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THE COMPARATIVE STATICS OF SHIFTS IN RISK

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Abstract In this paper, we consider the comparative statics of shifts in risk. First, we take up the stochastic order which appears in Landsberger and Meilijson [9] and also in Eeckhoudt and Gollier [2]. This order which we call, in this paper, the "monotone generalized probability ratio"(MGPR) order lies between the "monotone probability ratio"(MPR) order and the "second-degree stochastic dominance"(SSD) order. We examine the comparative statics of the MGPR shift in risk in the economic model used by Eeckhoudt and Gollier [2]. Second, we consider the "third-degree stochastic dominance"(TSD) shift and the TSD transformation and discuss their comparative statics in some economic models.

1. Introduction

Given two random variables X_1 and X_2 with distribution functions F_1 and F_2 whose supports are in $[a, b]$, respectively, there are well-known stochastic order relations between X_1 and X_2, that is, the "first-degree stochastic dominance" (FSD) order and the "second-degree stochastic dominance" (SSD) order. We say that $X_1 \succeq_{FSD} X_2$ if $F_1(x) - F_2(x) \leq 0$ for all $x \in [a, b]$ and that $X_1 \succeq_{SSD} X_2$ if $\int_a^x [F_1(y) - F_2(y)]dy \leq 0$ for all $x \in [a, b]$. Then it can be shown that $X_1 \succeq_{FSD} X_2$ if and only if all decision-makers with nondecreasing utility functions prefer X_1 to X_2 and that $X_1 \succeq_{SSD} X_2$ if and only if all decision-makers with nondecreasing concave utility functions prefer X_1 to X_2. See, for example, Kroll and Levy [8]. The terms "prefer" and "concave" are used in the nonstrict sense.

Besides these two stochastic orders, there is an important stochastic order, namely the "third-degree stochastic dominance" (TSD) order. We say that $X_1 \succeq_{TSD} X_2$ if $\int_a^x \int_a^y [F_1(z) - F_2(z)]dzdy \leq 0$ for all $x \in [a, b]$ and $E[X_1] \geq E[X_2]$. Fishburn [3] states that the condition that $E[X_1] \geq E[X_2]$ can be omitted. The TSD is first considered by Whitmore [17]. He shows that $X_1 \succeq_{TSD} X_2$ if and only if all decision-makers with increasing concave utility functions whose third derivatives are nonnegative prefer X_1 to X_2. Here "increasing" can be replaced with "nondecreasing". See, for example, Whitmore [18]. Whitmore [17] indicates that decreasing absolute risk aversion (DARA) implies nonnegative third derivatives of utility functions. Here the term "decreasing" is used in the nonstrict sense. Hence, if a shift in risk is preferred by all TSD lovers, then this shift is also preferred by all decision-makers who have nondecreasing concave utility functions with decreasing absolute risk-aversion functions. Here we say that a decision-maker is a TSD lover when he or she has a nondecreasing concave utility function with nonnegative third derivative. As noted by Vickson [16], it is this observation that has led Whitmore [17] to consider TSD. Further, if the expected value is held constant by this shift, then the converse is also true. This is proved by Vickson [16]. See also Bawa [1], Kira and Ziemba [7], and Kroll and Levy [8].

When we consider portfolio problems where a riskless asset also exists, three equivalent relations stated above do not hold any longer. This paper concentrates on seeking the alternatives of the equivalent relation concerning TSD in various models. We would like to know what kind of relation between X_1 and X_2 exists if all decision-makers with nondecreasing concave utility functions whose third derivatives are nonnegative prefer X_1 to X_2. Also we would like to study the comparative statics concerning the TSD shift. For example, we would like to know that if $X_1 \succeq_{TSD} X_2$, then what kinds of decision-makers unambiguously prefer X_1 to X_2. Landsberger and Meilijson [9] show that the "monotone likelihood ratio" (MLR) shift in risk which refers to the ratio of density functions induces unambiguous preference over all risk averters in this context.

Eeckhoudt and Gollier [2] extend the results of Landsberger and Meilijson [9] by considering the "monotone probability ratio" (MPR) order. They indicate that the MPR order is an order which lies between the MLR order and the FSD order. Then, in the model used by Meyer and Ormiston [10], they show that all risk-averse individuals unambiguously prefer the MPR shift in their economic model which includes the standard portfolio model. As a matter of fact, the MPR order is conceptually identical with the reverse hazard rate order which is well-known in the area of applied probability. As for the reverse hazard rate order see, for example, Kijima and Ohnishi [5]. This order is a special case of an order which we call, in this paper, the "monotone generalized probability ratio" (MGPR) order.

MGPR appears in Landsberger and Meilijson [9]. MGPR also appears in Eeckhoudt and Gollier [2], but their concern is about MPR. They only use MGPR to prove the comparative statics concerning MPR. Since the MGPR order lies between the MPR order and the SSD order, it might be anticipated that all TSD lovers unambiguously prefer MGPR shifts in their model. In section 2, we explicitly consider MGPR and examine the comparative statics of the MGPR shift in the same model as used by Eeckhoudt and Gollier [2]. Regrettably we need one more condition besides the condition that the decision-maker is a TSD lover. But, when an MGPR shift preserves the mean of the distribution, we show that all TSD lovers prefer this MGPR shift.

In section 3, we consider the TSD shift. There are two motives. First, as stated above, TSD itself is an important concept to be investigated. Second, the analysis in section 2 motivates us. Eeckhoudt and Gollier [2] note that their model covers the portfolio problem with a risky asset and a riskless asset. In this portfolio problem, the demand for the risky asset corresponds to the optimal exposure to risk. The common feature of the decision-maker in this section is that he or she is a TSD lover. As is shown by Landsberger and Meilijson [9], MGPR implies SSD, which is well-known to imply TSD. Thus we consider, in section 3, the effect of the TSD shift on the TSD lover in the general portfolio setting with two stochastically independent risky assets. We give a necessary and sufficient condition for a TSD shift of one asset not to cause a decrease in the investment in that asset.

In section 4, we discuss related deterministic transformations. As for the deterministic transformations, Ormiston [13] discusses two types of deterministic transformations representing the FSD shift and the SSD shift and has considered their comparative statics in a model similar to that used by Meyer and Ormiston [10]. We discuss the TSD transformation and examine the comparative statics both in this model and in the two-period comsumption-investment model which is used by Sandmo [15]. The final section concludes.

2. The comparative statics of the MGPR shift in risk

Eeckhoudt and Gollier [2] consider three related stochastic orders, that is, the FSD order, the MLR order, and the MPR order. Among these orders, the MPR order is an order which they propose. Assume that the supports of two random variables X_1 and X_2 are contained in $[a, b]$. Denote the distribution functions of X_1 and X_2 by $F_1(x)$ and $F_2(x)$, respectively.

Then the definitions of $X_1 \succeq_{MLR} X_2$ and $X_1 \succeq_{MPR} X_2$ are as follows:

Definition 2.1. (MLR) $X_1 \succeq_{MLR} X_2$ if there exists a scalar $c \in [a, b]$ and a nonnegative nonincreasing function $l(x)$ such that $F_1(x) = 0$ for all $x < c$ and $F_2(x) = F_2(c) + \int_c^x l(y)dF_1(y)$ for all $x \geq c$.

Definition 2.2. (MPR) $X_1 \succeq_{MPR} X_2$ if there exists a scalar $c \in [a, b]$ and a nonnegative nonincreasing function $l(x)$ such that $F_1(x) = 0$ for all $x < c$ and $F_2(x) = l(x)F_1(x)$ for all $x \geq c$.

They show that MLR $\Rightarrow$ MPR $\Rightarrow$ FSD.[1]

We consider the stochastic order which appears in Landsberger and Meilijson [9]. As noted in Introduction, this stochastic order is also used to prove the result concerning MPR.[2] We call this stochastic order MGPR in this paper. The definition of $X_1 \succeq_{MGPR} X_2$ is given below so as to be in accordance with the definition of MPR given above.

Definition 2.3. (MGPR) $X_1 \succeq_{MGPR} X_2$ if there exists a scalar $c \in [a, b]$ and a nonnegative nonincreasing function $h(x)$ such that $F_1(x) = 0$ for all $x \leq c$ and $\int_a^x F_2(y)dy = h(x)\int_a^x F_1(y)dy$ for all $x > c$.

It is noted that $F_2(x) > 0$ is possible for $x < c$.[3]
As is shown in lemma 3.3 of Eeckhoudt and Gollier [2], it is true that MPR $\Rightarrow$ MGPR. Thus MGPR may be regarded as a natural extension of MPR. Also as is shown in Landsberger and Meilijson [9], MGPR $\Rightarrow$ SSD.

We now consider the comparative statics of the MGPR order in the model used by Eeckhoudt and Gollier [2]. The model is expressed as follows:

A decision-maker is assumed to select the level of exposure k_i to a specified risk X_i so as to maximize $E[u(z_0 + k_iX_i)] = \int_a^b u(z_0 + k_ix)dF_i(x)$. Here $u(\cdot)$ is his or her utility function, and z_0 is a scalar and fixed arbitrarily. The risk X_i is a random variable whose support is in $[a, b]$, with $a \leq 0 \leq b$. $F_i(x)$ is the distribution function of X_i. The optimal value of k_i is expressed as k_i^*.

According to Eeckhoudt and Gollier [2], this model is very general and has various applications. They indicate that the standard portfolio problem with one risky asset and one riskless asset can be regarded as an application of this model. In this portfolio problem, k_i and X_i represent the portfolio weight for the risky asset and the excess return of the risky asset over the riskless asset, respectively.

To give the results with regard to the comparative statics of MGPR, we need the following two lemmas one of which, that is, Lemma 1 corresponds to Lemma 3.3 of Eeckhoudt and Gollier [2].

Lemma 2.1. $X_1 \succeq_{MGPR} X_2$ implies that $E_1(x) \geq E_2(x)$ for all $x > c$. Here $E_i(x) = \frac{\int_a^x \left(\int_a^y z dF_i(z)\right)dy}{\int_a^x F_i(y)dy}$.

[1]The relation among them (MLR $\Rightarrow$ MPR $\Rightarrow$ FSD) is also shown in Kijima and Ohnishi [5].

[2]In Landsberger and Meilijson [9], it is shown that $h(x)$ is nonincreasing if and only if $E[X_2 \mid X_2 \leq x] \leq E[X_1 \mid X_1 \leq x]$. The former expression, that is, $h(x)$ being nonincreasing is used in Landsberger and Meilijson [9], while the latter expression (that is, $E[X_2 \mid X_2 \leq x] \leq E[X_1 \mid X_1 \leq x]$) is used in Eeckhoudt and Gollier [2].

[3]If we replace $x > c$ with $x \geq c$ in this definition, it is not possible that $F_2(x) > 0$ for $x < c$ because $\int_a^c F_2(x)dx = h(c)\int_a^c F_1(x)dx = 0$.

Proof. By integrating the numerator of $E_i(x)$ by parts twice, $E_1(x) \geq E_2(x)$ is expressed as

$$(1) \qquad -\frac{\int_a^x \left(\int_a^y F_2(z)dz\right) dy}{\int_a^x F_2(y)dy} \leq -\frac{\int_a^x \left(\int_a^y F_1(z)dz\right) dy}{\int_a^x F_1(y)dy}.$$

Since $\int_a^x F_2(y)dy = h(x)\int_a^x F_1(y)dy$, (1) is written as

$$(2) \qquad \begin{aligned} &\int_a^c \left(\int_a^y F_2(z)dz\right) dy + \int_c^x \left(\int_a^y F_2(z)dz\right) dy \\ &\geq h(x)\int_a^c \left(\int_a^y F_1(z)dz\right) dy + h(x)\int_c^x \left(\int_a^y F_1(z)dz\right) dy. \end{aligned}$$

The first term of the LHS of (2) is nonnegative, and the first term of the RHS is zero. Hence it suffices to prove that

$$(3) \qquad \int_c^x \left(\int_a^y F_2(z)dz\right) dy \geq h(x)\int_c^x \left(\int_a^y F_1(z)dz\right) dy.$$

Since $\int_a^y F_2(z)dz = h(y)\int_a^y F_1(z)dz$ for all $y > c$, (3) is written as

$$(4) \qquad h(x) \leq \frac{\int_c^x h(y)(\int_a^y F_1(z)dz)dy}{\int_c^x(\int_a^y F_1(z)dz)dy}.$$

Since $h(x)$ is nonincreasing in x, (4) holds. This completes the proof. □

Lemma 2.2. We assume that $X_1 \succeq_{MGPR} X_2$. Then $\alpha X_1 + \beta \succeq_{MGPR} \alpha X_2 + \beta$ for all $\alpha > 0$ and $\beta \in R$.

Proof. Let $Z_i = \alpha X_i + \beta$ and the distribution function of Z_i be F_{z_i}. Since $X_i \in [a, b]$, $Z_i \in [\alpha a + \beta, \alpha b + \beta]$. Now $X_1 \succeq_{MGPR} X_2$ implies that there exists a scalar $c \in [a, b]$ and a nonnegative nonincreasing function $h(x)$ such that $\int_a^x F_1(y)dy = 0$ for all $x \leq c$ and $\int_a^x F_2(y)dy = h(x)\int_a^x F_1(y)dy$ for all $x > c$. We consider z such that $x = \frac{z-\beta}{\alpha} > c$, that is, $z > c\alpha + \beta = c_z$. Then since $F_{z_i}(z) = F_i(\frac{z-\beta}{\alpha})$, for $z > c_z$ we have

$$(5) \qquad \int_{\alpha a+\beta}^{z} F_{z_2}(y)dy = h_z(z)\int_{\alpha a+\beta}^{z} F_{z_1}(y)dy,$$

where $h_z(z) = h(\frac{z-\beta}{\alpha})$.
Further, for all $z \leq c_z$, we have $F_{z_1}(z) = 0$. As $h_z'(z) = \frac{1}{\alpha}h'(\frac{z-\beta}{\alpha})$, $h_z(z)$ is nonincreasing in z. Also, by construction, $h_z(z)$ is nonnegative. Thus $Z_1 \succeq_{MGPR} Z_2$. □

This lemma shows that the stochastic ordering of MGPR is closed under multiplication of positive real numbers and addition of real numbers.

We now have the following result concerning the comparative statics of MGPR.

Theorem 2.1. Assume that the utility function u satisfies $u' \geq 0$, $u'' \leq 0$, $u''' \geq 0$. Then $k_1^* \geq k_2^*$ for any $X_1 \succeq_{MGPR} X_2$ if the derivative of $u'(z)z$ at $z = b$ is nonnegative.

Proof. The proof follows from the same reasoning as used in the proof of Proposition 3.1 of Eeckhoudt and Gollier [2].

As is assumed by Eeckhoudt and Gollier [2], we assume, without loss of generality, that $k_1^* = 1$ and, to simplify the notation, $z_0 = 0$. Also, thanks to Lemma 2.2, we can assume, without loss of generality, that $F_1(a) = F_2(a) = 0$. Since $u'' \leq 0$, $E[X_2u'(kX_2)]$ is a nonincreasing function of k. Also the first-order condition for k_2^* is written as $E[X_2u'(k_2^*X_2)] = 0$.

Hence, in order to prove $k_1^* \geq k_2^*$, it suffices to show that $E[X_2 u'(k_1^* X_2)] = E[X_2 u'(X_2)] \leq 0$. In Eeckhoudt and Gollier [2], it is shown that $E[X_2 u'(X_2)]$ can be written as

$$(6) \quad E[X_2 u'(X_2)] = u'(b)(E[X_2] - E[X_1]) + \int_a^b u''(x) \left[\int_a^x y dF_1(y) - \int_a^x y dF_2(y) \right] dx.$$

Applying integration by parts twice to the second term of (6), the second term is written as

$$\begin{aligned}
&\int_a^b u''(x) \left(\int_a^x y dF_1(y) - \int_a^x y dF_2(y) \right) dx \\
&= \left[u''(x) \cdot \int_a^x \left(\int_a^y (z dF_1(z) - z dF_2(z)) \right) dy \right]_a^b \\
&\quad - \int_a^b u'''(x) \left[\int_a^x \left(\int_a^y (z dF_1(z) - z dF_2(z)) \right) dy \right] dx \\
&= u''(b) \cdot \int_a^b \left(\int_a^y (z dF_1(z) - z dF_2(z)) \right) dy \\
&\quad - \int_a^b u'''(x) \left[\int_a^x \left(\int_a^y (z dF_1(z) - z dF_2(z)) \right) dy \right] dx \\
&= u''(b) \int_a^b (y F_1(y) - y F_2(y)) dy - u''(b) \int_a^b \left(\int_a^y (F_1(z) - F_2(z)) dz \right) dy \\
(7) \quad &\quad - \int_a^b u'''(x) \left[\int_a^x \left(\int_a^y (z dF_1(z) - z dF_2(z)) \right) dy \right] dx.
\end{aligned}$$

Since $X_1 \succeq_{SSD} X_2$, the second term of (7) is nonpositive.

In the following, we prove the first term of (7) plus the first term of (6) is nonpositive. Applying integration by parts to the first term of (7) gives

$$\begin{aligned}
&u''(b) \int_a^b (y F_1(y) - y F_2(y)) dy \\
&= u''(b) \left(\left[y \left(\int_a^y (F_1(z) - F_2(z)) dz \right) \right]_a^b - \int_a^b \left(\int_a^y (F_1(z) - F_2(z)) dz \right) dy \right) \\
(8) \quad &= u''(b) b (E[X_2] - E[X_1]) - u''(b) \int_a^b \left(\int_a^y (F_1(z) - F_2(z)) dz \right) dy.
\end{aligned}$$

Since the second term of (8) is equal to the second term of (7), this term is nonpositive. On the other hand, the first term of (8) plus the first term of (6) is written as $(u'(b) + u''(b)b)(E[X_2] - E[X_1])$. The derivative of $u'(z)z$ at $z = b$ is nonnegative, and $E[X_1] \geq E[X_2]$. Accordingly $(u'(b) + u''(b)b)(E[X_2] - E[X_1])$ is nonpositive. Hence the first term of (7) plus the first term of (6) is nonpositive. Therefore

$$(9) \quad E[X_2 u'(X_2)] \leq - \int_a^b u'''(x) \left[\int_a^x \left(\int_a^y (z dF_1(z) - z dF_2(z)) \right) dy \right] dx.$$

Thus, by using Lemma 2.1, the following inequality is shown to hold.

$$(10) \quad E[X_2 u'(X_2)] \leq - \int_c^b u'''(x) \cdot \int_a^x \left(\int_a^y z dF_1(z) \right) dy \cdot (1 - h(x)) dx,$$

which corresponds to the equation (14) of Eeckhoudt and Gollier [2]. If $f(x) = \int_a^x (\int_a^y z \, dF_1(z)) dy \leq 0$ for all $c < x \leq b$, the RHS of the inequality (10) is nonpositive, so the proof is completed.

Now assume that $f(x^*) > 0$ for some $c < x^* \leq b$. We set $x^{**} = \inf(x \mid f(x) > 0)$. As $c \leq 0$, x^{**} must be nonnegative. Thus $f'(x^{**})$ must be nonnegative, which implies $f'(x) > 0$ for all $x > x^{**}$. Therefore $f(x) > (\leq) 0$ for all $x > (<) x^{**}$. Thus the RHS of the inequality (10) is less than or equal to $-(1 - h(x^{**})) \int_c^b u'''(x) f(x) dx$. Then, by using the first-order condition that $E[X_1 u'(k_1^* X_1)] = E[X_1 u'(X_1)] = 0$, it can be shown that the RHS of the inequality (10) is less than or equal to $-(1-h(x^{**}))(-u'(b)E[X_1] + u''(b) \int_a^b (\int_a^x y dF_1(y)) dx)$. This is nonpositive because $E[X_1] = f'(b) > 0$ and $\int_a^b (\int_a^x y dF_1(y)) dx = f(b) > 0$. This proves the theorem. □

This theorem corresponds to Proposition 3.1 which is the main result of Eeckhoudt and Gollier [2]. The proposition shows that under an MPR shift in risk from X_2 to X_1, all risk averters unambiguously don't decrease the demand for the risky asset. Theorem 2.1 shows that under an MGPR shift in risk from X_2 to X_1, all TSD lovers with their relative risk aversion at b being less than or equal to unity don't decrease the demand for the risky asset.

Remarks.
(1) Hadar and Seo [4] obtain the following result concerning SSD (Theorem 2.3 of Hadar and Seo [4]).

Assume *a*) the utility function u satisfies $u' > 0$, $u'' \leq 0$, $u''' \geq 0$; *b*) X_i and Y are stochastically independent, $i = 1, 2$; *c*) $E[u(k_i X_i + (1 - k_i)Y)]$ is maximized at k_i^*. Then $k_1^* \geq k_2^*$ for any $X_1 \succeq_{SSD} X_2$ if and only if $u'(z)z$ is nondecreasing and concave.

Note that the condition which implies that $k_1^* \geq k_2^*$ for any $X_1 \succeq_{MGPR} X_2$ in Theorem 2.1 is less demanding than the conditions which imply $k_1^* \geq k_2^*$ for any $X_1 \succeq_{SSD} X_2$ in the above result obtained by Hadar and Seo [4]. From the viewpoint of our model as the standard portfolio model, this is a natural consequence because the model used in our case is a special case of the one used by Hadar and Seo [4] and MGPR is a stronger concept than SSD.[4]

(2) In Theorem 2.1, the conditions do not require the optimal value. In other words, there are no restrictions on the distributions of X_1 and X_2 except that $X_1 \succeq_{MGPR} X_2$. We only require restrictions on the utility function.[5]

When $E[X_1] = E[X_2]$, we have the following corollary.

Corollary 2.1. Assume that the utility function u satisfies $u' \geq 0$, $u'' \leq 0$, $u''' \geq 0$. Then $k_1^* \geq k_2^*$ for any $X_1 \succeq_{MGPR} X_2$.

The proof of this corollary is obvious from the proof of Theorem 2.1 combined with the fact that $E[X_1] = E[X_2]$.

Remarks
(1) When $E[X_1] = E[X_2]$, Landsberger and Meilijson [9] show that in the portfolio setting with one risky asset and one riskless asset, two conditions, one of which is MGPR, are necessary and sufficient for all risk averters not to decrease the demand for the risky asset. According to Corollary 2.1, when $E[X_1] = E[X_2]$, if we consider only an MGPR shift in risk from X_2 to X_1, all TSD lovers unambiguously prefer this MGPR shift.
(2) In the portfolio setting with one risky asset and one riskless asset (money), the cor-

[4]Consider the case of constant Y. Then X_i and z_0 in our model correspond to $X_i - Y$ and Y, respectively, in the model used by Hadar and Seo [4].

[5]As an example of a result concerning SSD which requires the optimal value to be used in the sufficient conditions, that is, which requires restrictions on the distributions of X_1 and X_2, the result of Kira and Ziemba [7] (Theorem 2.1 (b)) is noted. In this result, $u'''(x) \geq 0$ for any $x \in [a, b]$ is not required instead of the optimal value being required in the sufficient conditions.

responding sufficient conditions given by Rothschild and Stiglitz [14] in the case of mean-preserving increases in risk (the RS shift in risk) are $u' > 0$, $u'' < 0$, DARA, and nondecreasing relative risk aversion which is less than or equal to unity. As DARA implies $u''' \geq 0$, the sufficient conditions of Corollary 2.1 are less demanding than those of Rothschild and Stiglitz [14].

In Theorem 2.1 and Corollary 2.1, the common characteristics of the decision-maker is that his or her utility function satisfies $u' \geq 0$, $u'' \leq 0$, $u''' \geq 0$. On the other hand, since MGPR $\Rightarrow$ SSD, MGPR $\Rightarrow$ TSD. Now it is well-known that the decision-maker whose utility function satisfies $u' \geq 0$, $u'' \leq 0$, $u''' \geq 0$ prefers X_1 to X_2 if and only if $X_1 \succeq_{TSD} X_2$. Also, as noted in the footnote 4, the model in this section can be viewed as the special case of the portfolio problem with two stochastically independent risky assets. Thus it seems natural to expect that the equivalent relation stated above is true in this general portfolio problem. But, since the decision-maker whose utility function satisfies $u' \geq 0$, $u'' \leq 0$ may prefer X_2 to X_1 when $X_1 \succeq_{SSD} X_2$ in this general portfolio problem, this expectation is also not true. In the next section, we assume that the utility function of the decision maker satisfies $u' \geq 0$, $u'' \leq 0$, $u''' \geq 0$, $u'''' \leq 0$. Then we give a necessary and sufficient condition on the utility function for the decision-maker to prefer X_1 to X_2 when $X_1 \succeq_{TSD} X_2$ in this general portfolio problem.

3. Third-degree stochastic dominance

Hadar and Seo [4] examine the effects of the FSD shift, the "mean-preserving contraction" (MPC) shift, and the SSD shift in a return distribution on optimal portfolios. An MPC shift is a shift which decreases its risk while preserving the mean of the distribution. In the case of portfolios with two risky assets, Hadar and Seo [4] give conditions on the utility function which are necessary and sufficient for each of these shifts in the distribution of one asset not to cause a decrease in the investment in that asset. In the following, we assume that all returns X_1, X_2, and Y lie in the interval $[0, b]$ as Hadar and Seo [4] do. Also we assume that $E[u(k_iX_i + (1-k_i)Y)]$ is maximized at k_i^* where X_i and Y are stochastically independent ($i = 1, 2$). Following Hadar and Seo [4], we assume that $0 < k_i^* < 1$ and $\frac{d^2E[u(k_iX_i+(1-k_i)Y)]}{dk_i^2}\big|_{k_i=k_i^*} < 0$. Let $F_1(x)$, $F_2(x)$, and $G(y)$ be the distribution funcitons of X_1, X_2, and Y, respectively. We give a necessary and sufficient condition on the utility function as for the TSD shift in the next theorem.

Theorem 3.1. Assume that the utility function u satisfies $u' \geq 0$, $u'' \leq 0$, $u''' \geq 0$, $u'''' \leq 0$. Then $k_1^* \geq k_2^*$ for any $X_1 \succeq_{TSD} X_2$ if and only if $\phi'(z) \geq 0$, $\phi''(z) \leq 0$, $\phi'''(z) \geq 0$. Here $\phi(z) = u'(z)z$.

Proof. The proof follows from the same reasoning as used in the proof of Theorems 2.1 and 2.2 of Hadar and Seo [4].

Sufficiency. We define that $\eta(k) = \frac{dE[u(kX_1+(1-k)Y)]}{dk} - \frac{dE[u(kX_2+(1-k)Y)]}{dk}$. Then $\eta(k)$ is written as

$$\begin{aligned}\eta(k) &= \int_0^b \left(\int_0^b u'(kx + (1-k)y)(x-y)d(F_1(x) - F_2(x))\right) dG(y) \\ &= \int_0^b \left(\int_0^b \psi(x)d(F_1(x) - F_2(x))\right) dG(y). \qquad (11)\end{aligned}$$

Here $\psi(x : y, k) = u'(kx + (1-k)y)(x-y)$. From the assumptions, ϕ is nondecreasing in z and $u'' \leq 0$. Hence, from Lemma 2.1 of Hadar and Seo [4], ψ is nondecreasing in x for all

$y \geq 0$ and $0 < k < 1$. From the assumptions, ϕ is concave in z and $u''' \geq 0$. Hence, from Lemma 2.1 of Hadar and Seo [4], ψ is concave in x for all $y \geq 0$ and $0 < k < 1$. Further

$$\psi_{xxx}(x) = k^2[\phi'''(kx + (1-k)y) - yu''''(kx + (1-k)y)]. \tag{12}$$

As $\phi'''(z) \geq 0$ and $u'''' \leq 0$, so $\psi_{xxx} \geq 0$ for all $y \geq 0$ and $0 < k < 1$. Therefore the assumption that $X_1 \succeq_{TSD} X_2$ implies $\eta(k) \geq 0$ for all k. Thus $\eta(k_2^*) \geq 0$, which implies $k_1^* \geq k_2^*$.

Necessity. Since FSD $\Rightarrow$ TSD, the necessity part of the proof of Hadar and Seo [4] (Theorem 2.1) implies TSD $\Rightarrow \phi'(z) \geq 0$. Similarly, since MPC $\Rightarrow$ SSD $\Rightarrow$ TSD, the necessity part of the proof of Hadar and Seo [4] (Theorem 2.2) implies that TSD $\Rightarrow \phi''(z) \leq 0$.

In the following, we prove that $\phi'''(z) \geq 0$. We consider the following distributions of X_2, X_1, and Y.

	p	p	p	p	$1-4p$
X_2, Y	α_2	β_2	γ_2	δ_2	0
X_1	α_1	β_1	γ_1	δ_1	0

Here X_2 and Y are identically and independently distributed, and $0 < \alpha_2 < \alpha_1 < \beta_1 < \beta_2 < \gamma_1 < \gamma_2 < \delta_2 < \delta_1$, $\alpha_1 - \alpha_2 = \beta_2 - \beta_1 = \gamma_2 - \gamma_1 = \delta_1 - \delta_2$ and $\alpha_1 - \beta_1 = \gamma_2 - \delta_2$. By construction, $X_1 \succeq_{TSD} X_2$. It is noted that $E[X_1] = E[X_2]$. Since $k_1^* \geq k_2^*$, $E[u'((X_1+Y)/2)(X_1 - Y)] \geq 0$. Hence

$$\begin{aligned}
&\lim_{p\to 0} \frac{1}{p(1-4p)} E[u'((X_1+Y)/2)(X_1 - Y)] \\
&= u'(\alpha_1/2)\alpha_1 - u'(\alpha_2/2)\alpha_2 + u'(\beta_1/2)\beta_1 - u'(\beta_2/2)\beta_2 \\
&\quad + u'(\gamma_1/2)\gamma_1 - u'(\gamma_2/2)\gamma_2 + u'(\delta_1/2)\delta_1 - u'(\delta_2/2)\delta_2 \\
&\geq 0.
\end{aligned} \tag{13}$$

Therefore

$$\frac{\phi(\alpha_1/2) - \phi(\alpha_2/2)}{\alpha_1 - \alpha_2} - \frac{\phi(\gamma_2/2) - \phi(\gamma_1/2)}{\gamma_2 - \gamma_1} \geq \frac{\phi(\beta_2/2) - \phi(\beta_1/2)}{\beta_2 - \beta_1} - \frac{\phi(\delta_1/2) - \phi(\delta_2/2)}{\delta_1 - \delta_2}. \tag{14}$$

In (14), we let $\alpha_2 \to \alpha_1$, $\beta_2 \to \beta_1$, $\gamma_1 \to \gamma_2$, and $\delta_1 \to \delta_2$. Then we have

$$\phi'(\alpha_1/2) - \phi'(\gamma_2/2) \geq \phi'(\beta_1/2) - \phi'(\delta_2/2). \tag{15}$$

That is,

$$\phi'(\delta_2/2) - \phi'(\gamma_2/2) \geq \phi'(\beta_1/2) - \phi'(\alpha_1/2). \tag{16}$$

So that

$$\frac{\phi'(\delta_2/2) - \phi'(\gamma_2/2)}{\delta_2 - \gamma_2} \geq \frac{\phi'(\beta_1/2) - \phi'(\alpha_1/2)}{\beta_1 - \alpha_1}. \tag{17}$$

Letting $\gamma_2 \to \delta_2$ and $\beta_1 \to \alpha_1$ with keeping the relation $\delta_2 - \gamma_2 = \beta_1 - \alpha_1$, we have

$$\phi''(\delta_2/2) \geq \phi''(\alpha_1/2), \tag{18}$$

for all $\alpha_1 < \delta_2$. Hence $\phi'''(z) \geq 0$. This proves the theorem. □

Remark.

We denote, respectively, the absolute risk-aversion function and the relative risk-aversion function by $R_A(z)$ and $R_R(z)$ if these functions can be defined. Then, according to Hadar and Seo [4], $\phi'(z) \geq 0$ is equivalent to the statement that $[R_R(z+b) - bR_A(z+b)] \leq 1$ for all nonnegative z and b, which is also equivalent to the statement that $R_R(z) \leq 1$ for all $z \geq 0$. They also indicate that $\phi''(z) \leq 0$ is equivalent to the statement that $R_A(z)[R_R(z) - 1] \leq R'_R(z)$ for all z, which is also equivalent to the statement that $R_A(z)[R_R(z) - 2] \leq zR'_A(z)$ for all z.

We now denote the prudence function and the relative prudence function, respectively, by $P_A(z) \equiv -\frac{u'''(z)}{u''(z)}$ and $P_R(z) \equiv -z\frac{u'''(z)}{u''(z)}$ if these functions can be defined.[6] Then $\phi'''(z) \geq 0$ is equivalent to the statement that $P_A(z)[P_R(z) - 2] \leq P'_R(z)$ for all z, which is also equivalent to the statement that $P_A(z)[P_R(z) - 3] \leq zP'_A(z)$ for all z.

In the next section, we discuss related deterministic transformations.

4. The TSD transformation

Let us consider any random variable X contained in $[0, b]$. Let $F(x)$ be the distribution function of X. Ormiston [13] considers two kinds of deterministic transformations, that is, the FSD transformation and the SSD transformation. Let the transformation $t(x)$ be nondecreasing, continuous, and piecewise differentiable. Then the definitions of the FSD transformation and the SSD transformation are as follows:

Definition 4.1. (the FSD transformation) The random variable given by the transformation $t(X)$ first-degree stochastically dominates (FSD) X if the function $m(x) \equiv t(x) - x$ satisfies $m(x) \geq 0$ for all $x \in [0, b]$.

Definition 4.2. (the SSD transformation) The random variable given by the transformation $t(X)$ second-degree stochastically dominates (SSD) X if the function $m(x) \equiv t(x) - x$ satisfies $\int_0^x m(y)dF(y) \geq 0$ for all $x \in [0, b]$.

We hereafter assume that $t(x)$ is concave. We proceed one step further and define the TSD transformation as follows:

Definition 4.3. (the TSD transformation) The random variable given by the transformation $t(X)$ third-degree stochastically dominates (TSD) X if the function $m(x) \equiv t(x) - x$ satisfies the following conditions:

$$\text{(i)} \quad \int_0^b m(x)dF(x) \geq 0,$$

$$\text{(ii)} \quad \int_0^x \left(\int_0^y m(z)dF(z) \right) dy \geq 0,$$

for all $x \in [0, b]$.

The condition (i) states that the expectation of the random variable X after transformation is larger than or equal to that before transformation. Note that the definitions of the FSD transformation and the SSD transformation do not include assumptions concerning expectations, but the definition of the TSD transformation includes the one.

It can be shown that for nondecreasing $t(x)$, $m(x)$ satisfies the conditions in Definition 4.1 (in Definition 4.2, respectively) if and only if $t(X) \succeq_{FSD} X$ ($t(X) \succeq_{SSD} X$). See Theorem 1 and Theorem 2 of Meyer [11]. The reason why nondecreasing concave $t(x)$ is called the

[6]As for these functions, see Kimball [6].

TSD transformation in Definition 4.3 is similar. If, for nondecreasing concave $t(x)$, $m(x)$ satisfies the conditions in Definition 4.3, then it can be shown that $t(X) \succeq_{TSD} X$.[7]

In the following, we consider the comparative statics of the TSD transformation. First, we consider the same economic decision problem as studied in Section 3 in Ormiston [13]. Let k^* be the value of k which maximizes

$$E[u(z)] = \int_0^b u(z)dF(x). \tag{19}$$

Here $z \equiv z(k,x)$, and $z_{kk} < 0$. We assume that the utility function is four times continuously differentiable with $u'(z) \geq 0$ and $u''(z) \leq 0$. Also we assume that interior solutions to (19) exist. According to Ormiston [13], one condition which guarantees this is that $z_k = 0$ is satisfied for some finite k for all $x \in [0,b]$.[8] Ormiston [13] defines $\eta(k,\theta)$ by

$$\eta(k,\theta) = \int_0^b \phi(k, x + \theta m(x))dF(x), \tag{20}$$

where $0 \leq \theta \leq 1$ and $\phi \equiv u'(z)z_k$. Then $k = k(\theta)$ is defined by $\eta(k,\theta) = 0$. According to Ormiston [13], the sign of $\eta_\theta(k^*,0)$ is important. Concretely $\eta_\theta(k^*,0) \geq 0$ (≤ 0) implies $k'(0) \geq 0$ (≤ 0), which then implies that the optimal value of the choice variable increases (decreases).[9]

As for the TSD transformation, we have the following result.

Theorem 4.1. The optimal value of the choice variable increases (decreases) for any TSD transformation if $\phi_x(k,x) \geq 0$ (≤ 0), $\phi_{xx}(k,x) \leq 0$ (≥ 0), and $\phi_{xxx}(k,x) \geq 0$ (≤ 0) for all k and x.[10]

Proof. $\eta_\theta(k^*,0)$ is written as

$$\eta_\theta(k^*,0) = \int_0^b \phi_x(k^*,x)m(x)dF(x). \tag{21}$$

Integrating the RHS of (21) by parts twice gives

$$\begin{aligned} \eta_\theta(k^*,0) &= \phi_x(k^*,b)\int_0^b m(x)dF(x) - \phi_{xx}(k^*,b)\int_0^b \left(\int_0^x m(y)dF(y)\right)dx \\ &\quad + \int_0^b \phi_{xxx}(k^*,x)\left[\int_0^x \left(\int_0^y m(z)dF(z)\right)dy\right]dx. \end{aligned} \tag{22}$$

Since the TSD transformaiton implies that $\int_0^b m(x)dF(x) \geq 0$ and $\int_0^x(\int_0^y m(z)dF(z))dy \geq 0$ for all $x \in [0,b]$, it follows that $\phi_x \geq 0$ (≤ 0), $\phi_{xx} \leq 0$ (≥ 0), and $\phi_{xxx} \geq 0$ (≤ 0) imply $\eta_\theta(k^*,0) \geq 0$ (≤ 0). Hence the optimal value of the choice variable increases (decreases). This proves the theorem. □

This theorem indicates sufficient conditions concerning the utility function for the optimal value of the choice variable k to unambiguously increase (decrease) for the TSD transformation of risk.

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[7]The proof can be done similarly as in the case of the SSD transformation. See the proof of sufficiency in Theorem 2 of Meyer [11]. We need the assumption of concavity of $t(x)$ and have not succeeded in proving the converse.

[8]If we require $m(x)$ to be a monotonic function, then it may be called the simple TSD transformation. We don't consider the comparative statics of the simple TSD transformation in this paper.

[9]Hereafter the terms "increase" and "decrease" are used in the nonstrict sense.

[10]The concavity of $t(x)$ is not required to prove this theorem. Thus this theorem in fact holds with regard to a weaker ordering than the TSD transformation. This also holds with regard to Theorem 4.2.

Next, we examine the effect of the TSD transformation of future income on consumption in the case of Sandmo's two-period consumption-investment problem [15]. The support of future income is assumed to be contained in $[0, b]$. Then, in his model, the expected utility is written as

$$\begin{aligned} E[u] &= \int_0^b u(C_1, C_2)dF(Y_2) \\ &= \int_0^b u(C_1, Y_2 + (Y_1 - C_1)(1+r))dF(Y_2), \end{aligned} \tag{23}$$

where r is the rate of interest. C_1 is the first-period consumption, and C_2 is the second-period consumption. Y_1 is the fixed first-period income, and Y_2 is the uncertain second-period income whose distribution function is given by $F(Y_2)$. The utility function u is assumed to be four times continuously differentiable. Also we assume that u is strictly increasing with respect to both arguments.

We wish to maximize $E[u]$ with respect to C_1. Sandmo [15] shows the first-order condition, the second-order condition, and the expression of $\frac{\partial C_1}{\partial Y_1}$. These are as follows:
The first-order condition is

$$E[u_1 - (1+r)u_2] = 0. \tag{24}$$

The second-order condition is

$$D = E[u_{11} - 2(1+r)u_{12} + (1+r)^2 u_{22}] < 0. \tag{25}$$

From (24), it is seen that

$$\frac{\partial C_1}{\partial Y_1} = -(1+r)E[u_{12} - (1+r)u_{22}]/D. \tag{26}$$

We assume that $\frac{\partial C_1}{\partial Y_1} > 0$ under certainty as well as under uncertainty as Sandmo [15] does. According to Sandmo [15], this assumption implies that $u_{12} - (1+r)u_{22} > 0$ and $E[u_{12} - (1+r)u_{22}] > 0$. As for the TSD transformation concerning future income Y_2, we have the following comparative statics.

Theorem 4.2. Assume a) the absolute risk-aversion function, $-\frac{u_{22}}{u_2}$, is decreasing in C_2 and increasing in C_1; b) the absolute prudence function, $-\frac{u_{222}}{u_{22}}$, is decreasing in C_2 and increasing in C_1; c) as indicated above, $\frac{\partial C_1}{\partial Y_1} > 0$ under certainty as well as under uncertainty; d) $u_{22} < 0$. Then the first-period consumption increases for any TSD transformation concerning future income Y_2.

Proof. Define $\eta(C_1, \theta)$ by

$$\begin{aligned} \eta(C_1, \theta) &= E[u_1(C_1, Y_2 + \theta m(Y_2) + (Y_1 - C_1)(1+r)) \\ &\quad -(1+r)u_2(C_1, Y_2 + \theta m(Y_2) + (Y_1 - C_1)(1+r))]. \end{aligned} \tag{27}$$

From (27), $C_1 = C_1(\theta)$ is defined by $\eta(C_1, \theta) = 0$. Then $C_1'(0)$ is written as

$$C_1'(0) = \frac{-\eta_\theta(C_1^*, 0)}{\eta_{C_1}(C_1^*, 0)}. \tag{28}$$

Now

$$\eta_\theta(C_1, \theta) = E[(u_{12} - (1+r)u_{22})m(Y_2)]. \tag{29}$$

Therefore

$$\begin{aligned}
\eta_\theta(C_1^*,0) &= \int_0^b [u_{12}(C_1^*,Y_2+(Y_1-C_1^*)(1+r)) \\
&\quad -(1+r)u_{22}(C_1^*,Y_2+(Y_1-C_1^*)(1+r))]m(Y_2)dF(Y_2) \\
&= [u_{12}(C_1^*,b+(Y_1-C_1^*)(1+r)) \\
&\quad -(1+r)u_{22}(C_1^*,b+(Y_1-C_1^*)(1+r))]\int_0^b m(y_2)dF(y_2) \\
&\quad -\int_0^b\left[(u_{122}-(1+r)u_{222})\left(\int_0^{y_2} m(x)dF(x)\right)\right]dy_2, \qquad (30)
\end{aligned}$$

where C_1^* is the optimum value of C_1 which maximizes $E[u]$. Integrating the second term of the RHS of (30) by parts gives

$$\begin{aligned}
\eta_\theta(C_1^*,0) &= (u_{12}-(1+r)u_{22})\int_0^b m(y_2)dF(y_2) \\
&\quad -\left[(u_{122}-(1+r)u_{222})\left(\int_0^{y_2}\left(\int_0^x m(y)dF(y)\right)dx\right)\right]_0^b \\
&\quad +\int_0^b\left[\frac{\partial(u_{122}-(1+r)u_{222})}{\partial Y_2}\left(\int_0^{y_2}\left(\int_0^x m(y)dF(y)\right)dx\right)\right]dy_2. \qquad (31)
\end{aligned}$$

In the following, we prove that $\frac{\partial(u_{122}-(1+r)u_{222})}{\partial Y_2}$ is nonnegative. Since the absolute prudence function is assumed to be decreasing in C_2 and increasing in C_1, it is seen that

$$\frac{\partial}{\partial C_2}\left(\frac{u_{122}-(1+r)u_{222}}{u_{22}}\right)\le 0, \qquad (32)$$

where $C_2=Y_2+(Y_1-C_1)(1+r)$.[11]
On the other hand,

$$\begin{aligned}
\frac{\partial(u_{122}-(1+r)u_{222})}{\partial Y_2} &= \frac{\partial}{\partial C_2}\left(\frac{u_{122}-(1+r)u_{222}}{u_{22}}\cdot u_{22}\right) \\
&= \frac{\partial}{\partial C_2}\left(\frac{u_{122}-(1+r)u_{222}}{u_{22}}\right)\cdot u_{22}+\frac{u_{122}-(1+r)u_{222}}{u_{22}}\cdot u_{222}. \qquad (33)
\end{aligned}$$

The concavity of u with respect to C_2 and (32) give the first term of (33) is nonnegative. DARA of u with respect to C_2 implies $u_{222}\le 0$. Thus, if we show $u_{122}-(1+r)u_{222}\ge 0$, (33) is seen to be nonnegative. Now

$$u_{122}-(1+r)u_{222}=\frac{\partial(u_{12}-(1+r)u_{22})}{\partial C_2}=\frac{\partial}{\partial C_2}\left(\frac{u_{12}-(1+r)u_{22}}{u_2}\cdot u_2\right). \qquad (34)$$

As shown in Sandmo [15] and stated in the footnote 11, the assumption that a) the absolute risk-aversion function is decreasing in C_2 and increasing in C_1 implies $\frac{\partial}{\partial C_2}\left(\frac{u_{12}-(1+r)u_{22}}{u_2}\right)\le 0$. This result with $u_{22}<0$ and $u_{12}-(1+r)u_{22}>0$ gives $u_{122}-(1+r)u_{222}\le 0$. Thus $\frac{\partial(u_{122}-(1+r)u_{222})}{\partial Y_2}$ is shown to be nonnegative.

[11] This inequality can be derived along the lines of the proof in Sandmo [15] of the fact that the absolute risk-aversion function which is decreasing in C_2 and increasing in C_1 implies $\frac{\partial}{\partial C_2}\left(\frac{u_{12}-(1+r)u_{22}}{u_2}\right)\le 0$.

From c) and the definition of the TSD transformation, the first term of the RHS of (31) is nonnegative. These results with the assumption of the TSD transformation indicate that the second term and the third term of the RHS of (31) are nonnegative. Thus $\eta_\theta(C_1^*, 0)$ is nonnegative, which implies $C_1'(0) \geq 0$. Therefore the first-period consumption increases for any TSD transformation. This proves the theorem. □

The assumption a) is proposed by Sandmo [15]. He calls this hypothesis the hypothesis of decreasing temporal risk aversion. The assumption c) also appears in Sandmo [15]. He considers a combination of an additive shift and a multiplicative shift concerning future income Y_2. Then he shows that under assumptions a), c) and d), increased uncertainty of this type concerning future income decreases the first period consumption. This theorem indicates that in the case of the TSD transformation, if we add the assumption b) to the assumptions of Sandmo [15], we have the same result as Sandmo [15].

5. Conclusion

In this paper, we consider shifts in risk and their comparative statics. In section 2, we take up the stochastic order which apears in Landsberger and Meilijson [9] and also in Eeckhoudt and Gollier [2]. We call this stochastic order the MGPR order in this paper. We examine the comparative statics of the MGPR order in the economic model used by Eeckhoudt and Gollier [2]. That is, for the utility function u which satisfies $u' \geq 0$, $u'' \leq 0$, and $u''' \geq 0$, we give the sufficient condition concerning the utility function for an MGPR shift of risk to unambiguously increase the optimal exposure to the risk. It should be noted that we only require restrictions on the utility function. When $E[X_1] = E[X_2]$, for the utility function u which satisfies $u' \geq 0$, $u'' \leq 0$, and $u''' \geq 0$, we show that an MGPR shift unambiguously increases the optimal exposure to the risk. In section 3, we consider the TSD shift. In the case of portfolios with two stochastically independent risky assets, we provide a necessary and sufficient condition concerning the utility function which satisfies $u' \geq 0$, $u'' \leq 0$, $u''' \geq 0$, and $u'''' \leq 0$ for a TSD shift of one asset to cause an unambiguous increase in the investment in that asset. Meyer and Ormiston [12] extend the analysis of Hadar and Seo [4] to the dependent case. It would be possible to extend our analysis to the dependent case. Since, in Kijima and Ohnishi [5], some results obtained by Hadar and Seo [4] are proved by using the method of bivariate characterization, this case may also be analyzed effectively by using this method. Moreover it would be interesting to consider the TSD order more generally from the viewpoint of bivariate characterization.

In section 4, we discuss the TSD transformation. In a model similar to that used by Meyer and Ormiston [10], we show sufficient conditions concerning the utility function for the optimal value of the choice variable to unambiguously increase (decrease) for the TSD transformation of risk. Further, in two-period consumption-investment model, we give sufficient conditions for a TSD transformation concerning future income to unambiguously increase the first-period consumption. It is left to future research to consider the effects of the MPR shift in risk, the MGPR shift in risk, and the TSD shift in risk in this two-period consumption-investment model.

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