Journal of the Operations Research
Society of Japan
Vol. 40, No. 3, September 1997

BALANCED BISUBMODULAR SYSTEMS AND BIDIRECTED FLOWS

Kazutoshi Ando
University of Tsukuba

Satoru Fujishige
Osaka University

Takeshi Naitoh
Shiga University

(Received July 31, 1996; Revised November 5, 1996)

Abstract For a nonempty finite set V let 3^V be the set of all the ordered pairs of disjoint subsets of V, i.e., $3^V = \{(X, Y) \mid X, Y \subseteq V, X \cap Y = \emptyset\}$. We define two operations, reduced union $\sqcup$ and intersection $\sqcap$, on 3^V as follows: for each $(X_i, Y_i) \in 3^V$ $(i = 1, 2)$

$$\begin{aligned}(X_1, Y_1) \sqcup (X_2, Y_2) &= ((X_1 \cup X_2) - (Y_1 \cup Y_2), (Y_1 \cup Y_2) - (X_1 \cup X_2)),\\ (X_1, Y_1) \sqcap (X_2, Y_2) &= (X_1 \cap X_2, Y_1 \cap Y_2).\end{aligned}$$

Also, for a $\{\sqcup, \sqcap\}$-closed family $\mathcal{F} \subseteq 3^V$ a function $f : \mathcal{F} \to \mathbf{R}$ is called bisubmodular if for each $(X_i, Y_i) \in \mathcal{F}$ $(i = 1, 2)$ we have

$$f(X_1, Y_1) + f(X_2, Y_2) \geq f((X_1, Y_1) \sqcup (X_2, Y_2)) + f((X_1, Y_1) \sqcap (X_2, Y_2)).$$

For a $\{\sqcup, \sqcap\}$-closed family $\mathcal{F} \subseteq 3^V$ with $(\emptyset, \emptyset) \in \mathcal{F}$ and a so-called bisubmodular function $f : \mathcal{F} \to \mathbf{R}$ on $\mathcal{F}$ with $f(\emptyset, \emptyset) = 0$, the pair $(\mathcal{F}, f)$ is called a bisubmodular system on V.

In this paper we consider two classes of bisubmodular systems which are closely related to base polyhedra. The first one is the class of balanced bisubmodular systems. We give a characterization of balanced bisubmodular systems and show that their associated polyhedra are the convex hulls of reflections of base polyhedra. The second one is that of cut functions of bidirected networks. It is shown that the polyhedron determined by the cut function of a bidirected network is the set of the boundaries of flows in the bidirected network and is a projection of a section of a base polyhedron of boundaries of an associated ordinary network.

1. Introduction

For a nonempty finite set V let us consider a family $\mathcal{F} \subseteq 3^V \equiv \{(X, Y) | X, Y \subseteq V, X \cap Y = \emptyset\}$ which is closed with respect to the operations, *reduced union* $\sqcup$ and *intersection* $\sqcap$, defined as

$$(X_1, Y_1) \sqcup (X_2, Y_2) = ((X_1 \cup X_2) - (Y_1 \cup Y_2), (Y_1 \cup Y_2) - (X_1 \cup X_2)), \tag{1.1}$$
$$(X_1, Y_1) \sqcap (X_2, Y_2) = (X_1 \cap X_2, Y_1 \cap Y_2) \tag{1.2}$$

for each $(X_1, Y_1), (X_2, Y_2) \in 3^V$. Generalizing a theorem of Birkhoff [8] on representing a distributive lattice by a poset, Ando and Fujishige [2] showed that any such $\{\sqcup, \sqcap\}$-closed family $\mathcal{F}$ is represented as the set of ideals of a bidirected graph (see Theorem 2.1 in Section 2). The notions of bidirected graph, ideal etc. will be precisely defined later. A $\{\sqcup, \sqcap\}$-closed family is sometimes called a signed ring family.

For a $\{\sqcup, \sqcap\}$-closed family $\mathcal{F} \subseteq 3^V$ a function $f : \mathcal{F} \to \mathbf{R}$ is called *bisubmodular* if for each $(X_1, Y_1), (X_2, Y_2) \in \mathcal{F}$ we have

$$f(X_1, Y_1) + f(X_2, Y_2) \geq f((X_1, Y_1) \sqcup (X_2, Y_2)) + f((X_1, Y_1) \sqcap (X_2, Y_2)). \tag{1.3}$$

We call the pair $(\mathcal{F}, f)$ of a $\{\sqcup, \sqcap\}$-closed family $\mathcal{F} \subseteq 3^V$ with $(\emptyset, \emptyset) \in \mathcal{F}$ and a bisubmodular function $f: \mathcal{F} \to \mathbf{R}$ on $\mathcal{F}$ with $f(\emptyset, \emptyset) = 0$ a *bisubmodular system* on V. For a bisubmodular system $(\mathcal{F}, f)$ on V we associate a polyhedron $\mathrm{P}_*(f)$ defined as

$$\mathrm{P}_*(f) = \{x \mid x \in \mathbf{R}^V, \forall (X, Y) \in \mathcal{F}: x(X, Y) \leq f(X, Y)\}, \tag{1.4}$$

where $x(X,Y) = x(X) - x(Y)$ for each $(X, Y) \in 3^V$. We call $\mathrm{P}_*(f)$ the *bisubmodular polyhedron* associated with $(\mathcal{F}, f)$. When $\mathcal{F} = 3^V$, bisubmodular systems appear as a polypseudomatroid [11], a delta-matroid [9], a metroid [13], a ditroid [21], a universal polymatroid [20] and a special case of a jump system [10]. Bisubmodular polyhedra are also characterized as the polyhedra for which certain variant of *greedy algorithm* works, as was shown by Chandrasekaran and Kabadi [11]. Indeed, the class of bisubmodular polyhedra was first investigated as the class of such greedy polyhedra by Dunstan and Welsh [14]. A characterization of bisubmodular functions was given in [5]. General structures of bisubmodular polyhedra were studied in some depth in [3], based on the signed Birkhoff theorem.

In this paper we consider two special classes of bisubmodular systems, namely, *balanced* bisubmodular systems and bisubmodular systems associated with *cut functions* of bidirected networks. As we will see later, these two subclasses have a common feature: the bisubmodular polyhedra associated with these bisubmodular systems are represented in terms of base polyhedra.

In Section 2 we describe elementary notions of $\{\sqcup, \sqcap\}$-closed family, bisubmodular system and bidirected graph and give fundamental results to be used later.

In Section 3 we introduce the concept of balanced bisubmodular system. We give a characterization of balanced bisubmodular systems and it will turn out that the possibly unbounded bisubmodular polyhedron associated with such a balanced bisubmodular system is the convex hull of reflections of some base polyhedra.

In Section 4 we investigate cut functions of bidirected networks and associated polyhedra, where the reduction technique used in [6], [7] and [16] is fully exploited. We will see, as is the case for cut functions of ordinary directed networks, that the polyhedron determined by the cut function of a bidirected network is the set of the boundaries of flows in the bidirected network. Also, it will be shown that the polyhedron determined by the cut function of any bidirected network is a projection of the intersection of a base polyhedron and a subspace. Results obtained in Section 4 give a foundation for investigation of structures of bisubmodular polyhedra (see [3]).

It should be noted here that an interesting application of the bidirected network flow problem has recently been reported by Möhring, Müller-Hannemann and Weihe [19]. They considered the mesh refinement problem arising in a computer-aided design, where a desired mesh refinement was made by solving a sequence of bidirected flow problems.

This article is based on [4] and some parts of it are taken from K. Ando's dissertation [1].

2. Definitions and Preliminaries

We give basic definitions and some preliminaries to be employed in the subsequent sections.

Throughout this paper V denotes a nonempty finite set. $\mathbf{R}$, $\mathbf{R}_+$ and $\mathbf{Z}$, respectively, stand for the set of reals, the set of nonnegative reals and the set of integers. For any vector $x \in \mathbf{R}^V$ and any subset $U \subseteq V$ we define the *reflection* $x:U$ of x by U as

$$(x:U)(v) = \begin{cases} -x(v) & \text{if } v \in U \\ x(v) & \text{otherwise} \end{cases} \qquad (v \in V). \tag{2.1}$$

Also, for any $Q \subseteq \mathbf{R}^V$ we define the *reflection* $Q\!:\!U$ of Q by U as

$$Q\!:\!U = \{x\!:\!U \mid x \in Q\}. \tag{2.2}$$

2.1. $\{\sqcup, \sqcap\}$-closed families and bidirected graphs

Each element $(X, Y) \in 3^V$ is naturally made correspond to a $\{0, \pm 1\}$-vector $\chi_{(X,Y)}$, called *the characteristic vector* of (X, Y), as

$$\chi_{(X,Y)}(v) = \begin{cases} 1 & \text{if } v \in X \\ -1 & \text{if } v \in Y \\ 0 & \text{otherwise} \end{cases} \qquad (v \in V) \tag{2.3}$$

and we call each element of 3^V a *signed subset* of V. The *support* of a signed subset (X, Y) of V is $X \cup Y$. We sometimes identify a signed subset with its characteristic vector. For example, for a signed subset (X, Y) of V and a subset U of V, $(X, Y)\!:\!U$ denotes the signed subset corresponding to $\{0, \pm 1\}$-vector $\chi_{(X,Y)}\!:\!U$. A partial order $\sqsubseteq$ on 3^V is defined by

$$(X_1, Y_1) \sqsubseteq (X_2, Y_2) \iff X_1 \subseteq X_2, Y_1 \subseteq Y_2 \quad ((X_1, Y_1), (X_2, Y_2) \in 3^V). \tag{2.4}$$

A family $\mathcal{F} \subseteq 3^V$ is called $\{\sqcup, \sqcap\}$*-closed* if it is closed under the two operations, reduced union $\sqcup$ and intersection $\sqcap$, defined by (1.1) and (1.2). It is known that if $\mathcal{F}$ is $\{\sqcup, \sqcap\}$-closed, any maximal element (maximal with respect to the partial order $\sqsubseteq$) has the same support, which we call the *support* of $\mathcal{F}$ and denote it by $\mathrm{Supp}(\mathcal{F})$. If we have $\mathrm{Supp}(\mathcal{F}) = V$, we say $\mathcal{F}$ *spans* V (or $\mathcal{F}$ is *spanning*). A $\{\sqcup, \sqcap\}$-closed family $\mathcal{F} \subseteq 3^V$ with $(\emptyset, \emptyset) \in \mathcal{F}$ is called *simple* if for each distinct $v, w \in \mathrm{Supp}(\mathcal{F})$ there exists a signed subset $(X, Y) \in \mathcal{F}$ separating v and w, i.e., $|\{v, w\} \cap (X \cup Y)| = 1$.

A bisubmodular system $(\mathcal{F}, f)$ on V is called *simple* (or *spanning*) if $\mathcal{F}$ is simple (or spanning). It was shown in [3] that a bisubmodular system $(\mathcal{F}, f)$ is simple and spanning if and only if $\mathrm{P}_*(f)$ is pointed.

A *bidirected graph* is a graph $G = (V, A; \partial)$ with a *boundary operator* $\partial\colon A \to \mathbf{Z}^V$ such that for each arc $a \in A$ there exist two vertices $v, w \in V$ (possibly identical except for (3) below) we have

$$\begin{array}{lll} (1) & \partial a = v + w & \text{(arc } a \text{ has two tails } v \text{ and } w\text{),} \\ (2) & \partial a = -v - w & \text{(arc } a \text{ has two heads } v \text{ and } w\text{) or} \\ (3) & \partial a = v - w & \text{(arc } a \text{ has one head } v \text{ and one tail } w\text{),} \end{array} \tag{2.5}$$

where we regard $\mathbf{Z}^V$ as a free module with base V (see [15]). Figure 2.1 shows an example of a bidirected graph. We say an arc $a \in A$ is *of type* (i) for some $i \in \{1, 2, 3\}$ according as (i) in (2.5) holds. If all the arcs of the bidirected graph are of type (3), we call it a *directed graph*. An *ideal* of a bidirected graph $G = (V, A; \partial)$ is a signed subset $(X, Y) \in 3^V$ such that for any $a \in A$ we have

$$\langle \partial a, \chi_{(X,Y)} \rangle \leq 0, \tag{2.6}$$

where $\langle \cdot, \cdot \rangle$ is the canonical inner product and ∂a should be regarded as a vector in $\mathbf{Z}^V$ (see [22]). We denote by $\mathcal{I}(G)$ the set of all the ideals of G.

We have the following representation theorem for $\{\sqcup, \sqcap\}$-closed families.

Theorem 2.1 (Signed Birkhoff Theorem [2] (see also [22])): *For any $\{\sqcup, \sqcap\}$-closed family $\mathcal{F}$ on V with $(\emptyset, \emptyset) \in \mathcal{F}$ there exists a bidirected graph $G = (V, A; \partial)$ such that $\mathcal{F} = \mathcal{I}(G)$.* □

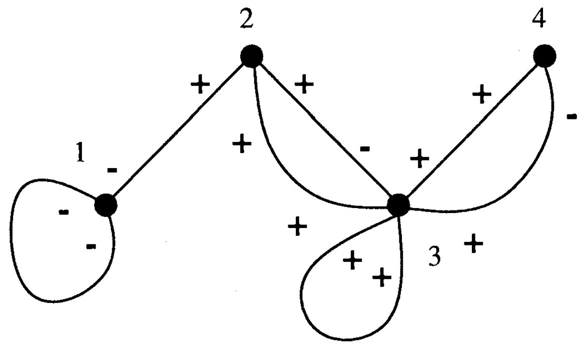

Figure 2.1: An example of a bidirected graph.

We call a pair $\mathcal{N} = (G = (V, A; \partial), c)$ of a bidirected graph $G = (V, A; \partial)$ and a function $c: A \rightarrow \mathbf{R}_+ \cup \{+\infty\}$ a *bidirected network*. The function c is called a *capacity* function. Given a bidirected network $\mathcal{N} = (G = (V, A; \partial), c)$ we define the following function $\hat{\kappa}_c : 3^V \rightarrow \mathbf{R}_+ \cup \{+\infty\}$ as

$$\hat{\kappa}_c(X, Y) = \sum\{\langle \partial a, \chi_{(X,Y)}\rangle c(a) \mid a \in A, \langle \partial a, \chi_{(X,Y)}\rangle > 0\}. \tag{2.7}$$

We call $\hat{\kappa}_c$ the *cut function* of $\mathcal{N}$. For a bidirected network $\mathcal{N} = (G = (V, A; \partial), c)$ a function $\varphi : A \rightarrow \mathbf{R}$ is called a *(feasible) flow* in $\mathcal{N}$ if $0 \leq \varphi(a) \leq c(a)$ for all $a \in A$. Then, the *boundary* $\partial\varphi : V \rightarrow \mathbf{R}$ of a flow φ is given by

$$\partial\varphi = \sum\{\varphi(a)\partial a \mid a \in A\}, \tag{2.8}$$

where ∂a is regarded as a vector in $\mathbf{R}^V$. Let us denote the set of all the boundaries of bidirected flows in $\mathcal{N}$ by $\partial\Phi$, i.e.,

$$\partial\Phi = \{\partial\varphi \mid \varphi \text{ is a feasible flow in } \mathcal{N}\}. \tag{2.9}$$

2.2. Submodular systems and bisubmodular systems

We call a family $\mathcal{D} \subseteq 2^V$ of subsets of V a *distributive lattice* if it is closed under set union $\cup$ and intersection $\cap$. A function $f : 2^V \rightarrow \mathbf{R} \cup \{+\infty\}$ is called *submodular* if for each $X, Y \in 2^V$ we have

$$f(X) + f(Y) \geq f(X \cup Y) + f(X \cap Y). \tag{2.10}$$

A pair $(\mathcal{D}, f)$ of a distributive lattice $\mathcal{D} \subseteq 2^V$ with $\emptyset, V \in \mathcal{D}$ and a submodular function $f : \mathcal{D} \rightarrow \mathbf{R}$ on $\mathcal{D}$ with $f(\emptyset) = 0$ is called a *submodular system* on V. The *submodular polyhedron* and the *base polyhedron* associated with $(\mathcal{D}, f)$ are, respectively, defined by

$$\mathrm{P}(f) = \{x \mid x \in \mathbf{R}^V, \forall X \in \mathcal{D}\colon x(X) \leq f(X)\}, \tag{2.11}$$

$$\mathrm{B}(f) = \{x \mid x \in \mathrm{P}(f), x(V) = f(V)\} \tag{2.12}$$

(see [17]).

A bidirected network $\mathcal{N} = (G = (V, A; \partial), c)$ is called a *directed network* if $G = (V, A; \partial)$ is a directed graph. For a directed network $\mathcal{N}$ the function $\kappa_c\colon 2^V \rightarrow \mathbf{R}_+ \cup \{+\infty\}$ defined as

$$\kappa_c(X) = \sum\{c(a) \mid a \in A, \langle \partial a, \chi_X\rangle > 0\} \quad (X \in 2^V) \tag{2.13}$$

is called the *cut function* of $\mathcal{N}$, where χ_X is the characteristic vector of set X defined by $\chi_X(v) = 1$ for $v \in X$ and $\chi_X(v) = 0$ for $v \in V - X$. It is well-known (see [17, Section 2.3]) that a cut function is a submodular function and the set $\partial\Phi$ of boundaries of feasible flows in $\mathcal{N}$ is given by

$$\partial\Phi = \mathrm{B}(\kappa_c), \tag{2.14}$$

where the right-hand side is the base polyhedron associated with submodular system $(\mathcal{D}, \kappa_c)$ on V with

$$\mathcal{D} = \{X \mid X \in 2^V, \kappa_c(X) < +\infty\}. \tag{2.15}$$

Let $(\mathcal{F}, f)$ be a bisubmodular system on V. We assume that $\mathcal{F}$ spans V. Then for each maximal element $(S,T) \in \mathcal{F}$ we have $S \cup T = V$. We call an element $(S,T) \in 3^V$ with $S \cup T = V$ an *orthant*. For each orthant $(S,T) \in \mathcal{F}$ define $\mathcal{F}^{(S,T)} \subseteq \mathcal{F}$ by

$$\mathcal{F}^{(S,T)} = \{(X,Y) \mid (X,Y) \in \mathcal{F}, (X,Y) \sqsubseteq (S,T)\}. \tag{2.16}$$

Note that $\mathcal{F}^{(S,T)}$ forms a distributive lattice with join $\sqcup$ and meet $\sqcap$ and that it has a unique maximal element (S,T) and a unique minimal element $(\emptyset, \emptyset)$.

Define for each orthant $(S,T) \in \mathcal{F}$

$$\mathrm{P}_{(S,T)}(f) = \{x \mid x \in \mathbf{R}^V, \forall (X,Y) \in \mathcal{F}^{(S,T)}: \ x(X,Y) \le f(X,Y)\}, \tag{2.17}$$

$$\mathrm{B}_{(S,T)}(f) = \{x \mid x \in \mathrm{P}_{(S,T)}(f), x(S,T) = f(S,T)\}. \tag{2.18}$$

We see that for any orthant $(S,T) \in \mathcal{F}$ the reflections $\mathrm{P}_{(S,T)}(f) : T$ and $\mathrm{B}_{(S,T)}(f) : T$ by T are, respectively, a submodular polyhedron and its corresponding base polyhedron (see [17]).

Since $\mathcal{F} = \bigcup\{\mathcal{F}^{(S,T)} \mid (S,T) \in \mathcal{F}, S \cup T = V\}$, the polyhedron $\mathrm{P}_*(f)$ of (1.4) is expressed as

$$\mathrm{P}_*(f) = \bigcap\{\mathrm{P}_{(S,T)}(f) \mid (S,T) \in \mathcal{F}, S \cup T = V\}. \tag{2.19}$$

We can show the following lemmas. We omit the proofs (cf. [17, Section 3.5(b)] and [2]).

Lemma 2.2: *For each orthant* $(S,T) \in \mathcal{F}$, $\mathrm{B}_{(S,T)}(f)$ *is a face of* $\mathrm{P}_*(f)$. □

Lemma 2.3: *For a simple and spanning bisubmodular system* $(\mathcal{F}, f)$ *on* V *we have*

$$\text{the set of extreme points of } \mathrm{P}_*(f) = \bigcup\{\text{the set of extreme points of } \mathrm{B}_{(S,T)} \mid (S,T) : \text{an orthant in } \mathcal{F}\}. \tag{2.20}$$

□

Lemma 2.3 follows from the greediness property of possibly unbounded bisubmodular polyhedra (see [3]).

3. Balanced Bisubmodular Systems

For a bidirected graph $G = (V, A; \partial)$ and a subset $U \subseteq V$ define the boundary operator $(\partial{:}U)\colon A \to \mathbf{Z}^V$ by

$$(\partial{:}U)(a) = (\partial a){:}U \quad (a \in A), \tag{3.1}$$

where we regard ∂a as a vector in $\mathbf{R}^V$. We call the bidirected graph $G{:}U = (V, A; \partial{:}U)$ the *reflection* of G by U. It should be noted here that (X,Y) is an ideal of G if and only if $(X,Y){:}U$ is an ideal of $G{:}U$.

A bidirected graph $G = (V, A; \partial)$ is said to be *balanced* if there exists a subset U of V such that the reflection $G{:}U$ of G by U is an ordinary directed graph (see [18]).

The balancedness is characterized in terms of ideals as follows.

Theorem 3.1: *A bidirected graph $G = (V, A; \partial)$ is balanced if and only if there exists an orthant $(S,T) \in 3^V$ such that both (S,T) and (T,S) are ideals of G.*
(Proof) If $G:U$ is an ordinary directed graph for some $U \subseteq V$, then, $(\emptyset, V)$ and $(V, \emptyset)$ are ideals of $G:U$. Hence, both $(U, V-U)$ and $(V-U, U)$ are ideals of G.

Conversely, suppose that $(U, V-U)$ and $(V-U, U)$ are ideals of a bidirected graph G. Then, $(\emptyset, V)$ and $(V, \emptyset)$ are ideals of $G:U$. This is possible only if there is neither arc of type (1) nor of type (2) in $G:U$, i.e., $G:U$ is a directed graph. □

A $\{\sqcup, \sqcap\}$-closed family $\mathcal{F}$ is called *balanced* if there exists an orthant (S,T) such that $(S,T) \in \mathcal{F}$ and $(T,S) \in \mathcal{F}$. We call a bisubmodular system $(\mathcal{F}, f)$ *balanced* if $\mathcal{F}$ is balanced.

Let $(\mathcal{F}, f)$ be a bisubmodular system and let $\mathcal{F} = \mathcal{I}(G)$ for some bidirected graph. If $\mathcal{F}$ is balanced, then by definition there exists an orthant $(S,T) \in 3^V$ such that $(S,T), (T,S) \in \mathcal{F}$. Then, for any bidirected graph $G = (V, A; \partial)$ representing $\mathcal{F}$ as $\mathcal{F} = \mathcal{I}(G)$, G must be balanced. Conversely, if $\mathcal{F}$ is represented as $\mathcal{F} = \mathcal{I}(G)$ for some balanced bidirected graph G, then by Theorem 3.1 we have $(S,T), (T,S) \in \mathcal{F}$ for some orthant $(S,T) \in 3^V$. Hence, $\mathcal{F}$ is balanced. Therefore, a bisubmodular system $(\mathcal{F}, f)$ is balanced if and only if $\mathcal{F}$ is represented by an ordinary directed graph with a possible reflection.

We also have another characterization of balancedness given below.

Theorem 3.2: *A bisubmodular system $(\mathcal{F}, f)$ on V is balanced if and only if for some orthant $(S,T) \in \mathcal{F}$ the characteristic cone of $\mathrm{B}_{(S,T)}(f)$ coincides with that of $\mathrm{P}_*(f)$.*
(Proof) First, note that for any bisubmodular system $(\mathcal{F}, f)$ on V and any orthant $(S,T) \in \mathcal{F}$ the characteristic cone of $\mathrm{B}_{(S,T)}(f)$ is a subset of the characteristic cone of $\mathrm{P}_*(f)$ due to Lemma 2.2.

The "only if" part: Suppose $(\mathcal{F}, f)$ is a balanced bisubmodular system on V. Then, by definition, there exists an orthant $(S,T) \in 3^V$ such that $(S,T), (T,S) \in \mathcal{F}$. Hence, for any vector x in the characteristic cone of $\mathrm{P}_*(f)$ we have $x(S,T) = 0$. Therefore, the characteristic cone of $\mathrm{P}_*(f)$ is a subset of the characteristic cone of $\mathrm{B}_{(S,T)}(f)$.

The "if" part: Conversely, suppose that a bisubmodular system $(\mathcal{F}, f)$ on V is not balanced. Let $G = (V, A; \partial)$ be a bidirected graph representing $\mathcal{F}$ as in Theorem 2.1. Then, for each orthant $(S,T) \in \mathcal{F} = \mathcal{I}(G)$ there is an arc $a_{(S,T)} \in A$ such that $\langle \partial a_{(S,T)}, \chi_{(T,S)} \rangle > 0$ since otherwise we would have $(T,S) \in \mathcal{F}$. This implies $\partial a_{(S,T)}$ is not in the characteristic cone of $\mathrm{B}_{(S,T)}(f)$. However, $\partial a_{(S,T)}$ is in the characteristic cone of $\mathrm{P}_*(f)$ by the definition of ideal. □

For any set $Q \subseteq \mathbf{R}^V$ let us denote the convex hull of Q by $\mathcal{H}(Q)$.

Now, we have the following theorem.

Theorem 3.3: *For a balanced bisubmodular system $(\mathcal{F}, f)$ on V we have*

$$\mathrm{P}_*(f) = \mathcal{H}(\bigcup\{\mathrm{B}_{(S,T)}(f) \mid (S,T) \in \mathcal{F}, S \cup T = V\}). \tag{3.2}$$

(Proof) It suffices to prove the theorem when $(\mathcal{F}, f)$ is simple (and spanning due to the balancedness of $(\mathcal{F}, f)$). This assumption guarantees that $\mathrm{P}_*(f)$ is pointed. From Lemma 2.2 we have

$$\mathrm{P}_*(f) \supseteq \mathcal{H}(\bigcup\{\mathrm{B}_{(S,T)}(f) \mid (S,T) \in \mathcal{F}, S \cup T = V\}). \tag{3.3}$$

The converse inclusion also holds by Lemma 2.3 and Theorem 3.2. □

It should be noted that (3.2) does not hold in general for non-balanced bisubmodular systems. For example, consider bisubmodular system $(\mathcal{F}, f)$ on $V = \{1, 2\}$, where

$$\mathcal{F} = \{(\emptyset, \emptyset), (\{1\}, \emptyset), (\{1,2\}, \emptyset), (\{2\}, \emptyset), (\{2\}, \{1\}), (\emptyset, \{1\})\}.$$

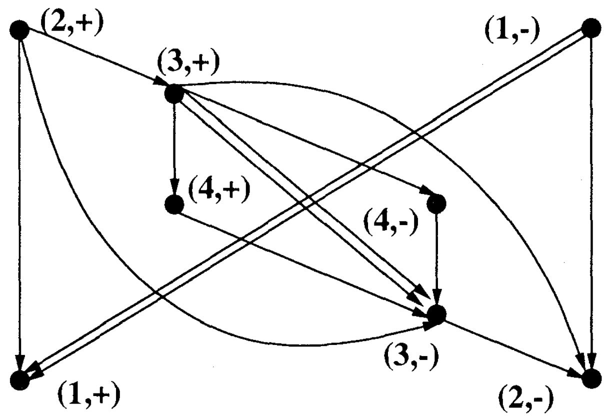

Figure 4.1: The signed covering graph of the bidirected graph in Figure 2.1.

Also, note that when $P_*(f)$ is bounded, i.e., $\mathcal{F} = 3^V$, bisubmodular system $(3^V, f)$ is balanced and we have (3.2) (as shown in [17]).

4. Flows in Bidirected Networks

For a bidirected graph $G = (V, A; \partial)$ we define an associated ordinary directed graph $\tilde{G} = (\tilde{V}, \tilde{A}; \tilde{\partial})$ as follows: $\tilde{V} = V \times \{+, -\}$, $\tilde{A} = A \times \{+, -\}$ and $\tilde{\partial}\colon \tilde{A} \to \mathbf{Z}^{\tilde{V}}$ is defined by

$$\begin{aligned} \tilde{\partial}(a^{(+)}) &= (v, \sigma) - (w, -\tau), \\ \tilde{\partial}(a^{(-)}) &= (w, \tau) - (v, -\sigma) \end{aligned} \tag{4.1}$$

for each $a \in A$ with $\partial a = \sigma v + \tau w$. The graph $\tilde{G}$ is called the *signed covering graph* of G ([23]). Figure 4.1 shows the signed covering graph of the bidirected graph in Figure 2.1.

With a bidirected network $\mathcal{N} = (G = (V, A; \partial), c)$ we associate a directed network $\tilde{\mathcal{N}} = (\tilde{G} = (\tilde{V}, \tilde{A}; \tilde{\partial}), \tilde{c})$ where $\tilde{G}$ is the signed covering graph of G and the capacity function $\tilde{c}\colon \tilde{A} \to \mathbf{R}_+ \cup \{+\infty\}$ is defined by

$$\tilde{c}(a^{(+)}) = \tilde{c}(a^{(-)}) = c(a) \quad (a \in A). \tag{4.2}$$

We call the directed network $\tilde{\mathcal{N}}$ the *signed covering network* of $\mathcal{N}$.

For each subset U of V we define $(U, +), (U, -) \subseteq \tilde{V}$ as

$$(U, +) = \{(v, +) \mid v \in U\}, \ (U, -) = \{(v, -) \mid v \in U\}. \tag{4.3}$$

The cut function $\hat{\kappa}_c : 3^V \to \mathbf{R}_+ \cup \{+\infty\}$ of a bidirected network $\mathcal{N} = (G = (V, A; \partial), c)$ can be represented in terms of the ordinary cut function of its associated signed covering network as follows.

Lemma 4.1: *Let $\mathcal{N} = (G = (V, A; \partial), c)$ be a bidirected network and $\tilde{\mathcal{N}} = (\tilde{G} = (\tilde{V}, \tilde{A}; \tilde{\partial}), \tilde{c})$ its signed covering network. Then we have*

$$\hat{\kappa}_c(X, Y) = \kappa_{\tilde{c}}((X, +) \cup (Y, -)) \tag{4.4}$$

for any $(X, Y) \in 3^V$.

(Proof) An elementary calculation yields (4.4). □

We obtain the bisubmodularity of the cut function $\hat{\kappa}_c$ from the submodularity of $\kappa_{\tilde{c}}$. We first show a preliminary lemma.

For a subset $\tilde{U}$ of $\tilde{V}$ define

$$\tilde{U}^* = \{(v, -\sigma) \mid (v, \sigma) \in \tilde{U}\}. \tag{4.5}$$

Lemma 4.2: *For any $\tilde{X} \subseteq \tilde{V}$ we have*

$$\kappa_{\tilde{c}}(\tilde{X}) \geq \kappa_{\tilde{c}}(\tilde{X} - \tilde{X}^*). \tag{4.6}$$

(Proof) For each $\tilde{X} \subseteq \tilde{V}$ we have

$$\kappa_{\tilde{c}}(\tilde{X}) = \kappa_{\tilde{c}}(\tilde{V} - \tilde{X}^*) \tag{4.7}$$

since the mapping defined by $\Delta^+\tilde{X} \ni a^{(\epsilon)} \mapsto a^{(-\epsilon)} \in \Delta^+(\tilde{V} - \tilde{X}^*)$ is a bijection and $\tilde{c}(a^{(\epsilon)}) = \tilde{c}(a^{(-\epsilon)})$ by definition, where $\Delta^+\tilde{Z}$ is the set of arcs leaving $\tilde{Z}$ for any $\tilde{Z} \subseteq \tilde{V}$, i.e., $\Delta^+\tilde{Z} = \{\tilde{a} \mid \tilde{a} \in \tilde{A}, \langle \tilde{\partial}\tilde{a}, \chi_{\tilde{Z}} \rangle > 0\}$.

Then from (4.7) and the submodularity of $\kappa_{\tilde{c}}$ we have

$$\begin{aligned} 2\kappa_{\tilde{c}}(\tilde{X}) &= \kappa_{\tilde{c}}(\tilde{X}) + \kappa_{\tilde{c}}(\tilde{V} - \tilde{X}^*) \\ &\geq \kappa_{\tilde{c}}(\tilde{X} - \tilde{X}^*) + \kappa_{\tilde{c}}(\tilde{V} - (\tilde{X}^* - \tilde{X})) \\ &= \kappa_{\tilde{c}}(\tilde{X} - \tilde{X}^*) + \kappa_{\tilde{c}}(\tilde{V} - (\tilde{X} - \tilde{X}^*)^*) \\ &= 2\kappa_{\tilde{c}}(\tilde{X} - \tilde{X}^*). \end{aligned} \tag{4.8}$$

□

Now, we have

Theorem 4.3: *For any bidirected network $\mathcal{N} = (G = (V, A; \partial), c)$ its cut function $\hat{\kappa}_c : 3^V \to \mathbf{R}_+ \cup \{+\infty\}$ is bisubmodular.*

(Proof) For any $(X_1, Y_1), (X_2, Y_2) \in 3^V$ we have from the submodularity of $\kappa_{\tilde{c}}$, Lemma 4.1 and Lemma 4.2 that

$$\begin{aligned} &\hat{\kappa}_c(X_1, Y_1) + \hat{\kappa}_c(X_2, Y_2) \\ &\quad = \kappa_{\tilde{c}}((X_1, +) \cup (Y_1, -)) + \kappa_{\tilde{c}}((X_2, +) \cup (Y_2, -)) \\ &\quad \geq \kappa_{\tilde{c}}((X_1 \cup X_2, +) \cup (Y_1 \cup Y_2, -)) + \kappa_{\tilde{c}}((X_1 \cap X_2, +) \cup (Y_1 \cap Y_2, -)) \\ &\quad \geq \kappa_{\tilde{c}}(((X_1 \cup X_2) - (Y_1 \cup Y_2), +) \cup ((Y_1 \cup Y_2) - (X_1 \cup X_2), -)) \\ &\qquad + \kappa_{\tilde{c}}((X_1 \cap X_2, +) \cup (Y_1 \cap Y_2, -)) \\ &\quad = \hat{\kappa}_c((X_1, Y_1) \sqcup (X_2, Y_2)) + \hat{\kappa}_c((X_1, Y_1) \sqcap (X_2, Y_2)). \end{aligned} \tag{4.9}$$

□

The bisubmodular polyhedron associated with the cut function of a bidirected network is related to the base polyhedron associated with the ordinary cut function of its signed covering network in an interesting way as the following theorem shows.

Theorem 4.4: *For any bidirected network $\mathcal{N} = (G = (V, A; \partial), c)$ we have*

$$\mathrm{P}_*(\hat{\kappa}_c) = \{x \mid x \in \mathbf{R}^V, \tilde{x} \in \mathrm{B}(\kappa_{\tilde{c}})\}, \tag{4.10}$$

where for each $x \in \mathbf{R}^V$ the vector $\tilde{x} \in \mathbf{R}^{\tilde{V}}$ is defined by

$$\tilde{x}(v, \sigma) = \sigma x(v) \quad ((v, \sigma) \in \tilde{V}). \tag{4.11}$$

(Proof) Consider any vector $x \in \mathbf{R}^V$ and its associated $\tilde{x} \in \mathbf{R}^{\tilde{V}}$ through (4.11). Then, $\tilde{x} \in \mathrm{B}(\kappa_{\tilde{c}})$ if and only if

$$\forall \tilde{X} \subseteq \tilde{V} : \tilde{x}(\tilde{X}) \leq \kappa_{\tilde{c}}(\tilde{X}). \tag{4.12}$$

Since $\tilde{x}(\tilde{X}) = \tilde{x}(\tilde{X} - \tilde{X}^*)$ and $\kappa_{\tilde{c}}(\tilde{X}) \geq \kappa_{\tilde{c}}(\tilde{X} - \tilde{X}^*)$ due to Lemma 4.2, this is equivalent to

$$\forall (X, Y) \in 3^V : \tilde{x}((X, +) \cup (Y, -)) \leq \kappa_{\tilde{c}}((X, +) \cup (Y, -)). \tag{4.13}$$

It follows from Lemma 4.1 that (4.13) is equivalent to

$$\forall (X, Y) \in 3^V : x(X, Y) \leq \hat{\kappa}_c(X, Y). \tag{4.14}$$

□

From Theorem 4.4 we have the following characterization of the boundaries of flows in bidirected networks.

Theorem 4.5: *For any bidirected network $\mathcal{N} = (G = (V, A; \partial), c)$ the set $\partial\Phi$ of the boundaries of flows in $\mathcal{N}$ is given by*

$$\partial\Phi = \mathrm{P}_*(\hat{\kappa}_c). \tag{4.15}$$

(Proof) The inclusion $\partial\Phi \subseteq \mathrm{P}_*(\hat{\kappa}_c)$ is clear. We prove the converse. Suppose $x \in \mathrm{P}_*(\hat{\kappa}_c)$. Then, by Theorem 4.4 $\tilde{x} \in \mathbf{R}^{\tilde{V}}$ defined by (4.11) is in the base polyhedron $\mathrm{B}(\kappa_{\tilde{c}})$ associated with the cut function $\kappa_{\tilde{c}}$ of the signed covering network $\tilde{\mathcal{N}} = (\tilde{G} = (\tilde{V}, \tilde{A}; \tilde{\partial}), \tilde{c})$. We know by (2.14) that $\mathrm{B}(\kappa_{\tilde{c}})$ is exactly the set of the boundaries of flows in $\tilde{\mathcal{N}}$. Hence, there exists a feasible flow $\tilde{\varphi}: \tilde{A} \to \mathbf{R}_+$ in $\tilde{\mathcal{N}}$ such that

$$\tilde{x} = \tilde{\partial}\tilde{\varphi}. \tag{4.16}$$

Define $\varphi: A \to \mathbf{R}_+$ by

$$\varphi(a) = \frac{1}{2}(\tilde{\varphi}(a^{(+)}) + \tilde{\varphi}(a^{(-)})) \quad (a \in A). \tag{4.17}$$

Clearly, φ is a feasible flow in $\mathcal{N}$. Furthermore, we have

$$\partial\varphi(v) = \frac{1}{2}(\tilde{\partial}\tilde{\varphi}(v, +) - \tilde{\partial}\tilde{\varphi}(v, -)). \tag{4.18}$$

It follows from (4.16) that $\partial\varphi = x$, i.e., $x \in \partial\Phi$. □

It should be noted that the degree sequence polyhedron of an undirected graph is a special case of (4.15), where all the arcs are of type (1) in the corresponding bidirected graph (see [12] and also [10] for its generalization and related topics).

For a bidirected network $\mathcal{N} = (G = (V, A; \partial), c)$ define a bidirected graph $G^\infty = (V, A^\infty; \partial^\infty)$ as $A^\infty = \{a \mid a \in A, c(a) = +\infty\}$ and ∂^∞ is the restriction of ∂ to A^∞. Define

$$\mathcal{F} = \{(X, Y) \mid (X, Y) \in 3^V, \kappa_c(X, Y) < +\infty\}. \tag{4.19}$$

Then, we have $\mathcal{I}(G^\infty) = \mathcal{F}$, and hence,

Theorem 4.6: *For the $\{\sqcup, \sqcap\}$-closed family $\mathcal{F}$ of (4.19), regarding $\hat{\kappa}_c$ as the restriction of $\hat{\kappa}_c$ to $\mathcal{F}$, $(\mathcal{F}, \hat{\kappa}_c)$ is a bisubmodular system on V. Also, $(\mathcal{F}, \hat{\kappa}_c)$ is a balanced bisubmodular system on V if and only if G^∞ is a balanced bidirected graph.* □

Acknowledgments

The authors are grateful to the referees for their useful comments. K. Ando's work was supported by a Research Fellowship of Japan Society for the Promotion of Science for Young Scientists and by a Grant-in-Aid of the Ministry of Education, Science, Sports and Culture of Japan. S. Fujishige's work is also supported by a Grant-in-Aid of the Ministry of Education, Science, Sports and Culture of Japan.

References

[1] K. Ando: *Bisubmodular Polyhedra and Bidirected Graphs*. Dissertation. Doctoral Program in Socio-Economic Planning, University of Tsukuba (January 1996).

[2] K. Ando and S. Fujishige: $\sqcup,\sqcap$-closed families and signed posets. Report No. 93813, Forschungsinstitut für Diskrete Mathematik, Universität Bonn (January 1994).

[3] K. Ando and S. Fujishige: On structures of bisubmodular polyhedra. *Mathematical Programming* **74** (1996) 293-318.

[4] K. Ando, S. Fujishige and T. Naitoh: Proper bisubmodular systems and bidirected flows. Discussion Paper No. 532, Institute of Socio-Economic Planning, University of Tsukuba (April 1993).

[5] K. Ando, S. Fujishige and T. Naitoh: A characterization of bisubmodular functions. *Discrete Mathematics* **148** (1996) 299-303.

[6] K. Ando, S. Fujishige and T. Nemoto: Decomposition of a bidirected graph into strongly connected components and its signed poset structure. *Discrete Applied Mathematics* **68** (1996) 237-248.

[7] K. Ando, S. Fujishige and T. Nemoto: The minimum-weight ideal problem for signed posets. *Journal of the Operations Research Society of Japan* **39** (1996) 558-565.

[8] G. Birkhoff: *Lattice Theory* (American Mathematical Colloquium Publications **25** (3rd ed.), Providence, R. I., 1967).

[9] A. Bouchet: Greedy algorithm and symmetric matroids. *Mathematical Programming* **38** (1987) 147-159.

[10] A. Bouchet and W. H. Cunningham: Delta-matroids, jump systems and bisubmodular polyhedra. *SIAM Journal on Discrete Mathematics* **8** (1995) 17-32.

[11] R. Chandrasekaran and S. N. Kabadi: Pseudomatroids. *Discrete Mathematics* **71** (1988) 205-217.

[12] W. H. Cunningham and J. Green-Krótki: b-matching degree-sequence polyhedra. *Combinatorica* **11** (1991) 219-230.

[13] A. Dress and T. F. Havel: Some combinatorial properties of discriminants in metric vector spaces. *Advances in Mathematics* **62** (1986) 285-312.

[14] F. D. J. Dunstan and D. J. A. Welsh: A greedy algorithm for solving a certain class of linear programmes. *Mathematical Programming* **62** (1973) 338-353.

[15] J. Edmonds and E. L. Johnson: Matching: a well-solved class of linear programs. In: *Combinatorial Structures and Their Applications* (R. Guy, H. Hanani, N. Sauer and J. Schönheim, eds., Gordon and Breach, New York, 1970), pp. 88-92.

[16] S. D. Fischer: *Signed Poset Homology and q-analog Möbius Functions.* Dissertation, University of Michigan, 1993.

[17] S. Fujishige: *Submodular Functions and Optimization* (North-Holland, Amsterdam, 1991).

[18] F. Harary: On the notion of balance of a signed graph. *Michigan Mathematical Journal* **2** (1953-1954) 143-146.

[19] R. H. Möhring, M. Müller-Hannemann and K. Weihe: Mesh refinement via bidirected flows: modeling, complexity, and computational results. Report No. 520/1996, Technische Universität Berlin (July 1996).

[20] M. Nakamura: A characterization of greedy sets: Universal polymatroids (I). Scientific Papers of the College of Arts and Sciences, The University of Tokyo, **38**-2 (1988) 155-167.

[21] L. Qi: Directed submodularity, ditroids and directed submodular flows. *Mathematical Programming* **42** (1988) 579-599.

[22] V. Reiner: Signed posets. *Journal of Combinatorial Theory*, Ser. A **62** (1993) 324-360.

[23] T. Zaslavsky: Orientation of signed graphs. *European Journal of Combinatorics* **12** (1991) 361-375.

Kazutoshi Ando:
Institute of Policy and Planning Sciences
University of Tsukuba
Tsukuba, Ibaraki 305, Japan
E-mail: ando@aries.sk.tsukuba.ac.jp

Satoru Fujishige:
Division of Systems Science
Department of Systems and Human Science
Graduate School of Engineering Science
Osaka University
Toyonaka, Osaka 560, Japan
E-mail: fujishig@sys.es.osaka-u.ac.jp

Takeshi Naitoh:
Faculty of Economics
Shiga University
Hikone, Shiga 522, Japan
E-mail: naitoh@biwako.shiga-u.ac.jp