], Bouchet–Dress–Havel [5], Chandrasekaran–Kabadi [6], Dress–Havel [7]) that a pair  $(V, \mathcal{F})$  (with  $\mathcal{F} \subseteq 2^V$ ) is called a delta-matroid if

**(DG)** For  $I, I' \in \mathcal{F}$  and for  $u \in I \Delta I'$ , there exists  $v \in I \Delta I'$  such that  $I \Delta \{u, v\} \in \mathcal{F}$ , and that a delta-matroid  $(V, \mathcal{F})$  is said to be even if  $|I \Delta I'|$  is even for all  $I, I' \in \mathcal{F}$ .

Hence, instead of starting with  $(V, \delta)$ , we may alternatively start with an even delta-matroid  $(V, \mathcal{F})$  and consider a function  $\delta : \mathcal{F} \rightarrow \mathbf{R}$ . In this way we can talk of a valuation of a given even delta-matroid  $(V, \mathcal{F})$ . That is, we say that  $\delta : \mathcal{F} \rightarrow \mathbf{R}$  is a valuation of  $(V, \mathcal{F})$  if the following property is satisfied:

**(DV)** For  $I, I' \in \mathcal{F}$  and for  $u \in I \Delta I'$ , there exists  $v \in (I \Delta I') - u$  such that  $I \Delta u \Delta v \in \mathcal{F}$ ,  $I' \Delta u \Delta v \in \mathcal{F}$  and

$$\delta(I) + \delta(I') \leq \delta(I \Delta u \Delta v) + \delta(I' \Delta u \Delta v).$$

A canonical combinatorial example of a valuated delta-matroid arises from the weighted (nonbipartite) matching problem. Let  $G = (V, E)$  be a graph with vertex set  $V$  and edge set  $E$ , and let  $w : E \rightarrow \mathbf{R}$  be a weight function. For a matching  $M \subseteq E$  we denote by  $\partial M \subseteq V$

the set of the vertices incident to  $M$ . It is well known that  $\mathcal{F} = \{\partial M \subseteq V \mid M : \text{matching}\}$  defines an even delta-matroid  $(V, \mathcal{F})$ . A novel observation here is that

$$\delta(I) = \max\{w(M) \mid M : \text{matching with } \partial M = I\}$$

gives a valuation of  $(V, \mathcal{F})$ , where the proof is by a standard augmenting path argument. It is remarked that the first example of a valuated delta-matroid due to Dress–Wenzel [12] is of algebraic nature and is related to determinants of a skew-symmetric matrix over a valuated field. The weighted matching example above can also be formulated in this algebraic setting using a weighted version of the Tutte matrix.

The objective of this paper is to give two alternative characterizations of a delta-matroid valuation, one in terms of the family of the maximizers and the other in terms of the local bisubmodularity of the conjugate function.

The first result (Theorem 2.2) reveals that  $\delta : \mathcal{F} \rightarrow \mathbf{R}$  is a valuation of an even delta-matroid  $(V, \mathcal{F})$  if and only if, for each linear weighting  $p : V \rightarrow \mathbf{R}$ , the maximizers of  $\delta + p$  form the family of feasible sets of an even delta-matroid. This fact allows us to regard a valuated delta-matroid as a collection of delta-matroids, just as we may regard a valuated matroid as a collection of matroids (see Corollary 2.3). This may also be compared with the fact that a delta-matroid can be characterized as a collection of matroids under twisting (Bouchet [2]).

The second result (Theorem 3.4) is concerned with the relationship among exchangeability (DV), bisubmodularity and convexity. It is well known (Bouchet–Cunningham [4], Chandrasekaran–Kabadi [6], Fujishige [19], Nakamura [33]) that the incidence vectors of the feasible sets of a delta-matroid agree with the vertices of a bisubmodular polytope contained in the unit hypercube in  $\mathbf{R}^V$  (see Lemma 3.2 for a precise statement). This means that, for an integral polytope contained in the unit hypercube in  $\mathbf{R}^V$ , the exchange property (DG) for the vertices is equivalent to the bisubmodularity for (the inequalities describing) the faces of the polytope. On the other hand, the property (DV) is a quantitative generalization of (DE) (a special case of (DG)). Then it is natural to ask for the generalization of bisubmodularity that corresponds to (DV):

$$\begin{array}{ccc} \text{(DG) / (DE)} & \implies & \text{(DV)} \\ \updownarrow & & \updownarrow \\ \text{Bisubmodularity} & \implies & ? \end{array} \quad (1.1)$$

An answer is given in Theorem 3.4, which shows that  $\delta$  satisfies (DV) if and only if its conjugate function is “locally bisubmodular” at each point.

## 2. Maximizers

### 2.1. Theorem

Let  $(V, \mathcal{F})$  be an even delta-matroid. We first note the fact that the defining exchange property (DV) for a valuation is equivalent to a seemingly weaker local exchange property. This is proven in Dress–Wenzel [13, Theorem 3.4] using the results on “matroids with coefficients”, whereas in this paper we give an alternative proof in Section 2.2.

**Lemma 2.1** *Let  $(V, \mathcal{F})$  be an even delta-matroid and  $\delta : \mathcal{F} \rightarrow \mathbf{R}$ . Then (DV) is equivalent to*

**(DV<sub>loc</sub>)** *For  $I, I' \in \mathcal{F}$  with  $|I \Delta I'| = 4$ , there exist distinct  $u, v \in I \Delta I'$  such that  $I \Delta u \Delta v \in \mathcal{F}$ ,  $I' \Delta u \Delta v \in \mathcal{F}$  and  $\delta(I) + \delta(I') \leq \delta(I \Delta u \Delta v) + \delta(I' \Delta u \Delta v)$ .  $\square$*

The first theorem of this paper is now stated, while the proof is given in Section 2.3. For  $\delta : \mathcal{F} \rightarrow \mathbf{R}$  and  $p : V \rightarrow \mathbf{R}$  define  $\delta[p] : \mathcal{F} \rightarrow \mathbf{R}$  and  $\mathcal{F}(\delta[p]) \subseteq 2^V$  by

$$\delta[p](I) = \delta(I) + \sum \{p(u) \mid u \in I\} \quad (I \in \mathcal{F}), \quad (2.1)$$

$$\mathcal{F}(\delta[p]) = \{I \in \mathcal{F} \mid \delta[p](I) \geq \delta[p](I') \ (\forall I' \in \mathcal{F})\}. \quad (2.2)$$

**Theorem 2.2** *Let  $(V, \mathcal{F})$  be an even delta-matroid and  $\delta : \mathcal{F} \rightarrow \mathbf{R}$ . Then  $\delta$  is a valuation (satisfying (DV)) if and only if*

**(MAX)**  $(V, \mathcal{F}(\delta[p]))$  is a delta-matroid for each  $p : V \rightarrow \mathbf{R}$ .

When  $\delta$  is integer-valued, we may restrict  $p$  to half-integer vectors in the “if” part.  $\square$

When specialized to a matroid valuation, this result yields the following, which is also a special case of the similar theorem [30, Theorem 4.4] for a function with the exchange property (EXC). [The integrality of  $p$  follows from the proof of Theorem 2.2, and not from the statement itself.]

**Corollary 2.3** *Let  $(V, \mathcal{B})$  be a matroid and  $\omega : \mathcal{B} \rightarrow \mathbf{R}$ . Then  $\omega$  is a valuation (satisfying (MV)) if and only if  $(V, \mathcal{B}(\omega[p]))$  is a matroid for each  $p : V \rightarrow \mathbf{R}$ , where*

$$\begin{aligned} \omega[p](B) &= \omega(B) + \sum \{p(u) \mid u \in B\} \quad (B \in \mathcal{B}), \\ \mathcal{B}(\omega[p]) &= \{B \in \mathcal{B} \mid \omega[p](B) \geq \omega[p](B') \ (\forall B' \in \mathcal{B})\}. \end{aligned}$$

When  $\omega$  is integer-valued, we may restrict  $p$  to integer vectors in the “if” part.  $\square$

**Remark 2.1** It is easy to see that for any  $I \in \mathcal{F}$  there exists  $p$  such that  $I \in \mathcal{F}(\delta[p])$ . However, the following statement is not true in general: for any  $I, I' \in \mathcal{F}$  there exists  $p$  such that  $\{I, I'\} \subseteq \mathcal{F}(\delta[p])$ .  $\square$

## 2.2. Proof of Lemma 2.1

We regard  $\delta : \mathcal{F} \rightarrow \mathbf{R}$  as  $\delta : 2^V \rightarrow \mathbf{R} \cup \{-\infty\}$  by setting  $\delta(I) = -\infty$  for  $I \in 2^V - \mathcal{F}$ , and define

$$\begin{aligned} \delta(I, u, v) &= \delta(I \Delta u \Delta v) - \delta(I) \quad (I \in \mathcal{F}, u \neq v), \\ \delta_p(I, u, v) &= \delta_p(I \Delta u \Delta v) - \delta_p(I) \quad (I \in \mathcal{F}, u \neq v), \end{aligned}$$

where  $\delta_p$  on the right hand side is an abbreviation of  $\delta[p]$  defined in (2.1). Note that  $\delta(I, u, v) = \delta(I, v, u)$  and  $\delta_p(I, u, v) = \delta_p(I, v, u)$ . If  $\{u, v\} \subseteq I \Delta I'$ ,  $I \in \mathcal{F}$  and  $I' \in \mathcal{F}$  we have

$$\begin{aligned} &\delta(I \Delta u \Delta v) + \delta(I' \Delta u \Delta v) - \delta(I) - \delta(I') \\ &= \delta(I, u, v) + \delta(I', u, v) \\ &= \delta_p(I, u, v) + \delta_p(I', u, v). \end{aligned} \quad (2.3)$$

First note the following fact.

**Lemma 2.4** *Let  $I \in \mathcal{F}$ ,  $I \Delta I' = \{v_1, v_2, v_3, v_4\}$  (with  $v_i$  being distinct) and  $p : V \rightarrow \mathbf{R}$ . If (DV<sub>loc</sub>) is satisfied, then*

$$\delta_p(I') - \delta_p(I) \leq \max\{\pi_{12} + \pi_{34}, \pi_{13} + \pi_{24}, \pi_{14} + \pi_{23}\}, \quad (2.4)$$

where  $\pi_{ij} = \delta_p(I, v_i, v_j)$  for  $i, j \in \{1, 2, 3, 4\}$ .

(Proof) First note that

$$I' \Delta v_i \Delta v_j = I \Delta v_k \Delta v_l \quad \text{if } \{i, j\} \cup \{k, l\} = \{1, 2, 3, 4\}. \quad (2.5)$$

From this and (2.3) we see that, if  $\{i, j\} \cup \{k, l\} = \{1, 2, 3, 4\}$ ,

$$\begin{aligned} \pi_{ij} + \pi_{kl} &= \delta_p(I, v_i, v_j) + \delta_p(I, v_k, v_l) \\ &= [\delta_p(I, v_i, v_j) + \delta_p(I', v_i, v_j)] + [\delta_p(I') - \delta_p(I)] \\ &= [\delta(I, v_i, v_j) + \delta(I', v_i, v_j)] + [\delta_p(I') - \delta_p(I)]. \end{aligned}$$

(DV<sub>loc</sub>) implies that  $[\delta(I, v_i, v_j) + \delta(I', v_i, v_j)] \geq 0$  for some  $(i, j)$  with  $i \neq j$ .  $\square$

Define

$$\mathcal{D} = \{(I, I') \mid I, I' \in \mathcal{F}, \exists u_* \in I \Delta I', \forall v \in (I \Delta I') - u_* : \delta(I) + \delta(I') > \delta(I \Delta u_* \Delta v) + \delta(I' \Delta u_* \Delta v)\},$$

which denotes the set of pairs  $(I, I')$  for which the exchangeability in (DV) fails. We want to show  $\mathcal{D} = \emptyset$  assuming (DV<sub>loc</sub>).

To the contrary suppose that  $\mathcal{D} \neq \emptyset$ , and take  $(I, I') \in \mathcal{D}$  such that  $|I \Delta I'|$  is minimum and let  $u_* \in I \Delta I'$  be as in the definition of  $\mathcal{D}$ . We have  $|I \Delta I'| > 4$ . Define  $p : V \rightarrow \mathbf{R}$  by

$$p(v) = \begin{cases} \delta(I, u_*, v) & (v \in (I - I') - u_*, I \Delta u_* \Delta v \in \mathcal{F}) \\ -\delta(I, u_*, v) & (v \in (I' - I) - u_*, I \Delta u_* \Delta v \in \mathcal{F}) \\ -\delta(I', u_*, v) - \varepsilon & (v \in (I - I') - u_*, I \Delta u_* \Delta v \notin \mathcal{F}, I' \Delta u_* \Delta v \in \mathcal{F}) \\ \delta(I', u_*, v) + \varepsilon & (v \in (I' - I) - u_*, I \Delta u_* \Delta v \notin \mathcal{F}, I' \Delta u_* \Delta v \in \mathcal{F}) \\ 0 & (\text{otherwise}) \end{cases}$$

with some  $\varepsilon > 0$  and consider  $\delta_p = \delta[p]$  defined in (2.1).

**Claim 1:**

$$\delta_p(I, u_*, v) = 0 \quad \text{if } v \in (I \Delta I') - u_*, I \Delta u_* \Delta v \in \mathcal{F}, \quad (2.6)$$

$$\delta_p(I', u_*, v) < 0 \quad \text{for } v \in (I \Delta I') - u_*. \quad (2.7)$$

The equality (2.6) follows from the definition of  $p$ , whereas the inequality (2.7) can be shown as follows. If  $I \Delta u_* \Delta v \in \mathcal{F}$ , we have  $\delta_p(I, u_*, v) = 0$  by (2.6) and

$$\delta_p(I, u_*, v) + \delta_p(I', u_*, v) = \delta(I, u_*, v) + \delta(I', u_*, v) < 0$$

by (2.3) and the definition of  $u_*$ . Otherwise we have  $\delta_p(I', u_*, v) = -\varepsilon$  or  $-\infty$  according to whether  $I' \Delta u_* \Delta v \in \mathcal{F}$  or not.

**Claim 2:** There exists  $\{u_0, v_0\} \subseteq (I \Delta I') - u_*$  such that  $u_0 \neq v_0$ ,  $I' \Delta u_0 \Delta v_0 \in \mathcal{F}$  and

$$\delta_p(I', u_0, v_0) \geq \delta_p(I', u, v) \quad (\{u, v\} \subseteq (I \Delta I') - u_*, u \neq v). \quad (2.8)$$

In fact, by (DE) for  $(I, I')$  we see  $I^* \equiv I \Delta u_* \Delta v_* \in \mathcal{F}$  for some  $v_* \in (I \Delta I') - u_*$ . Since  $I' \neq I^*$  and (DE) holds for  $(I', I^*)$  we have  $I' \Delta u_0 \Delta v_0 \in \mathcal{F}$  for some  $\{u_0, v_0\} \subseteq I' \Delta I^* = (I \Delta I') - \{u_*, v_*\}$  with  $u_0 \neq v_0$ . This shows that

$$\{\{u, v\} \mid \{u, v\} \subseteq (I \Delta I') - u_*, u \neq v, I' \Delta u \Delta v \in \mathcal{F}\}$$

is nonempty. Let  $\{u_0, v_0\}$  be a pair that maximizes  $\delta_p(I', u, v)$  among those pairs.

**Claim 3:**  $(I, I'') \in \mathcal{D}$  for  $I'' = I' \Delta u_0 \Delta v_0$ .

To prove this it suffices to show

$$\delta_p(I, u_*, v) + \delta_p(I'', u_*, v) < 0 \quad (v \in (I \Delta I'') - u_*).$$

We may restrict ourselves to  $v$  with  $I \Delta u_* \Delta v \in \mathcal{F}$ , since otherwise the first term  $\delta_p(I, u_*, v)$  is equal to  $-\infty$ . For such  $v$  the first term is equal to zero by (2.6). For the second term it follows from Lemma 2.4, (2.7) and (2.8) that

$$\begin{aligned} \delta_p(I'', u_*, v) &= \delta_p(I' \Delta \{u_0, u_*, v_0, v\}) - \delta_p(I' \Delta u_0 \Delta v_0) \\ &\leq \max[\delta_p(I', u_0, v_0) + \delta_p(I', u_*, v), \delta_p(I', u_0, v) + \delta_p(I', u_*, v_0), \\ &\quad \delta_p(I', v_0, v) + \delta_p(I', u_*, u_0)] - \delta_p(I', u_0, v_0) \\ &< \max[\delta_p(I', u_0, v_0), \delta_p(I', u_0, v), \delta_p(I', v_0, v)] - \delta_p(I', u_0, v_0) \\ &= 0. \end{aligned}$$

Since  $|I \Delta I''| = |I \Delta I'| - 2$ , Claim 3 contradicts our choice of  $(I, I') \in \mathcal{D}$ . Therefore we conclude  $\mathcal{D} = \emptyset$ .

**2.3. Proof of Theorem 2.2**

The “only if” part is easy to see. Take  $I, I' \in \mathcal{F}(\delta_p)$  and  $u \in I \Delta I'$ , where  $\delta_p = \delta[p]$  as before. Since  $\delta_p$  satisfies (DV), there exists  $v \in (I \Delta I') - u$  such that

$$2 \max \delta_p = \delta_p(I) + \delta_p(I') \leq \delta_p(I \Delta u \Delta v) + \delta_p(I' \Delta u \Delta v),$$

which shows  $I \Delta u \Delta v \in \mathcal{F}(\delta_p)$  and  $I' \Delta u \Delta v \in \mathcal{F}(\delta_p)$ . That is,  $\mathcal{F}(\delta_p)$  satisfies (DE).

The “if” part follows from Lemma 2.1 and the lemma below.

**Lemma 2.5** *Let  $I \in \mathcal{F}$  and  $I \Delta I' = \{v_1, v_2, v_3, v_4\}$  (with  $v_i$  being distinct). If (MAX) is satisfied, then*

$$\delta(I') - \delta(I) \leq \max\{\alpha_{12} + \alpha_{34}, \alpha_{13} + \alpha_{24}, \alpha_{14} + \alpha_{23}\}, \tag{2.9}$$

where  $\alpha_{ij} = \delta(I, v_i, v_j)$  for  $i, j \in \{1, 2, 3, 4\}$ . That is, (MAX) implies  $(DV)_{loc}$ .

(Proof) We may assume that  $I' \in \mathcal{F}$ , since otherwise (2.9) holds trivially with  $\delta(I') = -\infty$ . Denote by  $\gamma$  and  $\mu$  the left-hand side and the right-hand side of (2.9) respectively. We consider an undirected graph  $G = (\hat{V}, \hat{E})$  with  $\hat{V} = \{v_1, v_2, v_3, v_4\}$  and  $\hat{E} = \{(v_i, v_j) \mid I \Delta v_i \Delta v_j \in \mathcal{F}\}$ . The graph  $G$  has a perfect matching (of size 2) by Theorem 4.1 of Bouchet [3]. In addition we associate  $\alpha_{ij}$  with edge  $(v_i, v_j)$  as the weight. Then  $\mu$  is equal to the maximum weight of a perfect matching in  $G$ , and accordingly (cf., e.g., Lovász–Plummer [23]) there exists  $\hat{p} : \hat{V} \rightarrow \mathbf{R}$  such that

$$\hat{p}(v_i) + \hat{p}(v_j) \geq \alpha_{ij} \quad ((v_i, v_j) \in \hat{E}), \quad \sum_{i=1}^4 \hat{p}(v_i) = \mu. \tag{2.10}$$

(In fact,  $\hat{p}$  is the “optimal dual variable”; no “blossoms” are needed since  $|\hat{V}| = 4$ .)

To show  $\gamma \leq \mu$ , suppose that  $\gamma > \mu$ . Then we can modify  $\hat{p}$  to  $\bar{p} : \hat{V} \rightarrow \mathbf{R}$  such that

$$\bar{p}(v_i) + \bar{p}(v_j) \geq \alpha_{ij} \quad ((v_i, v_j) \in \hat{E}), \quad \sum_{i=1}^4 \bar{p}(v_i) = \gamma. \tag{2.11}$$

Using  $\bar{p}$  we define  $p : V \rightarrow \mathbf{R}$  by

$$p(v) = \begin{cases} +\bar{p}(v) & (v \in I - I') \\ -\bar{p}(v) & (v \in I' - I) \\ +M & (v \in I \cap I') \\ -M & (v \in V - (I' \cup I)) \end{cases}$$

where  $M > 0$  is a sufficiently large number.

For this  $p$  we have  $\{I, I'\} \subseteq \mathcal{F}(\delta_p)$ , i.e.,  $\delta_p(I) = \delta_p(I') \geq \delta_p(J) (\forall J \in \mathcal{F})$ . In fact, this is immediate from the following relations:

$$\begin{aligned} \delta_p(I') - \delta_p(I) &= [\delta(I') - \delta(I)] - \sum_{i=1}^4 \bar{p}(v_i) = 0, \\ \delta_p(I\Delta v_i\Delta v_j) - \delta_p(I) &= [\delta(I\Delta v_i\Delta v_j) - \delta(I)] + [p(I\Delta v_i\Delta v_j) - p(I)] \\ &= \alpha_{ij} - \bar{p}(v_i) - \bar{p}(v_j) \leq 0, \\ \delta_p(J) - \delta_p(I) &\leq \delta(J) - \delta(I) - M \leq 0 \quad \text{unless } I \cap I' \subseteq J \subseteq I \cup I', \end{aligned}$$

where it should be recalled that  $|J\Delta I|$  is even for  $J \in \mathcal{F}$ . By applying (DE) to  $(V, \mathcal{F}(\delta_p))$ , which is an even delta-matroid since it is a delta-matroid with  $\mathcal{F}(\delta_p) \subseteq \mathcal{F}$ , we see  $\delta_p(I\Delta v_i\Delta v_j) = \delta_p(I' \Delta v_i \Delta v_j) = \delta_p(I)$  for some  $\{i, j\} \subset \{1, 2, 3, 4\}$  with  $i \neq j$ . Putting  $\{k, l\} = \{1, 2, 3, 4\} - \{i, j\}$  and noting (2.5) we obtain

$$\begin{aligned} 0 &= \delta_p(I\Delta v_i\Delta v_j) + \delta_p(I\Delta v_k\Delta v_l) - 2\delta_p(I) \\ &= \delta(I\Delta v_i\Delta v_j) + \delta(I\Delta v_k\Delta v_l) - 2\delta(I) - \sum_{i=1}^4 \bar{p}(v_i) \\ &= \alpha_{ij} + \alpha_{kl} - \gamma \leq \mu - \gamma, \end{aligned}$$

a contradiction to  $\gamma > \mu$ . □

If  $\delta$  is integer-valued, we have  $\alpha_{ij} = \delta(I, v_i, v_j) \in \mathbf{Z}$  for all  $(v_i, v_j) \in \hat{E}$  in the proof in Lemma 2.4, and consequently, we can assume  $\hat{p}(v_i), \bar{p}(v_i) \in \frac{1}{2}\mathbf{Z}$  in (2.10) and (2.11). This completes the proof of Theorem 2.2. For the integrality assertion in Corollary 2.3, note that the graph  $G$  is bipartite in this case, and therefore we can assume  $\hat{p}(v_i), \bar{p}(v_i) \in \mathbf{Z}$  in (2.10) and (2.11).

### 3. Conjugate Function

We consider an extension of the correspondence between the exchangeability (DG) (or (DE)) and the bisubmodularity (see (1.1)) and conclude in Theorem 3.4 that (DV) for  $\delta$  is equivalent to “local bisubmodularity” of the concave conjugate function  $\delta^\circ$ .

#### 3.1. Concave conjugate function

In a manner compatible with the standard method in the convex analysis (Rockafellar [35], Stoer–Witzgall [37]), we introduce the concept of conjugate function of a set function, as has been done in Murota [28] in studying matroid valuations.

Let  $\mathcal{F} \subseteq 2^V$  be any family of subsets of  $V$ . Denote by  $\overline{\mathcal{F}}$  the convex hull of the characteristic vectors (incidence vectors) of the members of  $\mathcal{F}$ . That is,

$$\overline{\mathcal{F}} = \{b \in \mathbf{R}^V \mid b = \sum_{I \in \mathcal{F}} \lambda_I \chi_I, \lambda \in \Lambda(\mathcal{F})\}, \tag{3.1}$$

where  $\lambda = (\lambda_I \mid I \in \mathcal{F}) \in \mathbf{R}^{\mathcal{F}}$ ,  $\chi_I \in \mathbf{R}^V$  is the characteristic vector of  $I$  defined as

$$\chi_I(v) = \begin{cases} 1 & (v \in I) \\ 0 & (v \in V - I), \end{cases}$$

and

$$\Lambda(\mathcal{F}) = \{\lambda \in \mathbf{R}^{\mathcal{F}} \mid \sum_{I \in \mathcal{F}} \lambda_I = 1, \lambda_I \geq 0 (I \in \mathcal{F})\}.$$

For a function  $\delta : \mathcal{F} \rightarrow \mathbf{R}$  in general we define  $\delta^\circ : \mathbf{R}^V \rightarrow \mathbf{R}$  by

$$\delta^\circ(p) = \min\{\langle p, I \rangle - \delta(I) \mid I \in \mathcal{F}\}, \tag{3.2}$$

where  $\langle p, I \rangle = \sum\{p(u) \mid u \in I\}$ . We call  $\delta^\circ$  the concave conjugate function of  $\delta$ . Since  $|\mathcal{F}|$  is finite,  $\delta^\circ$  is a polyhedral concave function, taking finite values for all  $p$ . The function  $\delta$  is uniquely determined from the concave conjugate function  $\delta^\circ$  by

$$\delta(I) = \min\{\langle p, I \rangle - \delta^\circ(p) \mid p \in \mathbf{R}^V\}$$

(see [28, Lemma 3.2]).

For  $p : V \rightarrow \mathbf{R}$  (or  $p \in \mathbf{R}^V$ ) we define  $\delta[p] : \mathcal{F} \rightarrow \mathbf{R}$  by

$$\delta[p](I) = \delta(I) + \langle p, I \rangle \tag{3.3}$$

as in (2.1). It is easy to see  $(\delta[p_0])^\circ(p) = \delta^\circ(p - p_0)$ .

### 3.2. Exchangeability (DG) and bisubmodularity

A function  $f : 3^V \rightarrow \mathbf{R}$ , where  $3^V = \{(X, Y) \mid X \cap Y = \emptyset, X \subseteq V, Y \subseteq V\}$ , is said to be bisubmodular if

$$\begin{aligned} f(X, Y) + f(X', Y') & \\ \geq f((X \cup X') - (Y \cup Y'), (Y \cup Y') - (X \cup X')) + f(X \cap X', Y \cap Y') & \\ ((X, Y), (X', Y') \in 3^V). & \end{aligned} \tag{3.4}$$

A polyhedron defined by

$$P_*(f) = \{b \in \mathbf{R}^V \mid b(X) - b(Y) \leq f(X, Y) ((X, Y) \in 3^V)\}, \tag{3.5}$$

where  $b(X) = \sum_{v \in X} b(v)$ , is called a bisubmodular polyhedron if  $f$  is a bisubmodular function.

The following lemmas are immediate from the results in Chandrasekaran–Kabadi [6] (see also Bouchet–Cunningham [4], Fujishige [19], Nakamura [33] and Qi [34]). The former is concerned with the greedy algorithm for maximizing a linear function over a bisubmodular polyhedron, and the latter shows a characterization of a delta-matroid in terms of the polyhedral rank function. We use

$$\text{sign } \alpha = \begin{cases} 1 & (\alpha \geq 0) \\ -1 & (\alpha < 0) \end{cases}$$

in the expression (3.6) below.

**Lemma 3.1** Let  $V = \{v_1, v_2, \dots, v_n\}$  (where  $n = |V|$ ) and  $p \in \mathbf{R}^V$  be such that  $|p(v_1)| \geq |p(v_2)| \geq \dots \geq |p(v_n)|$ , and put  $p_j = p(v_j)$  ( $j = 1, \dots, n$ ),  $p_{n+1} = 0$  and

$$X_j = \{v_i \mid 1 \leq i \leq j, p(v_i) \geq 0\}, \quad Y_j = \{v_i \mid 1 \leq i \leq j, p(v_i) < 0\} \quad (j = 1, \dots, n).$$

If  $f : 3^V \rightarrow \mathbf{Z}$  satisfies the following three conditions, (R1)–(R3),

(R1)  $f(\emptyset, \emptyset) = 0$ ,

(R2)  $f(X + v, Y) - f(X, Y) \in \{0, 1\}$ ,  $f(X, Y + v) - f(X, Y) \in \{0, -1\}$   
for  $v \in V - (X \cup Y)$ ,

(R3)  $f$  is bisubmodular,

then  $\hat{b} \in \mathbf{R}^V$  defined by

$$\hat{b}(v_j) = (\text{sign } p_j)[f(X_j, Y_j) - f(X_{j-1}, Y_{j-1})] \quad (j = 1, \dots, n) \tag{3.6}$$

maximizes  $\langle p, b \rangle$  over  $P_*(f)$ , that is,

$$\hat{b} \in P_*(f), \quad \langle p, \hat{b} \rangle = \max\{\langle p, b \rangle \mid b \in P_*(f)\}.$$

□

**Lemma 3.2**

(1) If  $\mathcal{F} \subseteq 2^V$  satisfies (DG), then the function  $f : 3^V \rightarrow \mathbf{Z}$  defined by

$$f(X, Y) = \max\{|X \cap I| - |Y \cap I| \mid I \in \mathcal{F}\} \quad ((X, Y) \in 3^V) \tag{3.7}$$

satisfies (R1)–(R3); and moreover,  $\overline{\mathcal{F}}$  agrees with  $P_*(f)$ .

(2) If  $f : 3^V \rightarrow \mathbf{Z}$  satisfies (R1)–(R3), then the vertices of  $P_*(f)$  are  $\{0, 1\}$ -vectors and

$$\mathcal{F} = \{I \subseteq V \mid \chi_I \text{ is a vertex of } P_*(f)\} \tag{3.8}$$

satisfies (DG); and moreover, (3.7) holds true. □

The function  $f$  associated with a delta-matroid  $(V, \mathcal{F})$  by (3.7) is called the polyhedral rank function of  $(V, \mathcal{F})$ .

We recast the above facts into a form (Theorem 3.3) that is suitable for our subsequent extension. Define  $\psi^\circ : \mathbf{R}^V \rightarrow \mathbf{R}$  by

$$\psi^\circ(p) = \min\{\langle p, I \rangle \mid I \in \mathcal{F}\}. \tag{3.9}$$

Note that  $\psi^\circ$  is the concave conjugate function of  $\psi \equiv 0$  (on  $\mathcal{F}$ ) in the sense of (3.2), and also that

$$-\psi^\circ(-p) = \max\{\langle p, I \rangle \mid I \in \mathcal{F}\} \tag{3.10}$$

agrees with the support function of  $\overline{\mathcal{F}}$  as defined in [35], [37]. Obviously,  $\psi^\circ(p)$  is concave,  $\psi^\circ(0) = 0$ , and positively homogeneous, i.e.,  $\psi^\circ(\lambda p) = \lambda \psi^\circ(p)$  for  $\lambda > 0$ . Hence the hypograph

$$\text{Hyp}(\psi^\circ) = \{(p, q) \in \mathbf{R}^V \times \mathbf{R} \mid q \leq \psi^\circ(p)\} \tag{3.11}$$

is a convex cone.

Suppose  $\mathcal{F}$  satisfies (DG). We first observe

$$\begin{aligned} -\psi^\circ(\chi_Y - \chi_X) &= -\min\{|Y \cap I| - |X \cap I| \mid I \in \mathcal{F}\} \\ &= \max\{|X \cap I| - |Y \cap I| \mid I \in \mathcal{F}\} \\ &= f(X, Y) \quad ((X, Y) \in 3^V), \end{aligned}$$

where  $f$  is the polyhedral rank function of  $(V, \mathcal{F})$  defined by (3.7). Secondly, the value of  $\psi^\circ(p)$  at arbitrary  $p$  can be expressed as a linear combination of  $\psi^\circ(\chi_X - \chi_Y)$  ( $(X, Y) \in 3^V$ ). In fact, Lemma 3.1 (the greedy algorithm for the bisubmodular polyhedron) shows

$$\begin{aligned} \psi^\circ(p) &= \min\{\langle p, b \rangle \mid b \in P_*(f)\} \\ &= -\max\{\langle -p, b \rangle \mid b \in P_*(f)\} \\ &= -\sum_{j=1}^n (|p_j| - |p_{j+1}|) f(Y_j, X_j) \\ &= \sum_{j=1}^n (|p_j| - |p_{j+1}|) \psi^\circ(\chi_{X_j} - \chi_{Y_j}), \end{aligned}$$

where, for given  $p \in \mathbf{R}^V$ , we index the elements of  $V$  as  $V = \{v_1, v_2, \dots, v_n\}$  (with  $n = |V|$ ) in such a way that  $|p(v_1)| \geq |p(v_2)| \geq \dots \geq |p(v_n)|$ ; and we put  $p_j = p(v_j)$  ( $j = 1, \dots, n$ ) and  $p_{n+1} = 0$  and

$$X_j = \{v_i \mid 1 \leq i \leq j, p(v_i) \geq 0\}, \quad Y_j = \{v_i \mid 1 \leq i \leq j, p(v_i) < 0\} \quad (j = 1, \dots, n).$$

Conversely, suppose  $\psi^\circ$  defined from  $\mathcal{F}$  by (3.9) satisfies the two conditions:

(C1)  $f(X, Y) = -\psi^\circ(\chi_Y - \chi_X)$  satisfies (R1)–(R3).

(C2)  $\psi^\circ(p) = \sum_{j=1}^n (|p_j| - |p_{j+1}|) \psi^\circ(\chi_{X_j} - \chi_{Y_j})$ ,

where, for given  $p \in \mathbf{R}^V$ , we index the elements of  $V$  as  $V = \{v_1, v_2, \dots, v_n\}$  (with  $n = |V|$ ) in such a way that  $|p(v_1)| \geq |p(v_2)| \geq \dots \geq |p(v_n)|$ ; and we put  $p_j = p(v_j)$  ( $j = 1, \dots, n$ ),  $p_{n+1} = 0$ ,  $X_j = \{v_i \mid 1 \leq i \leq j, p(v_i) \geq 0\}$  ( $j = 1, \dots, n$ ) and  $Y_j = \{v_i \mid 1 \leq i \leq j, p(v_i) < 0\}$  ( $j = 1, \dots, n$ ).

Then Lemma 3.2 shows that  $\mathcal{F}$  satisfies (DG) and that  $f(X, Y) = -\psi^\circ(\chi_Y - \chi_X)$  is the polyhedral rank function of  $(V, \mathcal{F})$ .

We say that a positively homogeneous function  $h : \mathbf{R}^V \rightarrow \mathbf{R}$  is “delta-matroidal” if it satisfies (C1) and (C2) with  $\psi^\circ$  replaced by  $h$ . By a result of Qi [34] such  $h$  is necessarily concave. We also say that a cone is “delta-matroidal” if it is a hypograph of a “delta-matroidal”  $h$ .

With this terminology the above observations are summarized in the following theorem, which characterizes the exchange property of  $\mathcal{F}$  in the language of  $\psi^\circ$  (or the support function of  $\overline{\mathcal{F}}$ ). It is emphasized that this is a reformulation of known results.

**Theorem 3.3**  $\mathcal{F} \subseteq 2^V$  satisfies (DG) if and only if  $\psi^\circ$  is “delta-matroidal” (satisfying (C1) and (C2)). □

### 3.3. Exchangeability (DV) and bisubmodularity

We now assume that  $(V, \mathcal{F})$  is an even delta-matroid and consider the concave conjugate function

$$\delta^\circ(p) = \min\{\langle p, I \rangle - \delta(I) \mid I \in \mathcal{F}\} \tag{3.12}$$

of  $\delta : \mathcal{F} \rightarrow \mathbf{R}$ . Unlike  $\psi^\circ$ ,  $\delta^\circ$  is not a positively homogeneous function. Accordingly, the hypograph

$$\text{Hyp}(\delta^\circ) = \{(p, q) \in \mathbf{R}^V \times \mathbf{R} \mid q \leq \delta^\circ(p)\} \tag{3.13}$$

is not a cone but a polyhedron. Its characteristic cone (or recession cone) [35], [36], [37] is given by  $\text{Hyp}(\psi^\circ)$  of (3.11), and hence it is “delta-matroidal” by Lemma 3.2.

Since  $\delta^\circ$  is a concave function, we can think of its subdifferential in the ordinary sense in the convex analysis. Namely, the subdifferential of  $\delta^\circ$  at  $p_0 \in \mathbf{R}^V$ , denoted  $\partial\delta^\circ(p_0)$ , is defined by

$$\partial\delta^\circ(p_0) = \{b \in \mathbf{R}^V \mid \delta^\circ(p) - \delta^\circ(p_0) \leq \langle p - p_0, b \rangle \ (\forall p \in \mathbf{R}^V)\}. \tag{3.14}$$

Note that  $\partial\delta^\circ(p_0) \neq \emptyset$  for any  $p_0 \in \mathbf{R}^V$ , since  $\delta^\circ$  is a polyhedral concave function. Using this notion we define, as in [30], a positively homogeneous concave function  $\hat{L}(\delta^\circ, p_0) : \mathbf{R}^V \rightarrow \mathbf{R}$  by

$$\hat{L}(\delta^\circ, p_0)(p) = \inf\{\langle p, b \rangle \mid b \in \partial\delta^\circ(p_0)\}, \tag{3.15}$$

which we call the localization of  $\delta^\circ$  at  $p_0$ . Note that

$$\delta^\circ(p) \leq \delta^\circ(p_0) + \hat{L}(\delta^\circ, p_0)(p - p_0) \tag{3.16}$$

and that  $\delta^\circ(p)$  is equal to the right-hand side in the neighborhood of  $p_0$ . Also note the expression

$$\text{Hyp}(\hat{L}(\delta^\circ, p_0)) = \{(p, q) \in \mathbf{R}^V \times \mathbf{R} \mid q \leq \langle p, b \rangle, b \in \partial\delta^\circ(p_0)\}. \tag{3.17}$$

**Example 3.1** For an affine function  $\delta(I) = \alpha + \langle \eta, I \rangle$  for  $I \in \mathcal{F}$  we have

$$\delta^\circ(p) = \psi^\circ(p - \eta) - \alpha, \quad \hat{L}(\delta^\circ, \eta) = \psi^\circ,$$

where  $\psi^\circ$  is defined in (3.9). □

We are now in the position to state the second result of this paper. It establishes a link between (DV) and bisubmodularity, showing that (DV) for  $\delta$  is equivalent to the localization of  $\delta^\circ$  being “delta-matroidal” at each point. Recalling that the first condition (C1) for being “delta-matroidal” refers to bisubmodularity, while (C2) is related to greediness, which is somehow equivalent to bisubmodularity, we may say that the exchange property (DV) is nothing but “a collection of local bisubmodularity”, just as the exchange property (DG) corresponds to bisubmodularity. See [30, Theorem 5.3] for a similar result for matroid valuations.

**Theorem 3.4** *Let  $(V, \mathcal{F})$  is an even delta-matroid and  $\delta : \mathcal{F} \rightarrow \mathbf{R}$ . Then  $\delta$  is a valuation (satisfying (DV)) if and only if the localization  $\hat{L}(\delta^\circ, p_0)$  of  $\delta^\circ$  is “delta-matroidal” (satisfying (C1) and (C2)) at each point  $p_0$ .*

(Proof) A hyperplane  $\{(p, q) \mid q = \langle p, I \rangle - \delta(I)\}$  in the space of  $(p, q)$ , indexed by  $I \in \mathcal{F}$ , contains  $(p, q) = (p_0, \delta^\circ(p_0))$  if and only if

$$\langle p_0, I \rangle - \delta(I) = \delta^\circ(p_0) = \min\{\langle p_0, J \rangle - \delta(J) \mid J \in \mathcal{F}\},$$

which means  $I \in \mathcal{F}(\delta[-p_0])$  and  $\langle p, I \rangle - \delta(I) = \langle p - p_0, I \rangle + \delta^\circ(p_0)$  for such  $I$ . Therefore, in the neighborhood of  $p_0$ ,  $\delta^\circ(p)$  is equal to

$$\begin{aligned} & \min\{\langle p, I \rangle - \delta(I) \mid I \in \mathcal{F}(\delta[-p_0])\} \\ & = \min\{\langle p - p_0, I \rangle \mid I \in \mathcal{F}(\delta[-p_0])\} + \delta^\circ(p_0). \end{aligned}$$

This shows

$$\hat{L}(\delta^\circ, p_0)(p) = \min\{\langle p, I \rangle \mid I \in \mathcal{F}(\delta[-p_0])\}. \tag{3.18}$$

By Theorem 3.3, this is “delta-matroidal” if and only if  $\mathcal{F}(\delta[-p_0])$  satisfies (DG), whereas the latter condition for all  $p_0$  is equivalent to (DV) by Theorem 2.2. □

**Remark 3.1** It follows from Theorem 3.4 that the localization of a “delta-matroidal” function is again “delta-matroidal”. Therefore, it is sufficient in Theorem 3.4 to consider the localization of  $\delta^\circ$  at points  $p_0$  such that  $(p_0, \delta^\circ(p_0))$  lies in a minimal face of  $\text{Hyp}(\delta^\circ)$ .  $\square$

**Remark 3.2** By way of a concluding remark we indicate an implication of Theorem 3.4 in the context of discrete convex analysis [30, 31]. Let  $(V, \delta_1)$  and  $(V, \delta_2)$  be valuated delta-matroids and suppose we are interested in the separation of  $\delta_1$  and  $-\delta_2$  by an affine function. Theorem 3.4 shows that the separation of  $\delta_1$  and  $-\delta_2$  for a pair of general valuations  $(\delta_1, \delta_2)$  can be reduced to a special case where  $\delta_1$  and  $\delta_2$  are affine functions. This is due to the general observation in [30, §6.3] concerning separations.  $\square$

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