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THE GENERALIZED STABLE SET PROBLEM FOR PERFECT BIDIRECTED GRAPHS

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Abstract Bidirected graphs are a generalization of undirected graphs. For bidirected graphs, we can consider a problem whichi is a natural extension of the maximum weighted stable set problem for undirected graphs. Here we call this problem the generalized stable set problem. It is well known that the maximum weighted stable set problem is solvable in polynomial time for perfect undirected graphs. Perfectness is naturally extended to bidirected graphs in terms of polytopes. Furthermore, it has been proved that a bidirected graph is perfect if and only if its underlying graph is perfect. Thus it is natural to expect that the generalized stable set problem for perfect bidirected graphs can be solved in polynomial time. In this paper, we show that the problem for any bidirected graph is reducible to the maximum weighted stable set problem for a certain undirected graph is in time polynomial in the number of vertices, and moreover, prove that this reduction preserves perfectness. That is, this paper gives an affirmative answer to our expectation.

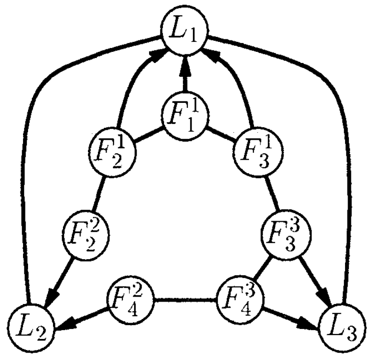
1 Introduction

Bidirected graphs, first introduced by Edmonds and Johnson [11] are a generalization of undirected graphs. A *bidirected graph* $G = (V, E)$ has a set of vertices V, and a set of edges E, in which each edge $e \in E$ has two vertices $i, j \in V$ as its ends and two associated signs at i and j. We say that an edge e is *incident to* $i, j \in V$ if e has i and j as its ends and that e is incident to i *with a plus* (or *minus*) *sign* if e has a plus (or minus) sign at i. We call e a *selfloop* if $i = j$. The edges are classified into three types: the $(+,+)$-edges are the edges with two plus signs at their ends, the $(-,-)$-edges are the edges with two minus signs, and the $(+,-)$-edges (and the $(-,+)$-edges) are the edges with one plus and one minus sign. Two vertices i and j are said to be *adjacent* if there is an edge incident to these. Undirected graphs may be interpreted as bidirected graphs with only $(+,+)$-edges.

By associating a variable x_i with each vertex $i \in V$, we may consider the following inequality system:

$$
\begin{array}{rl}
x_i+x_j\leq 1 & \text{for each } (+,+)\text{-edge incident to } i \text{ and } j,\\
-x_i-x_j\leq -1 & \text{for each } (-,-)\text{-edge incident to } i \text{ and } j,\\
x_i-x_j\leq 0 & \text{for each } (+,-)\text{-edge incident to } i \text{ and } j.
\end{array}
$$

Such systems are called *degree-two inequality systems*, and have been studied by Johnson and Padberg [21], Bourjolly [7], Ando, Fujishige and Nemoto [2], and Ando [1]. We will call the degree-two inequality system arising from G the *system* of G, and any solution to the system, a *solution of* G. We note here that besides having a natural correspondence with bidirected graphs, degree-two inequality systems may also be regarded as a *complete set of implicants with length at most two.* Studies from this approach include those by Hausmann



Facilities: $\mathcal{F} = \{F_1, F_2, F_3, F_4\}$,
Locations: $\mathcal{L} = \{L_1, L_2, L_3\}$,
Relations: $\mathcal{F}(L_1) = \{F_1, F_2, F_3\}$,
$\mathcal{F}(L_2) = \{F_2, F_4\}$,
$\mathcal{F}(L_3) = \{F_3, F_4\}$,
Constraints: $\mathcal{C}_F = \{(F_1, F_2), (F_1, F_3), (F_3, F_4)\}$,
$\mathcal{C}_L = \{(L_1, L_2), (L_1, L_3)\}$.

Figure 1: A small example of the facility location problem.

and Korte [17], and Ikebe and Tamura [19].

Here we consider an optimization problem over the 0−1 solutions of a given bidirected graph G as below:

$$\text{maximize}\{\sum_{i\in V} w_i x_i \mid \boldsymbol{x} = (x_i)_{i\in V} \text{ is a 0−1 solution of } G\}, \tag{1.1}$$

for a given integral weight vector $\boldsymbol{w} = (w_i)_{i\in V} \in \boldsymbol{Z}^V$. This problem includes the set packing problem, the maximum weighted stable set problem and so on. Here we call the problem the *generalized stable set problem.* The next facility location problem is an example which may be formulated as (1.1) and seems not to be easily formulated as the maximum weighted stable set problem. An instance consists of facilities $\mathcal{F} = \{F_1, \cdots, F_s\}$ and potential locations $\mathcal{L} = \{L_1, \cdots, L_t\}$. If facility F_i is built, it will make a profit of p_i in its durable years. On the other hand, each location L_j costs c_j and has a set $\mathcal{F}(L_j) \subseteq \mathcal{F}$ of facilities which can be built in L_j. The objective of the problem is to maximize the amount of gains under the following constraints:

(a) a set $\mathcal{C}_F$ of pairs of facilities such that both of each pair may not be built in the same location is given,
(b) a set $\mathcal{C}_L$ of pairs of locations such that both of each pair may not be bought is given,
(c) each facility is built in at most one location.

Constraint (c) is not fatal because we may assign different names to the same facility. This problem can be formulated in terms of bidirected graphs. Let us consider a vertex, denoted by L_j for convenience, for each location $L_j \in \mathcal{L}$ and a vertex, denoted by F_i^j, for each facility $F_i \in \mathcal{F}(L_j)$. Let G be the bidirected graph with these vertices and with edge set defined in the following way:

- join F_i^j and F_h^j by a $(+,+)$-edge if $F_i, F_h \in \mathcal{F}(L_j)$ and $\{F_i, F_h\} \in \mathcal{C}_F$, by (a),
- join L_j and L_k by a $(+,+)$-edge if $\{L_j, L_k\} \in \mathcal{C}_L$, by (b),
- join F_i^j and F_i^k by a $(+,+)$-edge if $F_i \in \mathcal{F}(L_j) \cap \mathcal{F}(L_k)$, by (c),
- join L_j and F_i^j by a $(-,+)$-edge.

The last construction means that F_i cannot be built in L_j if L_j is not bought. Then any 0−1 solution $\boldsymbol{x}$ of G is a feasible facility location and vice versa. (When $x_v = 1$, if v corresponds to a location then it is bought, and if it corresponds to a facility then it is built.) Assigning weight p_i for vertex F_i^j and $-c_j$ for vertex L_j, the problem is formulated in the form of (1.1). Figure 1 gives an example of the facility location problem. In the figure, we draw $(+,+)$-edges and $(+,-)$-edges by using ordinary undirected edges and directed edges respectively.

It is well known that the maximum weighted stable set problem for perfect graphs can be solved in polynomial time [13, 14, 15]. On the other hand, the concept of perfectness may be extended to bidirected graphs, see Section 2. Moreover, Ikebe and Tamura [20] proved that a bidirected graph G is perfect if and only if its underlying graph $\underline{G}$ is perfect, where $\underline{G}$ is defined as the undirected graph obtained by exchanging all edges for $(+,+)$-edges. From the above facts, one may naturally expect that the generalized stable set problem (1.1) can be solved in time polynomial for perfect bidirected graphs. The main aim of this paper is to verify the expectation. To do this, we prove that (1.1) for any perfect bidirected graph can be reduced to the maximum weighted stable set problem for a certain perfect undirected graph in time polynomial in the number of vertices. Combining this and the excellent method of Grötschel, Lovász and Schrijver [13, 14, 15], we attain our aim. For this reduction, a $(-,+)$-edge elimination, which will be defined in Section 3, plays an important role. We will show that the $(-,+)$-edge elimination preserves perfectness.

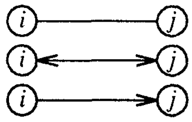
In Section 2, we introduce several definitions and results for bidirected graphs. Section 3 gives two proofs for which the $(-,+)$-edge elimination preserves the perfectness of bidirected graphs. In Section 4, we deal with polynomial time reducibility of the generalized stable set problem to the maximum weighted stable set problem by using the $(-,+)$-edge elimination.

2 Preliminaries

Johnson and Padberg [21] indicated that bidirected graphs which are simple and transitive are particular important in the following sense. A bidirected graph is called *transitive*, if whenever there are edges $e_1 = \{i,j\}$ and $e_2 = \{j,k\}$ with opposite signs at j, then there is also an edge $e_3 = \{i,k\}$ whose signs at i and k agree with those of e_1 and e_2. Interpreting this in terms of the inequality system, this simply says that any degree-two inequality which is implied by the existing inequalities must already be present. Thus, any bidirected graph and its transitive closure have the same solution set. Moreover, any bidirected graph can be transformed into its transitive closure in time polynomial in the number of vertices. We say that a bidirected graph is *simple* if it has no selfloop and if it has at most one edge for each pair of distinct vertices. Let $G = (V,E)$ be a transitive bidirected graph, and especially, let us consider the $0-1$ solutions of G. If there are a $(+,+)$-selfloop and a $(-,-)$-selfloop at some vertex i then G has no $0-1$ solution, because no $0-1$ vector satisfies the induced equality $x_i + x_i = 1$. We note that the converse is also true, see Theorem 2.1. If, for example, there is a $(+,+)$-selfloop at vertex i, then we must have the inequality $x_i + x_i \leq 1$, and hence x_i must be 0 and we may delete i from G, since x_i is $0-1$ valued and G is transitive. Suppose that there are a $(+,+)$-edge and a $(-,-)$-edge incident to distinct vertices i and j. Then the equality $x_i + x_j = 1$ must be satisfied. We can delete i because x_i is uniquely determined by x_j. Similarly, for any bidirected graph G, we can either determine that it has no $0-1$ solution, or reduce its vertex set to be simple and transitive without changing the $0-1$ solutions, by using such procedures in time polynomial in the number of vertices.

For our purpose, it is enough to deal with only simple and transitive bidirected graphs. We call such a bidirected graph *closed*. Note that any simple undirected graph is closed.

For a closed bidirected graph G, we denote an edge e as the pair $\langle i,j\rangle$, where i and j are the two ends of e, and we draw each edge $\langle i,j\rangle$ as below and consider an inequality according to its type as,



$$\begin{array}{rll} x_i+x_j \le 1 & \text{if } \langle i,j\rangle \text{ is a } (+,+)\text{-edge}, \\ -x_i-x_j \le -1 & \text{if } \langle i,j\rangle \text{ is a } (-,-)\text{-edge}, \\ x_i-x_j \le 0 & \text{if } \langle i,j\rangle \text{ is a } (+,-)\text{-edge}, \end{array}$$

where x_i is a variable corresponding to the vertex i. We remark that $\langle i,j\rangle$ and $\langle j,i\rangle$ denote the same edge, however, if this edge is a $(+,-)$-edge with a minus sign at j, we say that $\langle i,j\rangle$ is a $(+,-)$-edge and $\langle j,i\rangle$ a $(-,+)$-edge.

We now define the $0-1$ polytope $P_I(G)$ for G as

$$P_I(G) = \operatorname{conv}\{\boldsymbol{x} \mid \boldsymbol{x} \text{ is a } 0-1 \text{ solution of } G\},$$

the convex hull of all the $0-1$ solutions of G. As all stable set polytopes of graphs have full-dimension, $P_I(G)$ has also the same feature.

Theorem 2.1 ([21]): *For any closed bidirected graph G, $P_I(G)$ is full-dimensional.*

We next introduce bicliques, strong bicliques and corresponding valid inequalities for $P_I(G)$. For a subset C of vertices of G, let $G[C]$ denote the subgraph of G induced by C. A pair of disjoint subsets of vertices (C^+, C^-) is called a *biclique* if the following conditions hold:

(B1) there is an edge between any two vertices in $C^+ \cup C^-$,
(B2) for any edge e of $G[C^+ \cup C^-]$, if an end vertex i of e is in C^+ then e has a plus sign at i, and if $i \in C^-$ then e has a minus sign at i.

If a biclique $C = (C^+, C^-)$ has at least two vertices, i.e., $|C^+ \cup C^-| \ge 2$, then the partition is uniquely determined from $C^+ \cup C^-$ by

$$\begin{array}{rcl} C^+ &=& \{i \in C^+ \cup C^- \mid \text{ there is an edge of } G[C^+ \cup C^-] \text{ with a plus end at } i\}, \\ C^- &=& \{i \in C^+ \cup C^- \mid \text{ there is an edge of } G[C^+ \cup C^-] \text{ with a minus end at } i\}. \end{array}$$

Thus we will regard a biclique as the set of its vertices and use C as $C^+ \cup C^-$, whenever there is no confusion. For two bicliques $C = (C^+, C^-)$ and $D = (D^+, D^-)$, suppose that $C \triangle D$ means the symmetric difference $(C^+ \cup C^-) \triangle (D^+ \cup D^-)$. Analogously, $C \cup D$, $C \cap D$ and so on are defined.

A biclique $C = (C^+, C^-)$ is said to be *strong* if in addition, it satisfies

(B3) C is maximal with respect to (B1) and (B2), that is, there is no biclique $\widehat{C} = (\widehat{C}^+, \widehat{C}^-)$ such that $C^+ \subseteq \widehat{C}^+$, $C^- \subseteq \widehat{C}^-$ and $C \ne \widehat{C}$,
(B4) there is no vertex $u \in V \setminus C$ such that there are edges $\langle u,i\rangle$ with a plus sign at i for all $i \in C^+$, and edges $\langle u,i\rangle$ with a minus sign at i for all $i \in C^-$.

Note that (B4) implies (B3). Let us consider the bidirected graph of Figure 2. Sets $\{1,2,4\}$ and $\{1,2,3,4\}$ satisfy (B1) but are not bicliques. $\{2,3,4\}$ has properties (B1), (B2) and also (B3), i.e., this is a maximal biclique. This set, however, is not strong because the vertex 1 destroys the condition (B4) for $\{2,3,4\}$. For this instance, $\{1,2,3\}$, $\{1,4\}$ and $(\emptyset, \{3\})$ are examples of strong bicliques.

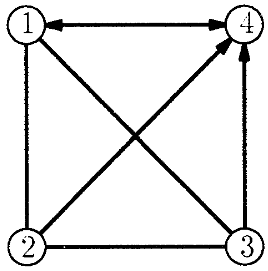


Figure 2: Bicliques and strong bicliques.

Bicliques are analogies of cliques of undirected graphs, and as such, have corresponding inequalities. For a given biclique $C = (C^+, C^-)$, the corresponding *biclique inequality* is

$$\sum_{i\in C^+} x_i + \sum_{i\in C^-} (1 - x_i) \leq 1.$$

It turns out that biclique inequalities satisfy properties similar to clique inequalities of undirected graphs.

Proposition 2.2 ([21]): *If G is a closed bidirected graph, then every* $0-1$ *solution satisfies the biclique inequalities of G.*

Theorem 2.3 ([21]): *Let G be a closed bidirected graph. Then, a biclique inequality induces a facet of $P_I(G)$ if and only if the corresponding biclique is strong.*

From the previous results we have

Theorem 2.4 ([21]): *For a closed bidirected graph G, the* $0-1$ *solutions of G are exactly the* $0-1$ *solutions of*

$$\sum_{i\in C^+} x_i + \sum_{i\in C^-} (1 - x_i) \leq 1 \qquad \text{for all strong bicliques } C \text{ of } G,$$

$$0 \leq x_i \leq 1 \qquad \text{for all } i \in V.$$

It may seem that the constraints $0 \leq x_i \leq 1$ for $i \in V$ are necessary. However, these are implied by the strong biclique inequalities from Proposition 2.5 below. For a vertex $v \in V$, we define $N_G^+(v)$ as the set of vertices adjacent to v by edges incident to v with plus signs, analogously define $N_G^-(v)$, and set $N_G(v) = N_G^+(v) \cup N_G^-(v)$.

Proposition 2.5 ([20]): *Let G be a closed bidirected graph and v a vertex of V.*

(a) *If $N_G^+(v) \neq \emptyset$, there is a strong biclique C with $v \in C^+$. Moreover, if $N_G^+(v) = \emptyset$, then $(\{v\}, \emptyset)$ is a strong biclique with the corresponding inequality $x_v \leq 1$.*

(b) *If $N_G^-(v) \neq \emptyset$, there is a strong biclique C with $v \in C^-$. Moreover, if $N_G^-(v) = \emptyset$, then $(\emptyset, \{v\})$ is a strong biclique with the corresponding inequality $x_v \geq 0$.*

We remark that Proposition 2.5 follows from Theorem 2.3 and from the fact that $(\{v\}, \emptyset)$ and $(\emptyset, \{v\})$ are bicliques for any vertex v.

Given a closed bidirected graph G, let the polytope $Q(G)$ be defined as

$$Q(G) = \{\boldsymbol{x} \in \boldsymbol{R}^V \mid \boldsymbol{x} \text{ satisfies all strong biclique inequalities of } G\}.$$

By Proposition 2.5, $Q(G)$ is bounded. For any simple undirected graph, the strong biclique inequalities are precisely the maximal clique inequalities and the nonnegativity inequalities.

The class of undirected graphs for which the strong biclique inequalities are the only facets, are the perfect graphs [8]. See also [4, 5, 12, 15] for details of perfect graphs. Following conventions for undirected graphs, we will say that a bidirected graph G is *perfect*, if it is closed and $P_I(G) = Q(G)$. Obviously, this perfectness is a natural extension of the perfectness of undirected graphs. Furthermore, the following interesting relation holds between a bidirected graph G and its underlying graph $\underline{G}$ which is obtained from G by changing all edges to $(+,+)$.

Theorem 2.6 ([20]): *A closed bidirected graph G is perfect if and only if $\underline{G}$ is.*

We add that the theorem can be also proved by using results in [16].

Bidirected graphs have generally three types of edges. However, we can eliminate all $(-,-)$-edges while preserving the polyhedral structures of the related polytopes. Consider a simple transformation for bidirected graphs as below:

(2.1) given a vertex v, reverse the signs of the v side of all edges incident to v.

This transformation is called the *reflection* of G at v. When we do reflection at v, the new 0−1 solutions can be obtained from the 0−1 solutions of G by simply reversing all the 0s and 1s in the vth coordinate. The facet inequalities and the objective functions of the problem (1.1) may also be produced by replacing all of x_v by $1 - x_v$. Hence, in polyhedral terms, reflection consists of only mirroring and translation, and essentially changes none of the structure of the associated polytope. Obviously, reflection preserves perfectness. Moreover, the new problem of type (1.1) is equivalent to the original one. Reflection, however, simplifies closed bidirected graphs as below.

Lemma 2.7 ([3, 22]): *All $(-,-)$-edges can be eliminated by a sequence of reflections.*

We remark that elimination of all $(-,-)$-edges is done in time polynomial in the number of vertices. A bidirected graph having no $(-,-)$-edge is called *pure*. Any biclique C of a pure bidirected graph has at most one vertex in its minus part, i.e., $|C^-| \leq 1$. In the sequel of the paper, we only consider pure and closed bidirected graphs.

3 An edge elimination for bidirected graphs

Let G be a pure and closed bidirected graph. For a vertex v with $N_G^-(v) \neq \emptyset$, let us denote by $G \not\mapsto v$ the graph obtained by deleting all edges incident to v with a minus sign. Obviously, $G \not\mapsto v$ is closed. We call this transformation the *$(-,+)$-edge elimination* at v. In this section, we will prove that if G is perfect then $G \not\mapsto v$ is also perfect. Here we assume that $N_G^+(v) \neq \emptyset$ since if $N_G^+(v)$ is vacant then $G \not\mapsto v$ is obviously perfect if G is, because $G \not\mapsto v$ consists of two disjoint perfect bidirected graphs $G[V \setminus \{v\}]$ and $G[\{v\}]$.

We first discuss how the $(-,+)$-edge elimination at v changes $P_I(G)$ and $Q(G)$. The degree-two inequality system of $G \not\mapsto v$ is a subsystem of the system of G. Thus, it is clear that $P_I(G) \subseteq P_I(G \not\mapsto v)$. Generally, new 0−1 solutions are created by a $(-,+)$-edge elimination.

Proposition 3.1. *Let $\boldsymbol{x}$ be a 0−1 solution of G with $x_v = 1$. Then the 0−1 vector $\hat{\boldsymbol{x}}$ obtained from $\boldsymbol{x}$ by exchanging 1 for 0 in the vth coordinate is a solution of $G \not\mapsto v$.*

Proof. If $x_i = 0$ for all $i \in N_G^-(v)$ then $\hat{\boldsymbol{x}}$ is a solution of G because there is no $(-,-)$-edge, that is, $\hat{\boldsymbol{x}}$ is also a solution of $G \not\mapsto v$. Otherwise, however, $\hat{\boldsymbol{x}}$ is also a solution of $G \not\mapsto v$, since there is no edge joining v and $i \in N_G^-(v)$ in $G \not\mapsto v$. ∎

We now examine how the set of strong bicliques changes with the $(-,+)$-edge elimination at v.

Proposition 3.2. *Let C be a strong biclique of G with $v \notin C$. Then C is also a strong biclique of $G \not\to v$. The converse also holds if in addition one of the following conditions $C \subseteq N_G^+(v)$, $C \subseteq N_G^-(v)$ or $C \not\subseteq N_G(v)$ holds.*

Proof. The set C satisfies (B1) and (B2) in $G \not\to v$ because $v \notin C$ and the edges which are eliminated are incident to v. Obviously, (B4) is preserved by any elimination of edges.

Suppose that C is a strong biclique of $G \not\to v$. Trivially, C is a biclique of G. If $C \subseteq N_G^+(v)$ then C is clearly strong in G. If $C \subseteq N_G^-(v)$, we assume on the contrary that C is not strong in G. Then v must break the strongness of C in G, and hence, $C \cup \{v\}$ must be a biclique of G. Since $N_G^+(v) \neq \emptyset$ and G is transitive, for any $u \in N_G^+(v)$, $C \cup \{u\}$ is a biclique of G (also of $G \not\to v$). However, this is a contradiction. Hence C is a strong biclique of G. Next suppose that $C \not\subseteq N_G(v)$. Then, if C is not strong in G, there is a vertex $u (\neq v)$ which destroys (B4) for C. However, this contradicts the fact that C is strong in $G \not\to v$. Hence the converse holds if $C \subseteq N_G^+(v)$ or $C \subseteq N_G^-(v)$ or $C \not\subseteq N_G(v)$. ∎

Proposition 3.3. *Let $C = (C^+, C^-)$ be a strong biclique of G with $v \in C^+$. Then C is also a strong biclique of $G \not\to v$. The converse also holds.*

Proof. Since $v \in C^+$, C is a biclique of $G \not\to v$. Obviously, (B4) is satisfied by C.

If C with $v \in C^+$ is a strong biclique of $G \not\to v$ then C is a biclique of G. Assume on the contrary that C is not strong in G. However, this immediately implies that C is not strong in $G \not\to v$ from (B4). Hence the converse holds. ∎

Every biclique of G containing v in its minus part disappears in $G \not\to v$ if it has at least two vertices. Furthermore, some bicliques not strong in G may become strong in $G \not\to v$. We recall that a biclique $C = (C^+, C^-)$ may be interpreted as its vertex set $C = C^+ \cup C^-$.

Proposition 3.4. *Let $C = (C^+, C^-)$ and $D = (D^+, D^-)$ be bicliques of G with $v \in C^+$ and $v \in D^-$. Then $\widehat{C} = C \Delta D$ is a biclique of $G \not\to v$. Furthermore, C and D are strong if $\widehat{C}$ is.*

Proof. We note that $\widehat{C} = (C \cup D) \setminus \{v\}$ because the facts that $v \in C^+ \cap D^-$ and that C and D are bicliques imply $C \cap D = \{v\}$. Since G is transitive, there is an edge between any two vertices in $C \setminus \{v\}$ and $D \setminus \{v\}$, and the sign at each vertex agrees with the sign of the partition the vertex is in. Thus, $\widehat{C}$ is a biclique of $G \not\to v$.

Suppose that $\widehat{C}$ is strong. Assume on the contrary that D does not satisfy (B4). Then there is a vertex $u \notin D$ such that there are edges $\langle u, i \rangle$ with a plus end at i for each $i \in D^+$ and edges with a minus end at i for each $i \in D^-$, especially for $v \in D^-$. Then $u \notin C$. Since G is transitive, this means that for every $i \in C \setminus \{v\}$, there is an edge $\langle u, i \rangle$ whose sign of the i side agrees in the sense of (B4). However, this implies that u hinders $\widehat{C}$ from satisfying (B4). This is a contradiction. Similarly, one can prove that C is strong. ∎

We remark that every biclique of $G \not\to v$ is originally a biclique of G. For the strong bicliques of $G \not\to v$, the followings hold.

Lemma 3.5. *Let $\widehat{C}$ be any strong biclique of $G \not\to v$. Then one of the followings holds.*

(a) $v \notin \widehat{C}$ *and* $\widehat{C}$ *is also a strong biclique of* G,

(b) *there are strong bicliques* C *and* D *of* G *such that* $v \in C^+ \cap D^-$ *and* $\widehat{C} = C \triangle D$,

(c) $v \in \widehat{C}^+$, *and* $\widehat{C}$ *is also a strong biclique of* G,

(d) $\widehat{C} = (\emptyset, \{v\})$.

Proof. We consider the three cases: $v \in \widehat{C}^+$, $v \in \widehat{C}^-$ and $v \notin \widehat{C}$. In the first case, we can conclude (c) because $\widehat{C}$ is a biclique of G and must be strong from Proposition 3.3. In the second case, $\widehat{C}$ must be $(\emptyset, \{v\})$ because all edges incident to v have a plus sign at v in $G \not\mapsto v$. Suppose that the last case holds. Evidently $\widehat{C}$ is a biclique of G. If $\widehat{C} \subseteq N_G^+(v)$ or $\widehat{C} \subseteq N_G^-(v)$ or $\widehat{C} \not\subseteq N_G(v)$ then $\widehat{C}$ is strong in G from Proposition 3.2. In these subcases, (a) holds. Finally, let us consider the subcase when $\widehat{C} \not\subseteq N_G^+(v), N_G^-(v)$ and $\widehat{C} \subseteq N_G(v)$. Let $C = (\widehat{C} \cap N_G^+(v)) \cup \{v\}$ and $D = (\widehat{C} \cap N_G^-(v)) \cup \{v\}$. Then C and D have at least two vertices. For any $u \in C \setminus \{v\}$ and $w \in D \setminus \{v\}$, $\langle u, w\rangle$ and $\langle u, v\rangle$ ($\langle u, w\rangle$ and $\langle v, w\rangle$) have the same sign at u (at w) because G is closed. Thus C and D form bicliques in G. Obviously, $\widehat{C} = C \triangle D$. By Proposition 3.4, C and D must be strong, that is, (b) holds. ∎

We consider the relation between $Q(G)$ and $Q(G \not\mapsto v)$. It is obvious that $Q(G) \subseteq Q(G \not\mapsto v)$. For any vector $\boldsymbol{x}$ and any biclique C, let

$$C(\boldsymbol{x}) = \sum_{i \in C^+ \setminus \{v\}} x_i + \sum_{i \in C^- \setminus \{v\}} (1 - x_i).$$

That is, the biclique inequality corresponding to C is represented as $C(\boldsymbol{x}) + x_v \leq 1$ if $v \in C^+$, $C(\boldsymbol{x}) + 1 - x_v \leq 1$ if $v \in C^-$, otherwise $C(\boldsymbol{x}) \leq 1$. Suppose that $\{C_1, \cdots, C_s\}$ is the set of strong bicliques of G containing v in their plus parts, and that $\{D_1, \cdots, D_t\}$ is the set of those containing v in their minus parts.

Lemma 3.6. *For any* $\widehat{\boldsymbol{y}} \in Q(G \not\mapsto v)$, *there exists* $\boldsymbol{y} \in Q(G)$ *such that* $y_i = \widehat{y}_i$ *for all* $i \neq v$. *More exactly,* $\max_j\{D_j(\widehat{\boldsymbol{y}})\} \leq 1 - \max_i\{C_i(\widehat{\boldsymbol{y}})\}$ *holds, and* $\boldsymbol{y} \in Q(G)$ *if and only if*

$$\max_j\{D_j(\widehat{\boldsymbol{y}})\} \leq y_v \leq 1 - \max_i\{C_i(\widehat{\boldsymbol{y}})\}.$$

Proof. From Proposition 3.2, $\boldsymbol{y}$ satisfies all of the inequalities corresponding to all strong bicliques of G not containing v, regardless of the value of y_v. Hence, it suffices to determine the value of y_v so that all inequalities for strong bicliques of G containing v are satisfied. Since the symmetric difference of C_i and D_j $(i = 1, \cdots, s,\ j = 1, \cdots, t)$ is a biclique of $G \not\mapsto v$, we must have

$$C_i(\widehat{\boldsymbol{y}}) + D_j(\widehat{\boldsymbol{y}}) \leq 1, \quad (i = 1, \cdots, s,\ j = 1, \cdots, t).$$

Hence

$$\max_j\{D_j(\widehat{\boldsymbol{y}})\} \leq 1 - \max_i\{C_i(\widehat{\boldsymbol{y}})\}.$$

Since any strong biclique of G other than $C_1, \cdots, C_s, D_1, \cdots, D_t$ does not contain v, if y_v is in the range between these two values then $\boldsymbol{y} \in Q(G)$, and vice versa. ∎

We say that a vector $\boldsymbol{y}$ *lies on* a biclique inequality or the corresponding biclique if it satisfies the inequality with equality. Note that if we set $y_v = \max_j\{D_j(\widehat{\boldsymbol{y}})\}$, then $\boldsymbol{y}$ will lie on the biclique inequality for any D_j attaining the maximum, and if $y_v = 1 - \max_i\{C_i(\widehat{\boldsymbol{y}})\}$, then $\boldsymbol{y}$ will lie on any C_i attaining the maximum.

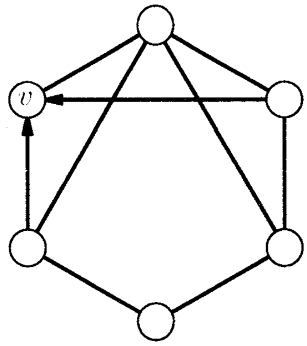


Figure 3: An imperfect bidirected graph G such that $G\not\to v$ is perfect.

Theorem 3.7. *For a closed bidirected graph G, if G is perfect then $G\not\to v$ is also perfect.*

Proof. Assume on the contrary that $Q(G\not\to v)$ has a non-integral extreme point $\hat{\boldsymbol{y}}$. We first suppose that $\hat{y}_v = 0$. By Lemma 3.6 we know that there is some $\boldsymbol{y} \in Q(G)$ with $y_i = \hat{y}_i$ for $i \neq v$. Since $\boldsymbol{y}$ is non-integral, and G is perfect, there must exist some $0-1$ solutions $\boldsymbol{x}^1, \cdots, \boldsymbol{x}^\ell$ of G such that $\boldsymbol{y}$ can be expressed as a convex combination $\boldsymbol{y} = \sum_{j=1}^{\ell} \lambda_j \boldsymbol{x}^j$ of these points. On the other hand, from Proposition 3.1, we also know that the vectors $\hat{\boldsymbol{x}}^j$, obtained by setting the vth coordinate to 0 in $\boldsymbol{x}^j$ are $0-1$ solutions of $G\not\to v$, implying that $\hat{\boldsymbol{y}} = \sum_{j=1}^{\ell} \lambda_j \hat{\boldsymbol{x}}^j$. But since $\hat{\boldsymbol{y}}$ is non-integral, it must be that $\hat{\boldsymbol{y}} \neq \hat{\boldsymbol{x}}^j$, which contradicts the assumption that $\hat{\boldsymbol{y}}$ is an extreme point of $Q(G\not\to v)$.

Now suppose $\hat{y}_v > 0$. Since $\hat{\boldsymbol{y}}$ is an extreme point, it must lie on some strong biclique $\widehat{C}$ such that $v \in \widehat{C}^+$. By Lemma 3.5, $\widehat{C}$ is also a strong biclique of G. From Lemma 3.6, $\hat{\boldsymbol{y}}$ is also contained in $Q(G)$. Since $\hat{\boldsymbol{y}}$ is non-integral and G is perfect, $\hat{\boldsymbol{y}}$ can be represented by a convex combination of some $0-1$ solutions $\boldsymbol{x}^1, \cdots, \boldsymbol{x}^\ell$ of G. Obviously, $\boldsymbol{x}^1, \cdots, \boldsymbol{x}^\ell$ are solutions of $G\not\to v$ and $\hat{\boldsymbol{y}} \neq \boldsymbol{x}^i$ $(i = 1, \cdots, \ell)$. This is a contradiction. ∎

Unfortunately, the converse of Theorem 3.7 is not true. For example, the closed bidirected graph in Figure 3 is imperfect because of Theorem 2.6 and of the fact that its underlying graph contains the 5-hole as an induced subgraph. However, $G\not\to v$ is a perfect graph.

Several transformations of graphs preserving perfectness, for example, complements of graphs, multiplications of vertices, substitutions and compositions of graphs, have been studied [23, 6, 9, 10, 18]. Our $(-,+)$-edge elimination also indicates an edge-transformation preserving perfectness. Let H be a simple undirected graph and let v be a vertex of H. Now we consider a partition $S \cup T$ of the neighbor $N_H(v)$ of v (we assume that $S, T \neq \emptyset$.) It is easy to show that the bidirected graph obtained from H by replacing all edges joining v and vertices of S for $(-,+)$-edges is closed if and only if any $i \in S$ and any $j \in T$ are adjacent. Then, the next lemma directly follows from Theorem 3.7.

Lemma 3.8. *Let H be a perfect graph and let v be a vertex of H. For any partition $S \cup T$ of $N_H(v)$ with $S, T \neq \emptyset$, if any $i \in S$ and $j \in T$ are adjacent, then the graph H' obtained by eliminating all edges joining v and $i \in S$ is also perfect.*

Conversely, by combining Theorem 2.6 and Lemma 3.8, we can easily prove Theorem 3.7. Before doing this, we verify Lemma 3.8 in terms of graphs.

Proof of Lemma 3.8. Let $\omega(H)$ and $\chi(H)$ denote the maximum clique size of H

and the chromatic number of H, respectively. It is enough to show that $\omega(H') = \chi(H')$ since for any induced subgraph of H, the construction of Lemma 3.8 is defined similarly. If there is a maximum clique of H which does not contain v, then $\omega(H') = \omega(H)$ and $\chi(H') = \chi(H)$, obviously. Now suppose that all the maximum cliques of H contain v. Let us consider the induced subgraph $H'' = H[V \setminus \{v\}]$. Since H'' is also perfect, $\omega(H'') = \omega(H) - 1 = \chi(H) - 1 = \chi(H'')$. By the assumption of Lemma 3.8, there is a vertex $u \in S$ whose color is distinct from all colors of the vertices of T. If we paint v the same color as u, we obtain a vertex coloring of H'. Hence

$$\omega(H'') \leq \omega(H') \leq \chi(H') = \chi(H'') = \omega(H'')$$

holds. That is, H' is perfect. ∎

By using this, we give a short proof of Theorem 3.7.

Proof of Theorem 3.7. Let $H = \underline{G}$, $S = N_G^-(v)$ and $T = N_G^+(v)$. The 'only if' part of Theorem 2.6 implies that H is perfect. Since G is transitive, H, v, S and T satisfy the condition of Lemma 3.8. Then it follows from Lemma 3.8 that H' is perfect. Obviously, $H' = \underline{G \not\to v}$. From the 'if' part of Theorem 2.6, $G \not\to v$ is also perfect. ∎

The second proof is very short even though the proof of Lemma 3.8 is contained in it. However, this proof uses Theorem 2.6. In order to prove this, discussions more difficult and detailed than the first proof are necessary. Furthermore, from the second proof, we do not obtain anything of essential properties of $(-,+)$-edge eliminations. It seems that the first proof of Theorem 3.7 is not unnecessary.

4 A reduction of the generalized stable set problem

We recall the generalized stable set problem for a given closed bidirected graph G and for a given integral weight vector $\boldsymbol{w} \in \boldsymbol{Z}^V$:

$$\text{(4.1)} \qquad \text{maximize}\{\sum_{i \in V} w_i x_i \mid \boldsymbol{x} \in P_I(G)\}.$$

For the maximum weighted stable set problem, we may assume that $w_i > 0$ for all $i \in V$, because if $w_v \leq 0$ for some $v \in V$ then there is a maximum weighted stable set not containing v, that is, we can delete all vertices with nonpositive weights. In this section, we introduce a reduction of the problem (4.1) according to the sign pattern of the weight vector $\boldsymbol{w}$. However, the reduction is not so trivial as the case of the maximum weighted stable set problem. We will use $(-,+)$-edge eliminations.

Let G be a pure and closed bidirected graph and let $\boldsymbol{w} \in \boldsymbol{Z}^V$. We call a vertex *positive* if there is no edge incident to it with a minus sign, otherwise *nonpositive*. If, for example, a positive vertex v has a nonpositive weight w_v, then there is an optimal 0−1 solution $\boldsymbol{x}$ with $x_v = 0$, because for any 0−1 solution $\boldsymbol{y}$ of G, the 0−1 vector obtained from $\boldsymbol{y}$ by replacing y_v for 0 is also a solution of G having an objective value greater than or equal to $\sum_{i \in V} w_i y_i$. Thus, we can delete v from G. Let us next assume that a nonpositive vertex v has a nonnegative weight w_v. Then the next lemma holds.

Lemma 4.1. *If a nonpositive vertex v has a nonnegative weight w_v, then*

$$\max\{\sum_{i \in V} w_i x_i \mid \boldsymbol{x} \in P_I(G)\} = \max\{\sum_{i \in V} w_i x_i \mid \boldsymbol{x} \in P_I(G \not\to v)\}.$$

Proof. Since $P_I(G) \subseteq P_I(G \not\mapsto v)$,

$$\max\{\sum_{i \in V} w_i x_i \mid \boldsymbol{x} \in P_I(G)\} \leq \max\{\sum_{i \in V} w_i x_i \mid \boldsymbol{x} \in P_I(G \not\mapsto v)\}.$$

Let $\widehat{\boldsymbol{x}} \in P_I(G \not\mapsto v)$ be an optimal 0–1 solution of the right-hand side problem. If $\widehat{x}_v = 1$ then $\widehat{\boldsymbol{x}}$ is also a solution of G which attains the maximum of the left-hand side problem. Suppose that $\widehat{x}_v = 0$. If $\widehat{x}_i = 0$ for all $i \in N_G^-(v)$ then $\widehat{\boldsymbol{x}}$ is a solution of G, and hence, the equation holds. If the 0–1 vector $\widehat{\boldsymbol{x}}'$ obtained from $\widehat{\boldsymbol{x}}$ by replacing $\widehat{x}_v$ for 1 is a solution of $G \not\mapsto v$ then it is also an optimal solution of both problems. Assume on the contrary that $\widehat{x}_i = 1$ for some $i \in N_G^-(v)$ and $\widehat{\boldsymbol{x}}'$ is not a solution of $G \not\mapsto v$. This implies that there is either $u \in N_G^{++}(v)$ with $\widehat{x}_u = 1$ or $u \in N_G^{+-}(v)$ with $\widehat{x}_u = 0$. Here $N_G^{+-}(v)$ is defined as the set of vertices adjacent to v by a $(+,-)$-edge incident to v with a plus sign, $N_G^{++}(v)$ is defined analogously. In the former case there is a $(+,+)$-edge joining i and u ; in the latter case, a $(+,-)$-edge, since G is transitive. However, in both cases, $\widehat{\boldsymbol{x}}$ does not satisfy the inequality corresponding to the edge $\langle i,u \rangle$. This is a contradiction. Hence there is a 0–1 vector which is optimal for both problems. ∎

Lemma 4.1 guarantees that if all nonpositive vertices have nonnegative weights then the original problem (4.1) can be reduced to the maximum weighted stable set problem. Furthermore, from Theorem 3.7, if the original bidirected graph is perfect then we can solve the problem (4.1) in polynomial time by using the method of Grötschel, Lovász and Schrijver [13, 14, 15], for such special weight vectors. Their method gives not only the optimal value but also a maximum weighted stable set for perfect graphs.

However, for any integral weight vector, the problem (4.1) is reducible to the maximum weighted stable set problem. The idea is very simple. From the above discussion, we can at least assume that $w_i > 0$ for all positive vertices i and $w_i < 0$ for all nonpositive vertices i. One can easily prove that there is a nonpositive vertex v such that all edges incident to v with plus signs are $(+,+)$-edges (see [20, Proposition 2.10]). Now we apply the reflection (2.1) of G at v. This transformation does not create any $(-,-)$-edge, does not change the optimization problem (4.1) essentially, and transforms the objective function $\sum_{i \in V} w_i x_i$ to $\sum_{i \neq v} w_i x_i + (-w_v)x_v + w_v$. Then, in the new problem, the nonpositive vertex v has a positive weight. Performing the $(-,+)$-edge elimination at v, we can decrease the number of nonpositive vertices. By repeating such procedures, the problem (4.1) for any closed bidirected graph can be reduced to the maximum weighted stable set problem for some undirected graph. Obviously, this reduction is done in time polynomial in the number of vertices, and preserves perfectness. We remark that each maximal stable set of the final graph corresponds to a 0–1 solution of the original bidirected graph from the proof of Lemma 4.1, and that an optimal 0–1 solution of the original problem can be reconstructed from the maximum weighted stable set (see the example below). Hence we obtain our main theorem.

Theorem 4.2. *The generalized stable set problem for any bidirected graph may be reduced to the maximum weighted stable set problem for a certain undirected graph in time polynomial in the number of vertices. Furthermore, this reduction preserves perfectness. Hence, if a given bidirected graph is perfect then the problem is solvable in time polynomial in the encoding length of the problem.*

We finally explain the procedure proposed in this paper by using an example of the facility location problem in Figure 1. Here we assume that p_i, $c_j > 0$ and that variable

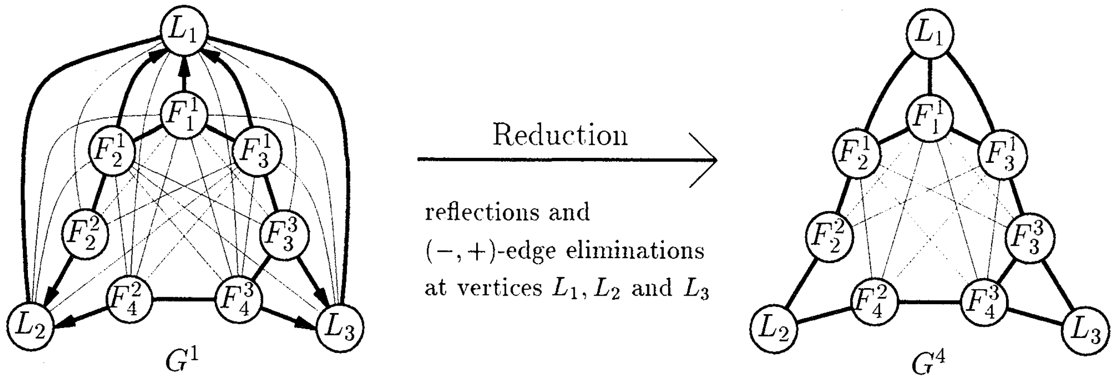


Figure 4: An example of the reduction.

x_i^j is assigned to vertex F_i^j and y_j to vertex L_j. Then the objective function is

$$p_1x_1^1 + p_2(x_2^1 + x_2^2) + p_3(x_3^1 + x_3^3) + p_4(x_4^2 + x_4^3) - c_1y_1 - c_2y_2 - c_3y_3.$$

Step 1. Construction of the transitive closure: Since the bidirected graph of Figure 1 is not transitive, let us make its transitive closure, the left-hand side bidirected graph G^1 in Figure 4, where added edges are drawn by thin lines.

Step 2. Reduction to a closed bidirected graph: In this case, G^1 is simple, i.e., closed. Furthermore G^1 is perfect because the complement of its underlying graph is easily checked to be perfect.

Step 3. Reduction to a pure bidirected graph: There is nothing to do because G^1 is also pure.

Step 4. Reduction to an undirected graph: We first select a nonpositive vertex L_1 because $-c_1 < 0$ and there is no $(+,-)$-edge incident to L_1 with a plus sign. Let $G^1{:}L_1$ denote the bidirected graph obtained from G^1 by the reflection at L_1. The $(-,+)$-edge elimination at L_1 for $G^1{:}L_1$ eliminates edges joining L_1 and $\{L_2, L_3, F_2^2, F_4^2, F_3^3, F_4^3\}$. Let $G^2 = (G^1{:}L_1)\not\to L_1$. Continuously, performing the reflections and the $(-,+)$-edge eliminations at L_2 and L_3, generates bidirected graphs $G^2{:}L_2$, $G^3 = (G^2{:}L_2)\not\to L_2$, $G^3{:}L_3$ and $G^4 = (G^3{:}L_3)\not\to L_3$ which is the right-hand side undirected graph in Figure 4. The objective function for G^4 is

$$p_1x_1^1 + p_2(x_2^1 + x_2^2) + p_3(x_3^1 + x_3^3) + p_4(x_4^2 + x_4^3) + c_1y_1 + c_2y_2 + c_3y_3 - (c_1 + c_2 + c_3).$$

Step 5. Optimal solution construction process: Let us find a maximum weighted stable set for G^4 and the weight vector. In this case all maximal stable sets of G^4 are listed below.

0−1 solutions of G^1	maximal stable sets of G^4	objective values
$\{L_1, F_1^1\}$	$\{F_1^1, L_2, L_3\}$	$p_1 - c_1$
$\{L_1, F_2^1, F_3^1\}$	$\{F_2^1, F_3^1, L_2, L_3\}$	$p_2 + p_3 - c_1$
$\{L_2, L_3, F_2^2, F_4^2, F_3^3\}$	$\{F_2^2, F_4^2, F_3^3, L_1\}$	$p_2 + p_3 + p_4 - c_2 - c_3$
$\{L_2, F_2^2, F_4^2\}$	$\{F_2^2, F_4^2, L_1, L_3\}$	$p_2 + p_4 - c_2$
$\{L_2, L_3, F_2^2, F_4^3\}$	$\{F_2^2, F_4^3, L_1\}$	$p_2 + p_4 - c_2 - c_3$
$\{L_3, F_3^3\}$	$\{F_3^3, L_1, L_2\}$	$p_3 - c_3$
$\{L_3, F_4^3\}$	$\{F_4^3, L_1, L_2\}$	$p_4 - c_3$
$\emptyset$	$\{L_1, L_2, L_3\}$	0

We assume that $p_i = c_j = 1$ for all i and j. For instance, $\{F_2^1, F_3^1, L_2, L_3\}$ is an optimal stable set of G^4. We can construct an optimal solution of the original problem from the stable set by reversely following up the reduction in the previous step. From the proof of Lemma 4.1, $\{F_2^1, F_3^1, L_2, L_3\}$ is an optimal solution for $G^3{:}L_3$. Then $\{F_2^1, F_3^1, L_2\}$ is an optimal solution of the problem for G^3. In the same way, $\{F_2^1, F_3^1, L_2\}$ and $\{F_2^1, F_3^1\}$ are optimal solutions for $G^2{:}L_2$ and G^2, respectively. From the maximality of $\{F_2^1, F_3^1, L_2, L_3\}$ in G^4, it can be shown that $\{F_2^1, F_3^1, L_1\}$ is not a solution of $G^2 = (G^1{:}L_1) \not\leftarrow L_1$, as follows: L_1 must be adjacent to a vertex of the maximal stable set in G^4; if L_1 is adjacent to F_2^1 (or F_3^1) then G^2 has a $(+,+)$-edge incident to L_1 and F_2^1 (or F_3^1); otherwise G^2 has a $(+,-)$-edge incident to L_1 and L_2 (or L_3) with a plus sign at L_1; in both cases, $\{F_2^1, F_3^1, L_1\}$ is not a solution of G^2. Then, from the proof of Lemma 4.1, $\{F_2^1, F_3^1\}$ is an optimal solution of the problem for $G^1{:}L_1$, and then $\{F_2^1, F_3^1, L_1\}$ is an optimal solution of the original problem. That is, an optimal solution for the original problem can be obtained from a maximum weighted stable set (must be inclusion-wise maximal) of the final undirected graph by reflections applied in the previous step (see the above list).

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Note added in proof. After submitting our paper, [24] was published. In this paper, a result equivalent to Theorem 2.6 was independently proved by using contexts similar to those of our paper, i.e.,the facts that the generalized stable set problem can be transformed to the maximum weighted stable set problem and that the perfectness is preserved by the transformation. However, the paper does not discuss the complexity of the generalized stable set problem for perfect bidirected graphs.

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