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TRANSIENT PROBABILITIES OF HOMOGENEOUS ROW-CONTINUOUS BIVARIATE MARKOV CHAINS WITH ONE OR TWO BOUNDARIES*

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Abstract This paper shows that row-continuous Markov chains with one or two boundaries have transient probabilities with matrix-geometric structure. Also explored is the relationship between the Green's function method and the matrix-geometric method of Neuts. A full probabilistic interpretation of transient rate matrices is given.

1. Introduction

A homogeneous row-continuous bivariate Markov chain with one or two boundaries is a natural extension of M/M/1 queueing systems with infinite or finite capacity. It is an extension in the sense that the transition rate matrix of a homogeneous row-continuous process is *block* tridiagonal while that of a birth-death process is tridiagonal. Because of this structural similarity, one may expect that a row-continuous Markov chain inherits some of the remarkably rich properties of the M/M/1 occupancy process.

Neuts (1978, 1981) has exploited the block structure of transition rate matrices to show that a homogeneous bivariate process with one boundary has a matrix geometric distribution, a bivariate analogue of the geometric distribution of the occupancy process $N(t)$ for M/M/1. For more recent works along this line, readers are referred to Ramaswami (1990) and Asmussen and Ramaswami (1990). Keilson and Zachmann (1981, 1988) employed the Green's function method to study the stationary behavior as well as some transient aspects of the homogeneous row-continuous process with one or two boundaries. The Green's function method, Keilson (1965), relates a process with boundaries to the corresponding homogeneous process with no boundaries. The method is applicable to both stationary analysis and transient analysis. Extensive discussion of the Green's function method and applications may be found in Graves and Keilson (1981), and Keilson (1965, 1979).

In this paper, the transient behavior of such processes with one or two boundaries is studied, and the matrix geometric structure of Neuts is established in the transient setting. The relationship between the Green function method and the matrix-geometric method of Neuts (1978, 1981) is also shown, providing a full probabilistic interpretation of transient rate matrices in terms of taboo probabilities. To be specific, consider the homogeneous row-continuous bivariate Markov chain with one boundary, $\boldsymbol{B}_{\mathrm{I}}(t) = (J_{\mathrm{I}}(t), N_{\mathrm{I}}(t))$, where $N_{\mathrm{I}}(t)$ is the occupancy process and $J_{\mathrm{I}}(t)$ represents the internal motion and let $\boldsymbol{p}_n(t) = [p_{n:j}(t)]$ be defined by $p_{n:j}(t) = \mathrm{P}\{\boldsymbol{B}_{\mathrm{I}}(t) = (j,n)\}$. Then, one finds that when $N_{\mathrm{I}}(0) = 0$

$$\boldsymbol{p}_n(t) = \int_0^t \boldsymbol{p}_0(t-x)\boldsymbol{R}_+^{(n)}(x)\mathrm{d}x, \qquad n \geq 0, \tag{1.1}$$

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where $\boldsymbol{R}_+(t)$ is a transient rate matrix and $\boldsymbol{R}_+^{(n)}(t)$ is the n-fold matrix convolution in time. The transient rate matrix $\boldsymbol{R}_+(t)$ is related to a taboo probability interpretation similar to that of Neuts. Both probabilistic and analytic proofs are provided for (1.1). For the two-boundary process $\boldsymbol{B}_{\mathrm{II}}(t) = (J_{\mathrm{II}}(t), N_{\mathrm{II}}(t))$ starting from the boundaries, the Laplace transform of the state probability vectors can be written as a sum of two matrix geometric terms. A well known matrix quadratic equation for the stationary rate matrix of Neuts is extended to a dynamic setting.

Also studied are some special cases of row-continuous Markov chains. When $\boldsymbol{B}_{\mathrm{I}}(t)$ with one boundary has the Markov modulated structure and $\boldsymbol{p}_0(0) = \boldsymbol{e}$ where $\boldsymbol{e}$ is the stationary probability vector of $J_{\mathrm{I}}(t)$, it can be shown that

$$\mathrm{P}\{N_{\mathrm{I}}(t) \geq n\} = \int_0^t \boldsymbol{e}\boldsymbol{R}_+^{(n)}(y)\mathbf{1}\mathrm{d}y. \tag{1.2}$$

Here $\mathbf{1}$ is the vector with all components equal to 1. This implies that, under the same condition, $\{N_{\mathrm{I}}(t)\}$ is stochastically increasing in t, which extends the corresponding result for M/M/1 system. We also re-examine M/M/1 occupancy process using the results regarding $\boldsymbol{B}_{\mathrm{I}}(t)$.

2. Dynamic Analysis of Homogeneous Row-Continuous Markov Chain with One or Two Boundaries

We first consider a bivariate (spatially) homogeneous row-continuous Markov chain $\boldsymbol{B}_{\mathrm{II}}(t) = (J_{\mathrm{II}}(t), N_{\mathrm{II}}(t))$ with two boundaries. The state space of $\boldsymbol{B}_{\mathrm{II}}(t)$ is given by $\mathbb{B} = \{(j,n) : 0 \leq j \leq J, 0 \leq n \leq N\}$. When the states for each row are treated as a block, the transition rate matrix $\boldsymbol{\nu}$ of $\boldsymbol{B}_{\mathrm{II}}(t)$ has the block tridiagonal form

$$\boldsymbol{\nu} = [\nu_{(i,m),(j,n)}] = \begin{bmatrix} \boldsymbol{\theta}_0 & \boldsymbol{\lambda} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} \\ \boldsymbol{\mu} & \boldsymbol{\theta} & \boldsymbol{\lambda} & \ddots & \vdots & \vdots \\ \mathbf{0} & \boldsymbol{\mu} & \boldsymbol{\theta} & \ddots & \mathbf{0} & \mathbf{0} \\ \vdots & \ddots & \ddots & \ddots & \boldsymbol{\lambda} & \mathbf{0} \\ \mathbf{0} & \cdots & \mathbf{0} & \boldsymbol{\mu} & \boldsymbol{\theta} & \boldsymbol{\lambda} \\ \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} & \boldsymbol{\mu} & \boldsymbol{\theta}_N \end{bmatrix} \tag{2.1}$$

where all elements are matrices of order $J+1$. Throughout the paper, $\boldsymbol{\nu}$ indicates a transition rate matrix with zero diagonal elements rather than an infinitesimal generator. The matrices $\boldsymbol{\theta}_0$ and $\boldsymbol{\theta}_N$ represent the possible irregular boundary behavior of the process. Simple retaining boundaries for which virtual transitions outside the state space are censored have $\boldsymbol{\theta}_0 = \boldsymbol{\theta}_N = \boldsymbol{\theta}$. Also considered in this section is the process $\boldsymbol{B}_{\mathrm{I}}(t) = (J_{\mathrm{I}}(t), N_{\mathrm{I}}(t))$ with one boundary at 0 and $N = \infty$. The associated spatially homogeneous process $\boldsymbol{B}_{\mathrm{H}}(t) = (J_{\mathrm{H}}(t), N_{\mathrm{H}}(t))$ is a process with state space $\mathbb{B}_{\mathrm{H}} = \{(j,n) : 0 \leq j \leq J, -\infty < n < \infty\}$, governed by the same set of transition rate matrices $\boldsymbol{\mu}, \boldsymbol{\lambda}$ and $\boldsymbol{\theta}$ but with no boundary at levels 0 and N of the occupancy process. The chain $\boldsymbol{B}_{\mathrm{H}}(t)$ is spatially homogeneous in the sense that the transition rates $\nu'_{(i,m),(j,n)}$ for the chain have the form

$$\nu'_{(i,m),(j,n)} = \mu_{ij}\delta_{m,n+1} + \theta_{ij}\delta_{mn} + \lambda_{ij}\delta_{m,n-1},$$

which depends only on $n - m$ for any (i,j). Here $\delta_{mn} = 1$ if $m = n$ and $\delta_{mn} = 0$ otherwise.

The analysis in this section is based on the results obtained in Keilson and Zachmann (1981, 1988), which are summarized in the following. The transient Green's function for the homogeneous process, $\boldsymbol{g}_n(t) = [g_{n:ij}(t)]$, $-\infty < n < \infty$, is defined by

$$g_{n:ij}(t) = \mathrm{P}\{J_{\mathrm{H}}(t) = j, N_{\mathrm{H}}(t) = n \mid J_{\mathrm{H}}(0) = i, N_{\mathrm{H}}(0) = 0\}, \quad 0 \le i, j \le J.$$

Because of the simple boundaryless structure of $\boldsymbol{B}_{\mathrm{H}}(t), \boldsymbol{g}_n(t)$ has a nice analytical structure. Specifically, by conditioning on the first visit to level n, it is seen in Keilson and Zachmann (1981, 1988) that

$$\boldsymbol{g}_n(t) = \begin{cases} \int_0^t \boldsymbol{s}_+^{(n)}(t-\tau)\boldsymbol{g}_0(\tau)\mathrm{d}\tau & \text{if } n \ge 0 \\ \int_0^t \boldsymbol{s}_-^{(|n|)}(t-\tau)\boldsymbol{g}_0(\tau)\mathrm{d}\tau & \text{if } n < 0. \end{cases} \tag{2.2}$$

Here $\boldsymbol{s}_\pm(t) = [s_{\pm:ij}(t)] = [s_{(i,m),(j,m\pm1)}(t)]$ is the upward (downward) matrix first passage time density $\boldsymbol{s}_{mn}(t) = [s_{(i,m),(j,n)}(t)]$ defined by

$$\begin{aligned} s_{(i,m),(j,n)}(t) &= \frac{\mathrm{d}}{\mathrm{d}t} S_{(i,m),(j,n)}(t) \\ S_{(i,m),(j,n)}(t) &= \mathrm{P}\{\text{the first visit of } \boldsymbol{B}_{\mathrm{H}}(\cdot) \text{ to row } n \text{ happens before time } t \\ &\qquad \text{with landing state } (j,n) \mid J_{\mathrm{H}}(0) = i,\ N_{\mathrm{H}}(0) = m\} \end{aligned}$$

and $\boldsymbol{s}_\pm^{(n)}(t)$ is the n-fold matrix convolution of $\boldsymbol{s}_\pm(t)$ with itself. The simplicity of (2.2) arises from the boundaryless structure for which $\boldsymbol{s}_{0n}(t) = \boldsymbol{s}_+^{(n)}(t)$, $n > 0$, and $\boldsymbol{s}_{0n}(t) = \boldsymbol{s}_-^{(|n|)}(t)$, $n < 0$. The compensation method relates the Green's functions $\boldsymbol{g}_n(t)$ of $\boldsymbol{B}_{\mathrm{H}}(t)$ and the state probabilities $\boldsymbol{p}_n(t) = (p_{n:j}(t))$ of the bounded process $\boldsymbol{B}_{\mathrm{II}}(t)$ where $p_{n:j}(t) = \mathrm{P}\{J_{\mathrm{II}}(t) = j, N_{\mathrm{II}}(t) = n\}$. Let the initial probability distribution of $\boldsymbol{B}_{\mathrm{II}}(t)$ be given by $(\boldsymbol{f}_n)_{n=0}^N$, i.e., $\boldsymbol{f}_n = (f_{n:j})$, $f_{n:j} = \mathrm{P}\{J_{\mathrm{II}}(0) = j,\ N_{\mathrm{II}}(0) = n\}$. For notational convenience we define $\boldsymbol{\mu}_D = [\delta_{ij}\sum_k \mu_{ik}]$ and $\boldsymbol{\lambda}_D = [\delta_{ij}\sum_k \lambda_{ik}]$. A similar notation will be used to represent other diagonal matrices. The relationship between the Green's functions and the state probabilities of the bounded process $\boldsymbol{B}_{\mathrm{II}}(t)$ is given in Keilson and Zachmann (1981, 1988) as

$$\begin{aligned} \boldsymbol{p}_n(t) &= \sum_{k=0}^{N} \boldsymbol{f}_k \boldsymbol{g}_{n-k}(t) + \sum_{k=-1}^{N+1} \boldsymbol{C}_k * \boldsymbol{g}_{n-k}(t) \\ &= \boldsymbol{C}_{-1} * \boldsymbol{g}_{n+1}(t) + \boldsymbol{C}_0 * \boldsymbol{g}_n(t) + \boldsymbol{C}_N * \boldsymbol{g}_{n-N}(t) + \boldsymbol{C}_{N+1} * \boldsymbol{g}_{n-(N+1)}(t) \\ &\quad + \sum_{m=0}^{N} \boldsymbol{f}_m \boldsymbol{g}_{n-m}(t) \end{aligned} \tag{2.3}$$

where $*$ indicates a convolution in time, and $\boldsymbol{C}_n(t) = (C_{n:j}(t))$, $-\infty < n < \infty$, are called compensation functions and are given as

$$\begin{aligned} \boldsymbol{C}_{-1}(t) &= -\boldsymbol{p}_0(t)\boldsymbol{\mu}, \quad \boldsymbol{C}_0(t) = \boldsymbol{p}_0(t)(\boldsymbol{\mu}_D + \boldsymbol{\theta}_D - \boldsymbol{\theta} + \boldsymbol{\theta}_0 - \boldsymbol{\theta}_{0D}), \\ \boldsymbol{C}_N(t) &= \boldsymbol{p}_N(t)(\boldsymbol{\lambda}_D + \boldsymbol{\theta}_D - \boldsymbol{\theta} + \boldsymbol{\theta}_N - \boldsymbol{\theta}_{ND}), \quad \boldsymbol{C}_{N+1}(t) = -\boldsymbol{p}_N(t)\boldsymbol{\lambda}, \quad \text{and} \\ \boldsymbol{C}_n(t) &= \boldsymbol{0} \qquad \text{for other } n. \end{aligned} \tag{2.4}$$

The matrix geometric structure of the transient behavior can best be observed in the Laplace transform domain. Let $\boldsymbol{\pi}_n(s) = \int_0^\infty e^{-st}\boldsymbol{p}_n(t)\mathrm{d}t$, $\gamma_n(s) = \int_0^\infty e^{-st}\boldsymbol{g}_n(t)\mathrm{d}t$, and $\boldsymbol{\sigma}_\pm(s) = \int_0^\infty e^{-st}\boldsymbol{s}_\pm(t)\mathrm{d}t$. It should be noted that the boundaryless process $J_{\mathrm{H}}(t)$ itself is Markov with transition rate matrix $\boldsymbol{\nu}_0 = \boldsymbol{\mu} + \boldsymbol{\theta} + \boldsymbol{\lambda}$. Let $\boldsymbol{e}$ be the stationary probability vector of $J_{\mathrm{H}}(t)$. It has been shown in Keilson and Zachmann (1981, 1988) that, if the process $\boldsymbol{B}_{\mathrm{H}}(t)$ has a bias toward the positive or negative side, i.e., $\boldsymbol{e}(\boldsymbol{\lambda}-\boldsymbol{\mu})\mathbf{1}\neq 0$, then $\gamma_0(s)$ is non-singular. Throughout the paper, we assume that $\boldsymbol{e}(\boldsymbol{\lambda}-\boldsymbol{\mu})\mathbf{1}\neq 0$. Then, one has the following theorem.

Theorem 2.1. For the two boundary process $\boldsymbol{B}_{\mathrm{II}}(t)$,

$$\boldsymbol{\pi}_n(s)=\boldsymbol{\pi}_0(s)\boldsymbol{D}_0(s)\boldsymbol{\rho}_+^n(s)+\boldsymbol{\pi}_N(s)\boldsymbol{D}_N(s)\boldsymbol{\rho}_-^{N-n}(s)+\sum_{m=0}^{N}\boldsymbol{f}_m\boldsymbol{\gamma}_{n-m}(s),\quad 0\le n\le N$$

where

$$\boldsymbol{\rho}_+(s)=\boldsymbol{\gamma}_0^{-1}(s)\boldsymbol{\gamma}_1(s),\qquad \boldsymbol{\rho}_-(s)=\boldsymbol{\gamma}_0^{-1}(s)\boldsymbol{\gamma}_{-1}(s),$$
$$\boldsymbol{D}_0(s)=(\boldsymbol{\mu}_D+\boldsymbol{\theta}_D-\boldsymbol{\theta}+\boldsymbol{\theta}_0-\boldsymbol{\theta}_{0D}-\boldsymbol{\mu}\boldsymbol{\sigma}_+(s))\boldsymbol{\gamma}_0(s),$$

and

$$\boldsymbol{D}_N(s)=(\boldsymbol{\lambda}_D+\boldsymbol{\theta}_D-\boldsymbol{\theta}+\boldsymbol{\theta}_N-\boldsymbol{\theta}_{ND}-\boldsymbol{\lambda}\boldsymbol{\sigma}_-(s))\boldsymbol{\gamma}_0(s).$$

Proof: From (2.3) and (2.4) one sees for $-\infty<n<\infty$ that

$$\begin{aligned}\boldsymbol{\pi}_n(s)&=\boldsymbol{\pi}_0(s)(\boldsymbol{\mu}_D+\boldsymbol{\theta}_D-\boldsymbol{\theta}+\boldsymbol{\theta}_0-\boldsymbol{\theta}_{0D})\boldsymbol{\gamma}_n(s)-\boldsymbol{\pi}_0(s)\boldsymbol{\mu}\boldsymbol{\gamma}_{n+1}(s)\\&\quad+\boldsymbol{\pi}_N(s)(\boldsymbol{\lambda}_D+\boldsymbol{\theta}_D-\boldsymbol{\theta}+\boldsymbol{\theta}_N-\boldsymbol{\theta}_{ND})\boldsymbol{\gamma}_{n-N}(s)-\boldsymbol{\pi}_N(s)\boldsymbol{\lambda}\boldsymbol{\gamma}_{n-(N+1)}(s)+\sum_{m=0}^{N}\boldsymbol{f}_m\boldsymbol{\gamma}_{n-m}(s).\end{aligned}$$

Since $\boldsymbol{\gamma}_n(s)=\boldsymbol{\sigma}_+^n(s)\boldsymbol{\gamma}_0(s)$, $n\ge 0$, and $\boldsymbol{\gamma}_n(s)=\boldsymbol{\sigma}_-^{|n|}(s)\boldsymbol{\gamma}_0(s)$ for $n\le 0$ (see (2.2)), one has for $0\le n\le N$

$$\begin{aligned}\boldsymbol{\pi}_n(s)&=\boldsymbol{\pi}_0(s)(\boldsymbol{\mu}_D+\boldsymbol{\theta}_D-\boldsymbol{\theta}+\boldsymbol{\theta}_0-\boldsymbol{\theta}_{0D}-\boldsymbol{\mu}\boldsymbol{\sigma}_+(s))\boldsymbol{\gamma}_0(s)\left(\boldsymbol{\gamma}_0^{-1}(s)\boldsymbol{\gamma}_1(s)\right)^n\\&\quad+\boldsymbol{\pi}_N(s)(\boldsymbol{\lambda}_D+\boldsymbol{\theta}_D-\boldsymbol{\theta}+\boldsymbol{\theta}_N-\boldsymbol{\theta}_{ND}-\boldsymbol{\lambda}\boldsymbol{\sigma}_-(s))\boldsymbol{\gamma}_0(s)\left(\boldsymbol{\gamma}_0^{-1}(s)\boldsymbol{\gamma}_{-1}(s)\right)^{N-n}\\&\quad+\sum_m\boldsymbol{f}_m\boldsymbol{\gamma}_{n-m}(s)\end{aligned}$$

completing the proof. □

The following statements establish the matrix geometric structure of row-continuous Markov chains with one or two boundaries in a transient setting.

Theorem 2.2. For the one boundary process $\boldsymbol{B}_{\mathrm{I}}(t)$ with $N_{\mathrm{I}}(0)=0$, $\boldsymbol{\pi}_n(s)=\boldsymbol{\pi}_0(s)\boldsymbol{\rho}_+^n(s)$, $n\ge 0$.

Proof: The transient probabilities of the one boundary case fall out from Theorem 2.1 by eliminating the second term corresponding to the reflected probability flow from the boundary at N. Hence, using $\boldsymbol{f}_n=\delta_{n0}\boldsymbol{f}_0$, one sees that for $n\ge 0$

$$\begin{aligned}\boldsymbol{\pi}_n(s)&=\boldsymbol{\pi}_0(s)\boldsymbol{D}_0(s)\boldsymbol{\rho}_+^n(s)+\boldsymbol{f}_0\boldsymbol{\gamma}_n(s)\\&=\boldsymbol{\pi}_0(s)\boldsymbol{D}_0(s)\boldsymbol{\rho}_+^n(s)+\boldsymbol{f}_0\boldsymbol{\gamma}_0(s)\boldsymbol{\rho}_+^n(s)\\&=(\boldsymbol{\pi}_0(s)\boldsymbol{D}_0(s)+\boldsymbol{f}_0\boldsymbol{\gamma}_0(s))\boldsymbol{\rho}_+^n(s).\end{aligned}$$

Thus, $\boldsymbol{\pi}_0(s)=\boldsymbol{\pi}_0(s)\boldsymbol{D}_0(s)+\boldsymbol{f}_0\boldsymbol{\gamma}_0(s)$, which implies $\boldsymbol{\pi}_n(s)=\boldsymbol{\pi}_0(s)\boldsymbol{\rho}_+^n(s)$, completing the proof. □

The proofs of the following statements are similar to that of Theorem 2.2 and are omitted.

Theorem 2.3. For the two boundary process $\boldsymbol{B}_{\mathrm{II}}(t)$ with all the initial support in the boundaries, i.e., $\boldsymbol{f}_n = \delta_{n0}\boldsymbol{f}_0 + \delta_{nN}\boldsymbol{f}_N$, $\boldsymbol{\pi}_n(s)$ has the following matrix geometric structure:

$$\boldsymbol{\pi}_n(s) = (\boldsymbol{\pi}_0(s)\boldsymbol{D}_0(s) + \boldsymbol{f}_0\gamma_0(s))\,\boldsymbol{\rho}_+^n(s) + (\boldsymbol{\pi}_N(s)\boldsymbol{D}_N(s) + \boldsymbol{f}_N\gamma_0(s))\,\boldsymbol{\rho}_-^{N-n}(s), \quad 0 \le n \le N.$$

Corollary 2.4. For the one boundary process $\boldsymbol{B}_{\mathrm{I}}(t)$,

$$\boldsymbol{\pi}_n(s) = \boldsymbol{\pi}_0(s)\boldsymbol{D}_0(s)\boldsymbol{\rho}_+^n(s) + \sum_m \boldsymbol{f}_m\gamma_{n-m}(s), \quad n \ge 0.$$

Corollary 2.5. For the one boundary process $\boldsymbol{B}_{\mathrm{I}}(t)$, if the initial distribution has bounded support with $\boldsymbol{f}_m = 0$, $m > M$, for some $M > 0$, $\boldsymbol{\pi}_n(s)$ for $n \ge M$ has the purely geometric structure:

$$\boldsymbol{\pi}_{M+n}(s) = \boldsymbol{\pi}_M(s)\boldsymbol{\rho}_+^n(s), \quad n \ge 0.$$

It is noted that the preceding results of the transient matrix geometric structure for the process with one boundary $\boldsymbol{B}_{\mathrm{I}}(t)$ are valid for stable as well as unstable systems.

As pointed out in Keilson and Zachmann (1981, 1988), the matrix geometric stationary probabilities of the one boundary process in Neuts (1978, 1981), can be obtained in terms of the ergodic Green's function $\boldsymbol{g}_{n:\infty} \equiv \int_0^\infty \boldsymbol{g}_n(t)\mathrm{d}t = \gamma_n(0)$. Here, we assume that $\boldsymbol{B}_{\mathrm{H}}(t)$ is transient having bias towards the negative side, $\boldsymbol{e}(\boldsymbol{\lambda} - \boldsymbol{\mu})\boldsymbol{1} < 0$, so that the ergodic Green's functions $\boldsymbol{g}_{n:\infty}$, $-\infty < n < \infty$, exist and $\boldsymbol{B}_{\mathrm{I}}(t)$ has stationary probabilities $\boldsymbol{e}_n$, $n \ge 0$. Since $\boldsymbol{g}_n(t) \to \boldsymbol{0}$ as $t \to \infty$ for every n, one sees from Corollary 2.4 that the stationary probabilities $\boldsymbol{e}_n = \lim_{t\to\infty}\boldsymbol{p}_n(t), 0 \le n \le N$, of the one boundary process $\boldsymbol{B}_{\mathrm{I}}(t)$ are given by

$$\boldsymbol{e}_n = \boldsymbol{e}_0\boldsymbol{\rho}_+^n(0) \quad \text{where} \quad \boldsymbol{\rho}_+(0) = \boldsymbol{g}_{0:\infty}^{-1}\boldsymbol{\sigma}_+(0)\boldsymbol{g}_{0:\infty}. \tag{2.5}$$

Also for the case of two boundary process $\boldsymbol{B}_{\mathrm{II}}(t)$, the stationary probabilities can be found from Theorem 2.1 as

$$\boldsymbol{e}_n = \boldsymbol{e}_0\boldsymbol{D}_0(0)\boldsymbol{\rho}_+^n(0) + \boldsymbol{e}_N\boldsymbol{D}_N(0)\boldsymbol{\rho}_-^{N-n}(0) \quad \text{where} \quad \boldsymbol{\rho}_-(0) = \boldsymbol{g}_{0:\infty}^{-1}\boldsymbol{\sigma}_-(0)\boldsymbol{g}_{0:\infty}, \tag{2.6}$$

see Keilson and Zachmann (1981, 1988).

3. Dynamic Quadratic Equations for Transient Rate Matrices and the Relation to Matrix-Geometric Method of Neuts.

In the setting of Neuts (1981), consider a discrete time row-continuous process $\boldsymbol{B}_{\mathrm{I}}^*(k) = (J_{\mathrm{I}}^*(k), N_{\mathrm{I}}^*(k))$ with one boundary governed by the upward transition probability matrix $\boldsymbol{a}_+$, transverse transition probability matrix $\boldsymbol{a}_0$, and downward transition probability matrix $\boldsymbol{a}_-$. Neuts (1981) shows that the stationary probabilities $(\boldsymbol{e}_n)_{n=0}^\infty$ of $\boldsymbol{B}_{\mathrm{I}}^*(k)$ have the matrix geometric form $\boldsymbol{e}_n = \boldsymbol{e}_0\boldsymbol{R}^{*n}$ with $\boldsymbol{R}^* = [R_{ij}^*]$ defined by

$$R_{ij}^* = \sum_{k=0}^\infty {}_m\mathrm{P}^{*(k)}_{(i,m)(j,m+1)}. \tag{3.1}$$

Here ${}_m\mathrm{P}^{*(k)}_{(i,m)(j,n)}$ is the taboo probability that, starting from the state (i,m) at time 0, the chain $\boldsymbol{B}_{\mathrm{I}}^*(k)$ is in (j,n) at time k without having returned to the level m in time interval $[1,k]$. It is also shown that $\boldsymbol{R}^*$ is the minimal nonnegative solution of a matrix quadratic equaiton:

$$\boldsymbol{R}^* = \boldsymbol{R}^{*2}\boldsymbol{a}_- + \boldsymbol{R}^*\boldsymbol{a}_0 + \boldsymbol{a}_+. \tag{3.2}$$

One may expect that $\boldsymbol{\rho}_+(0)$ corresponds to the rate matrix $\boldsymbol{R}^*$ of Neuts (1981), defined in terms of taboo probabilities, and is related to a matrix quadratic equation similar to (3.2). However, it is not apriori clear that $\boldsymbol{\rho}_+(0)$ corresponds to $\boldsymbol{R}^*$. In the following, it is shown that $\boldsymbol{R}_+(t)$ and $\boldsymbol{R}_-(t)$ defined by $\int_0^\infty e^{-st}\boldsymbol{R}_\pm(t)\mathrm{d}t = \boldsymbol{\rho}_\pm(s)$ are related to taboo probabilities and satisfy a dynamic form of the matrix quadratic equation (3.2). Some preliminary results are needed. Let $\boldsymbol{\nu}_D = [\delta_{ij}\sum_k(\mu_{ik}+\theta_{ik}+\lambda_{ik})]$ and $\boldsymbol{I}$ be the identity matrix.

Lamma 3.1.

$$\boldsymbol{\rho}_+(s) = \boldsymbol{\lambda}(s\boldsymbol{I}+\boldsymbol{\nu}_D-\boldsymbol{\theta}-\boldsymbol{\lambda}\boldsymbol{\sigma}_-(s))^{-1} \quad \text{and} \quad \boldsymbol{\rho}_-(s) = \boldsymbol{\mu}(s\boldsymbol{I}+\boldsymbol{\nu}_D-\boldsymbol{\theta}-\boldsymbol{\mu}\boldsymbol{\sigma}_+(s))^{-1}.$$

Proof: The forward Kolmogorov equation at level one for the homogeneous process is given by

$$\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{g}_1(t) = -\boldsymbol{g}_1(t)\boldsymbol{\nu}_D + \boldsymbol{g}_1(t)\boldsymbol{\theta} + \boldsymbol{g}_0(t)\boldsymbol{\lambda} + \int_0^t \boldsymbol{g}_1(x)\boldsymbol{\lambda}\boldsymbol{s}_-(t-x)\mathrm{d}x.$$

Taking the Laplace transform of the both sides and rearranging the terms, one has $\gamma_1(s) = \gamma_0(s)\boldsymbol{\lambda}(s\boldsymbol{I}+\boldsymbol{\nu}_D-\boldsymbol{\theta}-\boldsymbol{\lambda}\boldsymbol{\sigma}_-(s))^{-1}$. Thus, $\boldsymbol{\rho}_+(s) = \gamma_0^{-1}(s)\gamma_1(s) = \boldsymbol{\lambda}(s\boldsymbol{I}+\boldsymbol{\nu}_D-\boldsymbol{\theta}-\boldsymbol{\lambda}\boldsymbol{\sigma}_-(s))^{-1}$, proving the first equation. The second equation can be obtained from the Kolmogorov equation at level -1. □

Lemma 3.2.

$$\boldsymbol{\rho}_+(s)\boldsymbol{\mu} = \boldsymbol{\lambda}\boldsymbol{\sigma}_-(s) \quad \text{and} \quad \boldsymbol{\rho}_-(s)\boldsymbol{\nu} = \boldsymbol{\lambda}\boldsymbol{\sigma}_+(s).$$

Proof: Let

(3.3) $$\gamma^o(s) = (s\boldsymbol{I}+\boldsymbol{\nu}_D-\boldsymbol{\theta})^{-1},$$

corresponding to the state probabilities of Markov chain governed by the defective infinitesimal generator $-\boldsymbol{\nu}_D+\boldsymbol{\theta}$. As in Keilson and Zachmann (1981, 1988), the Laplace transforms of the first passage time density matrices satisfy

(3.4) $$\boldsymbol{\sigma}_+(s) = \gamma^o(s)\boldsymbol{\lambda} + \gamma^o(s)\boldsymbol{\mu}(\boldsymbol{\sigma}_+(s))^2 \quad \text{and} \quad \boldsymbol{\sigma}_-(s) = \gamma^o(s)\boldsymbol{\mu} + \gamma^o(s)\boldsymbol{\lambda}(\boldsymbol{\sigma}_-(s))^2,$$

the probabilistic meaning of which should be clear. Equation (3.3) and the second equation in (3.4) imply $(s\boldsymbol{I}+\boldsymbol{\nu}_D-\boldsymbol{\theta}-\boldsymbol{\lambda}\boldsymbol{\sigma}_-(s))\boldsymbol{\sigma}_-(s) = \boldsymbol{\mu}$. One also sees from Lemma 3.1 that $\boldsymbol{\rho}_+(s)(s\boldsymbol{I}+\boldsymbol{\nu}_D-\boldsymbol{\theta}-\boldsymbol{\lambda}\boldsymbol{\sigma}_-(s))\boldsymbol{\sigma}_-(s) = \boldsymbol{\lambda}\boldsymbol{\sigma}_-(s)$. Thus, $\boldsymbol{\rho}_+(s)\boldsymbol{\mu} = \boldsymbol{\lambda}\boldsymbol{\sigma}_-(s)$ follows from the last two equations. The second equation is immediate from the first equation by switching the upward motion and the downward motion. □

It should be noted that Lemma 3.2 relates the upward and downward first passage time density matrices $\boldsymbol{s}_+(t)$ and $\boldsymbol{s}_-(t)$ by

(3.5) $$\boldsymbol{\sigma}_+(s)\gamma_0(s)\boldsymbol{\mu} = \gamma_0(s)\boldsymbol{\lambda}\boldsymbol{\sigma}_-(s) \quad \text{and} \quad \boldsymbol{\sigma}_-(s)\gamma_0(s)\boldsymbol{\lambda} = \gamma_0(s)\boldsymbol{\mu}\boldsymbol{\sigma}_+(s).$$

We are now in a position to show a dynamic form of the quadratic equations for transient rate matrices $\boldsymbol{R}_+(t)$ and $\boldsymbol{R}_-(t)$ in the Laplace transform domain.

Theorem 3.3. $\boldsymbol{\rho}_+(s)$ and $\boldsymbol{\rho}_-(s)$ satisfies the matrix quadratic equations:
(a) $\boldsymbol{\lambda} + \boldsymbol{\rho}_+(s)(-s\boldsymbol{I}-\boldsymbol{\nu}_D+\boldsymbol{\theta}) + \boldsymbol{\rho}_+^2(s)\boldsymbol{\mu} = \boldsymbol{0}$, and
(b) $\boldsymbol{\mu} + \boldsymbol{\rho}_-(s)(-s\boldsymbol{I}-\boldsymbol{\nu}_D+\boldsymbol{\theta}) + \boldsymbol{\rho}_-^2(s)\boldsymbol{\lambda} = \boldsymbol{0}$, respectively.

Proof: Substituting the first equation of Lemma 3.2 into the first equation of Lemma 3.1, one has $\boldsymbol{\rho}_+(s) = \boldsymbol{\lambda}(s\boldsymbol{I}+\boldsymbol{\nu}_D-\boldsymbol{\theta}-\boldsymbol{\rho}_+(s)\boldsymbol{\mu})^{-1}$. The quadratic equation (a) for $\boldsymbol{\rho}_+(s)$ follows by rearranging the terms. Since the boundaryless process $\boldsymbol{B}_{\mathrm{H}}(t)$ is structurally symmetric with respect to $\boldsymbol{\lambda}$ and $\boldsymbol{\mu}$, the quadratic equation (b) for $\boldsymbol{\rho}_{-}(s)$ follows immediately from (a). □

Clearly Theorem 3.3 implies that the stationary rate matrix $\boldsymbol{\rho}_{+}(0)$ satisfies the quadratic equation, $\boldsymbol{R}^2\boldsymbol{\mu} + \boldsymbol{R}(\boldsymbol{\theta} - \boldsymbol{\nu}_D) + \boldsymbol{\lambda} = \mathbf{0}$, in Neuts (1981).

The following theorem provides the probabilistic interpretation of the transient rate matrices.

Theorem 3.4.

$$\begin{aligned} R_{+:ij}(t) &= \sum_k \lambda_{ik} \cdot {}_{n-1}\mathrm{P}_{(k,n)(j,n)}(t), \\ R_{-:ij}(t) &= \sum_k \mu_{ik} \cdot {}_{n+1}\mathrm{P}_{(k,n)(j,n)}(t) \end{aligned}$$

where ${}_{n\pm1}\mathrm{P}_{(i,n)(j,n)}(t)$ is the taboo probability that, starting from the state (i,n) at time 0, the homogeneous chain $\boldsymbol{B}_{\mathrm{H}}(t)$ is in (j,n) at time t without having visited to the level $n \pm 1$ in time interval $(0,t]$.

Proof: Consider a defective Markov chain $\boldsymbol{B}^o(t)$ in $\{(j,n) : 1 \le j \le J, 1 \le n\}$ by censoring all transitions of $\boldsymbol{B}_{\mathrm{H}}(t)$ from level 0 to level 1. Then it is clear that

$${}_{n-1}\mathrm{P}_{(k,n)(j,n)}(t) = \mathrm{P}\{\boldsymbol{B}^o(t) = (j,1) \mid \boldsymbol{B}^o(0) = (k,1)\}.$$

Let ${}_{n-1}\boldsymbol{\Pi}_{nn}(s) = \int_0^\infty e^{-st}{}_{n-1}\mathbf{P}_{nn}(t)\mathrm{d}t$ where ${}_{n-1}\mathbf{P}_{nn}(t) = [{}_{n-1}\mathrm{P}_{(k,n)(j,n)}(t)]$. By conditioning on the first transition of $\boldsymbol{B}^o(t)$, it can be seen that

$${}_{n-1}\boldsymbol{\Pi}_{nn}(s) = (s\boldsymbol{I} + \boldsymbol{\nu}_D)^{-1} + (s\boldsymbol{I} + \boldsymbol{\nu}_D)^{-1}\boldsymbol{\theta}_{n-1}\boldsymbol{\Pi}_{nn}(s) + (s\boldsymbol{I} + \boldsymbol{\nu}_D)^{-1}\boldsymbol{\lambda}\boldsymbol{\sigma}_{-}(s)_{n-1}\boldsymbol{\Pi}_{nn}(s).$$

Rearranging the terms,

$$\begin{aligned} {}_{n-1}\boldsymbol{\Pi}_{nn}(s) &= \left(\boldsymbol{I} - (s\boldsymbol{I} + \boldsymbol{\nu}_D)^{-1}(\boldsymbol{\theta} + \boldsymbol{\lambda}\boldsymbol{\sigma}_{-}(s))\right)^{-1}(s\boldsymbol{I} + \boldsymbol{\nu}_D)^{-1} \\ &= (s\boldsymbol{I} + \boldsymbol{\nu}_D - \boldsymbol{\theta} - \boldsymbol{\lambda}\boldsymbol{\sigma}_{-}(s))^{-1}. \end{aligned}$$

Thus one sees from Lemma 3.1 that

$$\boldsymbol{\rho}_{+}(s) = \boldsymbol{\lambda}(s\boldsymbol{I} + \boldsymbol{\nu}_D - \boldsymbol{\theta} - \boldsymbol{\lambda}\boldsymbol{\sigma}_{-}(s))^{-1} = \boldsymbol{\lambda}_{n-1}\boldsymbol{\Pi}_{nn}(s),$$

which implies the first equation. The case of $\boldsymbol{R}_{-}(t)$ can be proved similarly. □

Theorem 3.4 clarifies the probabilistic meaning of previous results given in terms of Laplace transforms. In particular, for the one boundary process with $N(0) = 0$, Theorem 2.2 is equivalent to

$$\boldsymbol{p}_n(t) = \int_0^t \boldsymbol{p}_0(x)\boldsymbol{R}_{+}^{(n)}(t-x)\mathrm{d}x, \quad n \ge 0.$$

Theorem 3.4 actually provides an alternative probabilistic proof for Theorem 2.2. Consider the one boundary process $\boldsymbol{B}_{\mathrm{I}}(t)$ with $N(0) = 0$. Then, by conditioning on the state of the last visit to level n from below, one sees from Theorem 3.4 that for every $n \ge 0$

$$\begin{aligned} \boldsymbol{p}_n(t) &= \int_0^t \boldsymbol{p}_{n-1}(x)\boldsymbol{\lambda}_{n-1}\mathbf{P}_{nn}(t-x)\mathrm{d}x \\ &= \boldsymbol{p}_0 * (\boldsymbol{\lambda}_{n-1}\mathbf{P}_{nn})^{(n)}(t), \end{aligned} \qquad (3.6)$$

which proves Theorem 2.2. When the process does not start from level 0, e.g. when $N(0) = M > 0$, $\boldsymbol{B}_{\mathrm{I}}(t)$ may reach level n without hitting level 0 with positive probability. This breaks the purely geometric structure of transient state probabilities. The argument used in (3.6), however, shows that $\boldsymbol{p}_n(t)$ for $n \geq M$, has the following geometric structure: $\boldsymbol{p}_n(t) = \int_0^t \boldsymbol{p}_M(x)\boldsymbol{R}_+^{(n-M)}(t-x)\mathrm{d}x$.

It is instructive to exhibit the discrete time version of Theorem 3.4. Consider the discrete time row-continuous process $\boldsymbol{B}_{\mathrm{I}}^*(k) = (J_{\mathrm{I}}^*(k), N_{\mathrm{I}}^*(k))$, $k = 0, 1, \cdots$, with one boundary discussed at the beginning of this section. Let $\boldsymbol{\pi}_n^*(u)$, $n \geq 0$, be the generating function of the state probability vector of $\boldsymbol{B}_{\mathrm{I}}^*(k)$, and let $\boldsymbol{\gamma}_n^*(u)$, $-\infty < n < \infty$, be the generating functions of the Green's function associated with the discrete time process $\boldsymbol{B}_{\mathrm{H}}^*(k)$ without boundaries. If $\boldsymbol{B}_{\mathrm{I}}^*(k)$ starts from the boundary, then it can be shown that $\boldsymbol{\pi}_n^*(u)$ has a matrix geometric structure. That is, $\boldsymbol{\pi}_n^*(u) = \boldsymbol{\pi}_0^*(u)\boldsymbol{\rho}_+^{*n}(u)$ where $\boldsymbol{\rho}_+^*(u)$ is given by

$$\boldsymbol{\rho}_+^*(u) = u\boldsymbol{a}_+\left(\boldsymbol{I} - u\boldsymbol{a}_0 - u\boldsymbol{a}_+\boldsymbol{\sigma}_+^*(u)\right)^{-1},$$

which is the discrete time analogue of Lemma 3.1. Here $\boldsymbol{\sigma}_+^*(u)$ is the generating function of the upward first passage time distribution of $\boldsymbol{B}_{\mathrm{H}}^*(k)$. Also, using the argument similar to the proof of Theorem 3.4, the generating function of the taboo probability matrix can be shown to satisfy ${}_{n-1}\boldsymbol{\Pi}_{nn}^*(u) = \left(\boldsymbol{I} - u\boldsymbol{a}_0 - u\boldsymbol{a}_+\boldsymbol{\sigma}_+^*(u)\right)^{-1}$. Thus, one sees that

$$\boldsymbol{\rho}_+^*(u) = u\boldsymbol{a}_+\,{}_{n-1}\boldsymbol{\Pi}_{nn}^*(u) = {}_n\boldsymbol{\Pi}_{n,n+1}^*(u),$$

where the obvious notation has been used. This equation, after setting $u = 1$, agrees with (3.1) as expected. A similar equation can be obtained for $\boldsymbol{\rho}_-^*(u)$ as well.

4. Markov Modulated Process and M/M/1 System

In this section, we investigate some special cases of the process with one boundary considered in the previous section. Specifically, the Markov modulated process and the dynamics of M/M/1 are studied.

Let $\boldsymbol{\lambda} = \boldsymbol{\lambda}_D$, $\boldsymbol{\mu} = \boldsymbol{\mu}_D$, $\boldsymbol{\theta}_0 = \boldsymbol{\theta}$ and $N = \infty$ in (2.1). One then has the Markov modulated process as a special case of the row-continuous Markov chain $\boldsymbol{B}_{\mathrm{I}}(t) = (J_{\mathrm{I}}(t), N_{\mathrm{I}}(t))$ with one boundary defined in Section 2. For this case, it is clear that $J_{\mathrm{I}}(t)$ itself is Markov with transition rate matrix $\boldsymbol{\nu}_0$ and infinitesimal generator $\boldsymbol{Q}_0$ given by

$$\boldsymbol{\nu}_0 = \boldsymbol{\theta}, \quad \boldsymbol{Q}_0 = \boldsymbol{\theta} - \boldsymbol{\theta}_D. \tag{4.1}$$

Let $\boldsymbol{e}$ be the stationary probability vector of $J_{\mathrm{I}}(t)$, i.e. $\boldsymbol{e}\boldsymbol{Q}_0 = \boldsymbol{0}$ and $\boldsymbol{e}\boldsymbol{1} = 1$. Also denote the busy period of $\boldsymbol{B}_{\mathrm{I}}(t)$ (the first passage time of $N_{\mathrm{I}}(t)$ from 1 to 0) by T_B.

Theorem 4.1.
(a) For the Markov modulated process case, the one boundary process $\boldsymbol{B}_{\mathrm{I}}(t) = (J_{\mathrm{I}}(t), N_{\mathrm{I}}(t))$ with $\boldsymbol{f}_n = \delta_{n0}\boldsymbol{f}_0$ has the state probabilities

$$\boldsymbol{\pi}_n(s) = \boldsymbol{f}_0(s\boldsymbol{I} - \boldsymbol{Q}_0)^{-1}(\boldsymbol{I} - \boldsymbol{\rho}_+(s))\boldsymbol{\rho}_+^n(s), \quad n \geq 0.$$

(b) In particular, if $\boldsymbol{f}_n = \delta_{n0}\boldsymbol{e}$,

$$\mathrm{P}\{N_{\mathrm{I}}(t) \geq n\} = \int_0^t \boldsymbol{e}\boldsymbol{R}_+^{(n)}(y)\boldsymbol{1}\mathrm{d}y \quad \text{and}$$

(c) $\mathrm{E}[N_{\mathrm{I}}(t)] = \sum_i \sum_j e_i \lambda_{ij} \int_0^t \mathrm{P}\{T_B > y \mid J_{\mathrm{I}}(0) = j,\ N_{\mathrm{I}}(0) = 1\} \mathrm{d}y.$

Proof: Since $J_{\mathrm{I}}(t)$ itself is Markov with the infinitesimal generator $\boldsymbol{Q}_0$, $\boldsymbol{\pi}(s) \equiv \sum_0^\infty \boldsymbol{\pi}_n(s) = \boldsymbol{f}_0(s\boldsymbol{I} - \boldsymbol{Q}_0)^{-1}$. One also sees from Theorem 2.2 that $\boldsymbol{\pi}(s) = \boldsymbol{\pi}_0(s)(\boldsymbol{I} - \boldsymbol{\rho}_+(s))^{-1}$. Thus

(4.2) $$\boldsymbol{\pi}_0(s) = \boldsymbol{f}_0(s\boldsymbol{I} - \boldsymbol{Q}_0)^{-1}(\boldsymbol{I} - \boldsymbol{\rho}_+(s)).$$

This proves (a) using Theorem 2.2 again. If $\boldsymbol{f}_n = \delta_{n0}\boldsymbol{e}$, (a) implies that

$$\boldsymbol{\pi}_n(s)\mathbf{1} = \boldsymbol{e}(s\boldsymbol{I} - \boldsymbol{Q}_0)^{-1}(\boldsymbol{I} - \boldsymbol{\rho}_+(s))\boldsymbol{\rho}_+^n(s)\mathbf{1} = \frac{\boldsymbol{e}}{s}(\boldsymbol{I} - \boldsymbol{\rho}_+(s))\boldsymbol{\rho}_+^n(s)\mathbf{1}.$$

Thus, $\sum_{k=n}^\infty \boldsymbol{\pi}_k(s)\mathbf{1} = \boldsymbol{e}\boldsymbol{\rho}_+^n(s)\mathbf{1}/s$, which proves (b). For (c), first note that the Kolmogorov equation of $\boldsymbol{B}_{\mathrm{I}}(t)$ at row 0 is $\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{p}_0(t) = -\boldsymbol{p}_0(t)(\boldsymbol{\theta}_D + \boldsymbol{\lambda}_D) + \boldsymbol{p}_0(t)\boldsymbol{\theta} + \int_0^t \boldsymbol{p}_0(x)\boldsymbol{\lambda}_D\boldsymbol{s}_-(t-s)\mathrm{d}x$. Thus,

(4.3) $$\boldsymbol{\pi}_0(s) = \boldsymbol{f}_0\left(s\boldsymbol{I} - \boldsymbol{Q}_0 + \boldsymbol{\lambda}_D(\boldsymbol{I} - \boldsymbol{\sigma}_-(s))\right)^{-1}.$$

Since (4.2) and (4.3) are valid for all probability vectors $\boldsymbol{f}_0$ and all the matrices involved are nonsingular for $\mathrm{Re}(s) > 0$,

(4.4) $$(\boldsymbol{I} - \boldsymbol{\rho}_+(s))^{-1} = (s\boldsymbol{I} + \boldsymbol{\lambda}_D + \boldsymbol{\nu}_{0D} - \boldsymbol{\nu}_0 - \boldsymbol{\lambda}_D\boldsymbol{\sigma}_-(s))\,(s\boldsymbol{I} - \boldsymbol{Q}_0)^{-1}.$$

Thus,

$$\begin{aligned}
\int_0^\infty e^{-st}\mathrm{E}[N_{\mathrm{I}}(t)]\mathrm{d}t &= \sum_{n=0}^\infty \int_0^\infty e^{-st}\mathrm{P}\{N_{\mathrm{I}}(t) > n\}\mathrm{d}t \\
&= \frac{\boldsymbol{e}}{s}\boldsymbol{\rho}_+(s)\sum_0^\infty \boldsymbol{\rho}_+^n(s)\mathbf{1} && \text{(via (b))} \\
&= \frac{\boldsymbol{e}}{s}(\boldsymbol{I} - \boldsymbol{\rho}_+(s))^{-1}\boldsymbol{\rho}_+(s)\mathbf{1} \\
&= \frac{\boldsymbol{e}}{s}(\boldsymbol{I} - \boldsymbol{\rho}_+(s))^{-1}\mathbf{1} - \frac{1}{s} \\
&= \frac{\boldsymbol{e}}{s}(s\boldsymbol{I} + \boldsymbol{\lambda}_D + \boldsymbol{\nu}_{0D} - \boldsymbol{\nu}_0 - \boldsymbol{\lambda}_D\boldsymbol{\sigma}_-(s))\,(s\boldsymbol{I} - \boldsymbol{Q}_0)^{-1}\mathbf{1} - \frac{1}{s} && \text{(via (4.4))} \\
&= \frac{\boldsymbol{e}}{s^2}\boldsymbol{\lambda}_D(\boldsymbol{I} - \boldsymbol{\sigma}_-(s))\mathbf{1}.
\end{aligned}$$

The last equality follows from $(s\boldsymbol{I} - \boldsymbol{Q}_0)^{-1}\mathbf{1} = \frac{1}{s}\mathbf{1}$. Hence one sees that $\frac{\mathrm{d}}{\mathrm{d}t}\mathrm{E}[N_{\mathrm{I}}(t)] = \sum_i \sum_j e_i\lambda_{ij}\mathrm{P}\{T_B > t \mid J_{\mathrm{I}}(0) = j,\ N_{\mathrm{I}}(0) = 1\}$, proving (c). ▯

We note in the preceding analysis that the Markov property of $J_{\mathrm{I}}(t)$ is essential. Thus, the fact that $\boldsymbol{\lambda}$ and $\boldsymbol{\mu}$ are diagonal is a critical assumption.

It is known that the M/M/1 occupancy process $N(t)$ given $N(0) = 0$ is stochastically increasing in t, see Keilson (1979). The following statement extends this monotonicity result to the Markov modulated process. Since Theorem 3.4 implies $\boldsymbol{R}_+(t) \geq \mathbf{0}$ for every $t \geq 0$, one sees from Theorem 4.1 (b) that

Corollary 4.2. For the Markov modulated process with $\boldsymbol{f}_n = \delta_{n0}\boldsymbol{e}$, the occupancy process $N_{\mathrm{I}}(t)$ is stochastically increasing in t.

We now consider the following special case of the Markov modulated process $\boldsymbol{B}_{\mathrm{I}}(t)$. Suppose that $\boldsymbol{\lambda} = \boldsymbol{\lambda}_{\mathrm{D}}$ and $\boldsymbol{\mu} = \mu\boldsymbol{I}$. That is, only the arrival rate is dependent on $J_{\mathrm{I}}(t)$. It should be noted that $\boldsymbol{s}_-(t)$ corresponds to $\boldsymbol{s}_{\mathrm{BP}}(t)$ where $\boldsymbol{s}_{\mathrm{BP}}(t)$ is the matrix density corresponding to the busy period T_B of $\boldsymbol{B}_{\mathrm{I}}(s)$. Then one has

Theorem 4.3. For Markov modulated process $\boldsymbol{B}_{\mathrm{I}}(t)$ with $\boldsymbol{\lambda} = \boldsymbol{\lambda}_{\mathrm{D}}$, $\boldsymbol{\mu} = \mu\boldsymbol{I}$, and $N_{\mathrm{I}}(0) = 0$,

$$\boldsymbol{p}_n(t) = \int_0^t \boldsymbol{p}_0(t-x)\left(\frac{\boldsymbol{\lambda}_D}{\mu}\boldsymbol{s}_{\mathrm{BP}}(x)\right)^{(n)} \mathrm{d}x, \quad n \geq 0.$$

Proof: From Theorem 2.1 and (3.5), it can be seen that $\boldsymbol{\rho}_+(s) = \gamma_0^{-1}(s)\gamma_1(s) = \gamma_0^{-1}(s) \times\boldsymbol{\sigma}_+(s)\gamma_0(s) = \gamma_0^{-1}(s)\gamma_0(s)\boldsymbol{\lambda}\boldsymbol{\sigma}_-(s)\boldsymbol{\mu}^{-1} = \boldsymbol{\lambda}_{\mathrm{D}}\boldsymbol{\sigma}_-(s)/\mu$. Thus the result follows from Theorem 2.2. □

We next re-examine the M/M/1 occupancy process using the previous results. Let $N(t)$ be the number of customers in the system with upward and downward transition rate λ and μ, respectively, and let $p_n(t) = \mathrm{P}\{N(t) = n\}$. Also let $s_+(t)$ and $s_-(t)$ be the upward and the downward first passage time density, respectively, of the homogeneous boundaryless process $N_{\mathrm{H}}(t)$ corresponding to the M/M/1. It should be noted from Lemma 3.2 and (3.5) that

(4.5)
$$\rho_+(s) = \rho\sigma_-(s) = \sigma_+(s)$$

where $\rho = \lambda/\mu$. Thus the geometric structure of $N(t)$ with $N(0) = 0$ can be seen from Theorem 2.2 and Theorem 3.4 as

(4.6)
$$p_n(t) = \int_0^t p_0(t-y)s_+^{(n)}(y)\mathrm{d}y = \int_0^t p_0(t-y)\lambda^n \cdot {}_0\mathrm{P}_{11}^{(n)}(y)\mathrm{d}y$$

where ${}_0\mathrm{P}_{11}(t)$ is the taboo probability of $N_{\mathrm{H}}(t)$ being at 1 without having returned to 0 starting from 0. Note that (4.6) implies that

(4.7)
$$s_+(t) = \lambda \cdot {}_0\mathrm{P}_{11}(t).$$

The state probabilities $p_n(t)$ can be written in terms of more familiar entities than (4.6). Let $S_{\mathrm{BP}}(t)$ and $s_{\mathrm{BP}}(t)$ be the busy period distribution of M/M/1 system and its density.

Proposition 4.4. For the M/M/1 occupancy process with $N(0) = 0$,

(a) $p_n(t) = \rho^n \int_0^t p_0(t-x)s_{\mathrm{BP}}^{(n)}(x)\mathrm{d}x$,

(b) $\mathrm{P}\{N(t) \geq n\} = \rho^n S_{\mathrm{BP}}^{(n)}(t)$.

Proof: Part (a) immediately follows from Theorem 4.3. Part (b) follows from Theorem 4.1 (b) and (4.5). □

5. Concluding Remarks

The structural properties of row-continuous Markov chains exhibited in this paper can possibly be exploited for the numerical evaluation of transient probabilities of such processes. Once the transient probabilities at boundaries are obtained, all the other probabilities would be calculated fairly easily because of the geometric structure. Thus, a development of numerical method for the evaluation of probabilities at boundaries is of considerable interest.

This paper deals with bivariate Markov chains that are skip-free on both sides. It is shown in Neuts (1981) that the matrix geometric structure of stationary probabilities holds for Markov chains that are skip-free only on one side. Of interest is an extension of the present analysis to this case.

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