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OPTIMAL MINIMAL-REPAIR AND REPLACEMENT PROBLEM UNDER AVERAGE COST CRITERION: OPTIMALITY OF (t, T)-POLICY

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Abstract The optimal minimal-repair and replacement problem of a reliability system under the average cost criterion is formulated as a semi-Markov decision process, and it is shown that there is a (t, T)-policy which is optimal among all allowable policies, under the assumption that the failure rate of the system monotonically increases to infinity. A (t, T)-policy implies that a failure before age t is minimally repaired, but the system is replaced when age T is reached or when the first failure after age t occurs whichever comes first.

1. Introduction

Minimal-repair and replacement are often used as practical maintenance activities of real reliability systems. A minimal repair is the maintenance activity to repair the failed system so that its function is recovered, without changing its age, while a replacement restores the entire system into the new condition so that it behaves as a new system. Further, replacement is classified into preventive replacement or failure (or corrective) replacement according as whether the system is in operation or in failure.

In the past three decades, vast literature has discussed various maintenance problems with the above maintenance activities. A pioneering work on the maintenance problem with minimal repair was done by Barlow and Hunter [2] in 1960 (see also Chapter 4 of Barlow and Proschan [3]). They proposed a periodic replacement policy with minimal repairs between replacements, and discussed the problem of determining an optimal preventive replacement age T to minimize the long-run average expected cost per unit time over the infinite horizon (the average cost in short). Since then, this basic model has been generalized and modified by many authors to handle more practical situations, as summarized in Ascher and Feingold [1], Nakagawa [5], and Valdez-Flores and Feldman [12]. The common question here is when to replace the system instead of performing minimal repair.

It is noted that Barlow and Hunter [2] considered only minimal repair and preventive replacement as maintenance activities. Phelps [6] introduced failure replacement as a maintenance activity, and discussed an optimal maintenance problem with minimal repair and failure replacement under the average cost criterion (since it was assumed in this model that the required costs for preventive and failure replacements are equal, the system should be replaced only when it is failed). He formulated the problem as a semi-Markov decision process, and, assuming that the failure time distribution has increasing failure rate (IFR), showed that the optimal policy has the following form: there exists a threshold age t such that a failure before age t is minimally repaired, but the system is replaced at the first failure after age t. Tahara and Nishida [11] discussed the maintenance problem with both preventive replacement and failure replacement which have different costs. Under the criterion of minimizing the expected total discounted cost over the infinite horizon (the discounted cost in short), and under the assumption that failure rate function monotonically increases to infinity, they showed, by using the technique of continuous-time stochastic dynamic programming, that there is a (t,T)-policy which is optimal. That is, there exists a pair of ages (t,T) such that a failure before age t is minimally repaired, but the system is replaced when age T is reached or when the first failure after age t occurs whichever comes first. Under the average cost criterion, however, they did not show the optimality of (t,T)-policy, though they characterized the optimal values of (t,T) in case that only (t,T)-policies are allowed.

It is known in the field of stochastic dynamic programming that there is substantial difference between discounted cost and average cost. In order to apply a certain limiting argument of the discounted cost problem to the average cost problem, we need to examine delicate conditions, such as the structures of state transitions and cost functions, and so on.

In this paper we discuss the above problem of finding optimal policies with minimal-repair and both types of replacements under the average cost criterion. We formulate the problem as a semi-Markov decision process, and prove the optimality of a (t,T)-policy, under the assumption that the failure rate function monotonically increases to infinity. In the theory of semi-Markov decision processes, average cost problems are usually addressed through discounted cost problems or finite horizon problems by using certain limiting arguments. In this paper, however, we apply a rather direct approach as it will be seen later, which is its most distinctive feature.

2. Optimal Minimal-Repair and Replacement Problem

We consider a binary-state reliability system described as follows.

(1) The system takes one of the two states: "1" and "0", where state 1 denotes the operating state (or normal state) and state 0 denotes the failed state (or malfunctioning state).

(2) The cumulative distribution function of the failure time X is $F(x)$ and it has a continuous density function $f(x)$. We use the following notations:

$\overline{F}(x) := 1 - F(x)$: the reliability (or survival) function,

$h(x) := \dfrac{f(x)}{\overline{F}(x)}$: the failure rate (or hazard rate) function,

$\dfrac{1}{\mu} := \displaystyle\int_0^{+\infty} x\mathrm{d}F(x) = \int_0^{+\infty} xf(x)\mathrm{d}x = \int_0^{+\infty} \overline{F}(x)\mathrm{d}x$: the mean time to failure (MTTF in short).

(3) The following three maintenance activities are used.

(f) failure replacement to replace a failed system with a new one,

(p) preventive replacement to replace preventively an operating system with a new one,

(m) minimal repair to repair a failed system to recover its function without changing its age.

For notational convenience, we refer these activities as f, p, and m, respectively. It is assumed that the time required for performing these activities is negligible but they incur the expected costs C_f, C_p, and C_m, respectively.

Our problem is to find a policy that minimizes the average cost, i.e., the sum of the expected costs per unit time for minimal-repair, preventive replacement, and failure replacement costs, averaged over the infinite time horizon. The following assumptions, which are reasonable and adopted in most of the literature, are made throughout this paper.

Assumption 2. 1

(A1)

$$C_f > C_p > 0,\ C_m > 0,\ C_m + C_p > C_f. \tag{2.1}$$

(A2) The failure rate function monotonically increases to infinity, i.e.,

$$h(x_1) < h(x_2) \ \text{ for } \ 0 \le x_1 < x_2 < +\infty, \tag{2.2}$$

and

$$\lim_{x \to +\infty} h(x) = +\infty. \tag{2.3}$$

□

3. Formulation as Semi–Markov Decision Process

The system at time t (≥ 0) is completely described by $S(t) := (X(t), Y(t))$, where

$$\begin{aligned} X(t) &:= [\text{the age of the system at time } t], \\ Y(t) &:= \begin{cases} 1 & \text{if the system is operating at time } t, \\ 0 & \text{if the system is in failure at time } t. \end{cases} \end{aligned}$$

Given the system history $(S(u); 0 \le u \le t)$, our problem is to determine, at time t,

$$\begin{array}{ll} \text{whether to execute activity } p \text{ or not} & \text{if } Y(t) = 1, \\ \text{which of the activities } f \text{ and } m \text{ to execute} & \text{if } Y(t) = 0, \end{array}$$

in order to minimize the average cost (defined at the end of previous section).

Tahara and Nishida [11] discussed the same maintenance problem under the criterion of minimizing the discounted cost by using the technique of continuous–time stochastic dynamic programming. However, this approach cannot be applied to our average cost problem directly because supporting theory is not sufficient for continuous–time and average cost problems so far. On the other hand, noting that an event occurs only when a failure of the system occurs or an activity p takes place, we can formulate this problem as the following semi–Markov decision process, in which decisions on maintenance activities are made just after the decision epochs at which system events occur.

[Formulation as Semi–Markov Decision Process]

<u>State Space:</u>

$$S := \{(0,1)\} \cup \{(x,0) : 0 < x < +\infty\}, \tag{3.1}$$

where

$(0,1)$: the state just after the system is preventively replaced to be new,

$(x,0)$: the failed state of the system with age x (> 0).

<u>Action Space:</u> The set $A(s)$ of allowable actions in state s $(\in S)$ is given by:

$$A(0,1) := \{T : 0 < T \le +\infty\}, \tag{3.2}$$

$$A(x,0) := \{(m,T) : x \le T \le +\infty\} \cup \{(f,T) : 0 < T \le +\infty\}, \tag{3.3}$$

where these actions have the following interpretations:

T : the age when the next preventive replacement is planned, where $T = +\infty$ means that no activity is taken until the next failure occurs,

(m,T) : the failed system is minimally repaired, and the next preventive replacement is planned at age T,

(f,T) : the failed system is replaced to be new, and the next preventive replacement is planned at age T.

The overall action space is denoted by

$$A := \bigcup_{s\in S} A(s).$$

□

To complete the formulation as a semi-Markov decision process, we must derive, for each s ($\in S$) and a ($\in A(s)$), the state transition probability measure $p(\cdot|s,a)$, the expected time $t(s,a)$ and the expected cost $c(s,a)$ till the next decision epoch. However, we omit the details since they are directly derived from the above interpretation.

Next, we define the sample space, histories, and policies.

$\Omega := S \times (A \times \mathcal{R}_+ \times S)^\infty$: the sample space, where $\mathcal{R}_+$ denotes the set of nonnegative reals.

The sample space Ω up to the k-th component has the following interpretation.

$\Omega_k := S \times (A \times \mathcal{R}_+ \times S)^k$: the set of all possible histories prior to the k-th decision epoch, i.e., $h_k \in \Omega_k$ has the form

$$\begin{aligned} h_0 &= (s_0) \text{ for } k=0, \\ h_k &= (s_0, a_0, u_1, s_1, a_1, u_2, s_2, \cdots, a_{k-1}, u_k, s_k) \\ &= (h_{k-1}, a_{k-1}, u_k, s_k) \text{ for } k \geq 1, \end{aligned} \tag{3.4}$$

where

t_i ($\in \mathcal{R}_+$) : the i-th decision epoch, where $t_0 := 0$,

$u_i := t_i - t_{i-1}$: the i-th decision interval,

s_i ($\in S$) : the state at t_i,

a_i ($\in A(s_i)$) : the action taken at t_i.

A policy π is now defined as a sequence

$$\pi := (\pi_0, \pi_1, \cdots, \pi_k, \cdots),$$

where π_k $(k = 0, 1, \cdots)$ is a transition probability measure from Ω_k to A, i.e., $\pi_k(B|h_k)$ for each set B ($\subset A(s_k)$) gives the conditional probability that action a_k is chosen at the k-th decision epoch t_k from set B, given the past history h_k ($\in \Omega_k$). Of course $\pi_k(A(s_k)|h_k) = 1$ holds if the last component of h_k is s_k. The set of all allowable policies is denoted by Π. An important class of policies is that of stationary policies. A policy $\pi \in \Pi$ is called stationary if there is a function $u : S \to A$ independent of k such that

$$u(s) \in A(s) \text{ for all } s \in S,$$

and

$$\pi_k(\{u(s_k)\}|h_k) = 1$$

for all $h_k \in \Omega_k$ and $k = 0, 1, \cdots$. That is, action $a_k = u(s_k)$ is always taken with probability 1 if the state at t_k is s_k. Such a stationary policy defined by u is denoted as $\pi = u^\infty$.

The average cost which we want to minimize is now described as follows:

$$g_\pi(s) := \limsup_{N\to+\infty} \frac{\sum_{k=0}^{N} E_\pi[c(S_k, A_k)|S_0 = s]}{\sum_{k=0}^{N} E_\pi[t(S_k, A_k)|S_0 = s]} \text{ for } s \in S, \tag{3.5}$$

where $E_\pi[\cdot]$ denotes the expectation operator under policy π, and S_k and A_k are random variables representing the k–th state and action, respectively.

The following theorem is known for a semi–Markov decision process (e.g., Theorem 2 of Ross [8]; see also Chapter 7 of Ross [7]).

Theorem 3. 1 If there exist a bounded function $v^* : S \to \mathcal{R}$ and a constant g^* satisfying the following optimality equations:

$$v^*(0,1) = \inf_{0<T\le+\infty}\left\{C_p\overline{F}(T) - g^*\int_0^T \overline{F}(u)\mathrm{d}u + \int_0^T v^*(u,0)\mathrm{d}F(u) + \overline{F}(T)v^*(0,1)\right\} \quad (3.6)$$

$$\begin{aligned} &v^*(x,0)\\ &= \min\Bigg[\inf_{0<T\le+\infty}\left\{C_f + C_p\overline{F}(T) - g^*\int_0^T \overline{F}(u)\mathrm{d}u + \int_0^T v^*(u,0)\mathrm{d}F(u) + \overline{F}(T)v^*(0,1)\right\},\\ &\inf_{x\le T\le+\infty}\left\{C_m + C_p\frac{\overline{F}(T)}{\overline{F}(x)} - g^*\int_x^T \frac{\overline{F}(u)}{\overline{F}(x)}\mathrm{d}u + \int_x^T v^*(u,0)\frac{1}{\overline{F}(x)}\mathrm{d}F(u) + \frac{\overline{F}(T)}{\overline{F}(x)}v^*(0,1)\right\}\Bigg]\\ &\text{for } 0 < x < +\infty, \end{aligned} \quad (3.7)$$

then the following properties hold.

(1)

$$\inf_{\pi\in\Pi} g_\pi(s) = g^* \quad \text{for all} \quad s \in S.$$

(2) For an arbitrary small ε (> 0), there exists a stationary policy $\pi(\varepsilon)$ such that

$$g_{\pi(\varepsilon)}(s) \le \inf_{\pi\in\Pi} g_\pi(s) + \varepsilon = g^* + \varepsilon \quad \text{for all} \quad s \in S.$$

Such a policy $\pi(\varepsilon)$ is called an ε–optimal stationary policy.

(3) If, for each s $(\in S)$, there exists a minimizer of the right hand side of (3.6) or (3.7), the stationary policy $\pi(0)$ composed of such actions is 0–optimal (or simply optimal). □

The function $v^*(\cdot)$ in the optimality equations (3.6) and (3.7) is called the relative cost function or the differential cost function. The value $v^*(s_1) - v^*(s_2)$ could be regarded as the difference between the expected total costs over a sufficient large planning horizon under an optimal policy due to starting from states s_1 and s_2. Because $v^*(\cdot)$ is determined only within an additive constant, we normalize, without any loss of generality, as:

$$v^*(0,1) = 0.$$

In this case, (3.6) and (3.7) become:

$$v^*(0,1) = \inf_{0<T\le+\infty}\left\{C_p\overline{F}(T) - g^*\int_0^T \overline{F}(u)\mathrm{d}u + \int_0^T v^*(u,0)\mathrm{d}F(u)\right\} = 0, \quad (3.8)$$

$$v^*(x,0) = \min\left[C_f, \inf_{x\le T\le+\infty}\left\{C_m + C_p\frac{\overline{F}(T)}{\overline{F}(x)} - g^*\int_x^T \frac{\overline{F}(u)}{\overline{F}(x)}\mathrm{d}u + \int_x^T v^*(u,0)\frac{1}{\overline{F}(x)}\mathrm{d}F(u)\right\}\right]$$
$$\text{for } 0 < x < +\infty. \quad (3.9)$$

4. (t,T)–Policy

Definition 4. 1 For $0 \leq t \leq T \leq +\infty$, a stationary policy u^{∞} defined by $u : S \to A$ is called (t,T)–policy if

$$u(s) = \begin{cases} T & \text{for } s = (0,1), \\ (m,T) & \text{for } s = (x,0) \text{ with } 0 < x < t, \\ (f,T) & \text{for } s = (x,0) \text{ with } t \leq x. \end{cases} \tag{4.1}$$

□

In this section, we confine ourselves to the class of (t,T)–policies, and find a pair (t^*,T^*) that minimizes the average cost in this class.

According to the well–known result in renewal reward process theory (e.g., Section 3.9 of Ross [8]), the average cost under (t,T)–policy is independent of the initial state and is given by the expected cost divided by the expected time duration between two successive replacements:

$$g_{(t,T)} := \frac{C_m \int_0^t h(u)\mathrm{d}u + \dfrac{C_f\left(\overline{F}(t) - \overline{F}(T)\right) + C_p\overline{F}(T)}{\overline{F}(t)}}{t + \displaystyle\int_t^T \frac{\overline{F}(u)}{\overline{F}(t)}\mathrm{d}u}. \tag{4.2}$$

We now introduce for $0 \leq x \leq t \leq T \leq +\infty$ the following functions:

$$A((0,1),t,T) := t + \int_t^T \frac{\overline{F}(u)}{\overline{F}(t)}\mathrm{d}u, \tag{4.3}$$

$$A((x,0),t,T) := t - x + \int_t^T \frac{\overline{F}(u)}{\overline{F}(t)}\mathrm{d}u, \tag{4.4}$$

$$B((0,1),t,T) := C_m \int_0^t h(u)\mathrm{d}u + \frac{C_f\left(\overline{F}(t) - \overline{F}(T)\right) + C_p\overline{F}(T)}{\overline{F}(t)}, \tag{4.5}$$

$$B((x,0),t,T) := C_m + C_m \int_x^t h(u)\mathrm{d}u + \frac{C_f\left(\overline{F}(t) - \overline{F}(T)\right) + C_p\overline{F}(T)}{\overline{F}(t)}. \tag{4.6}$$

$A(s,t,T)$ and $B(s,t,T)$ respectively represent the expected time and cost till the first replacement, when we start from state s ($\in S$) under (t,T)–policy. Using these functions, the above $g_{(t,T)}$ becomes

$$g_{(t,T)} = \frac{B((0,1),t,T)}{A((0,1),t,T)} \quad \text{for } 0 \leq t \leq T \leq +\infty.$$

Furthermore, let us denote the optimal average cost due to a (t,T)–policy by

$$g := \inf_{0 \leq t \leq T \leq +\infty} g_{(t,T)}, \tag{4.7}$$

and define

$$G((0,1),t,T) := B((0,1),t,T) - gA((0,1),t,T), \tag{4.8}$$

$$G((x,0),t,T) := B((x,0),t,T) - gA((x,0),t,T). \tag{4.9}$$

The following is well–known in fractional programming (e.g., Schaible and Ibaraki [9]).

Theorem 4. 1 Consider the following two problems:

$$\textbf{P1:} \qquad \inf_{0\le t\le T\le +\infty} \frac{B((0,1),t,T)}{A((0,1),t,T)}, \tag{4.10}$$

$$\textbf{P2:} \qquad \inf_{0\le t\le T\le +\infty} G((0,1),t,T). \tag{4.11}$$

A solution (t^*, T^*) is optimal in P1 if and only if it is optimal in P2. Furthermore such an optimal solution (t^*, T^*) satisfies

$$G((0,1),t^*,T^*) = 0. \tag{4.12}$$

□

Based on this theorem, we characterize (t^*, T^*) by examining problem P2. We start with a preliminary lemma, which is easy to show (see Appendices).

Lemma 4. 1

$$(C_f - C_p)h(0) < g \le C_f\mu. \tag{4.13}$$

□

Theorem 4. 2 There exists unique (t^*, T^*) attaining

$$g = g_{(t^*,T^*)} = \min_{0\le t\le T\le +\infty} g_{(t,T)} \tag{4.14}$$

such that

(1)

$$0 \le t^* < T^* < +\infty, \tag{4.15}$$

(2)

$$g = (C_f - C_p)h(T^*). \tag{4.16}$$

Furthermore, if $t^* > 0$ then

(3)

$$C_m - (C_f - C_p)\frac{\overline{F}(T^*)}{\overline{F}(t^*)} - g\int_{t^*}^{T^*} \frac{\overline{F}(u)}{\overline{F}(t^*)}\mathrm{d}u = 0, \tag{4.17}$$

(4)

$$C_m h(t^*) \le g, \tag{4.18}$$

(5)

$$(C_m - C_f) - C_m\int_0^{t^*} h(u)\mathrm{d}u + gt^* = 0. \tag{4.19}$$

Proof. By Theorem 4.1, we will find (t, T) that minimizes $G((0,1),t,T)$. To this end, note that

$$\frac{\partial}{\partial T}G((0,1),t,T) = \frac{\overline{F}(T)}{\overline{F}(t)}\{(C_f - C_p)h(T) - g\}. \tag{4.20}$$

Since $(C_f - C_p)h(0) < g$ (by Lemma 4.1), $C_f - C_p > 0$ (by Assumption 2.1 (A1)), and $h(T)$ monotonically increases to infinity (by Assumption 2.1 (A2)), we conclude that

$$(C_f - C_p)h(T) - g$$

changes its sign from $-$ to $+$ only once at T^* as T moves from 0 to $+\infty$, i.e.,

$$g = (C_f - C_p)h(T^*).$$

Next we consider the one-dimensional minimization of $G((0,1),t,T^*)$ over $t \in [0,T^*]$.

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}G((0,1),t,T^*) \\
&= h(t)\left\{C_m - (C_f - C_p)\frac{\overline{F}(T^*)}{\overline{F}(t)} - g\int_t^{T^*}\frac{\overline{F}(u)}{\overline{F}(t)}\mathrm{d}u\right\} \\
&= \frac{h(t)}{\overline{F}(t)}\left\{C_m\overline{F}(t) - (C_f - C_p)\overline{F}(T^*) - g\int_t^{T^*}\overline{F}(u)\mathrm{d}u\right\} \\
&= h(t)\left\{B((0,1),t,T^*) - gA((0,1),t,T^*) + (C_m - C_f) - C_m\int_0^t h(u)\mathrm{d}u + gt\right\} \quad (4.21)
\end{aligned}$$

Note that

$$\left.\frac{\mathrm{d}}{\mathrm{d}t}G((0,1),t,T^*)\right|_{t=T^*} = (C_m + C_p - C_f)h(T^*) > 0$$

by Assumption 2.1 (A1). Further we have

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\left\{C_m - (C_f - C_p)\frac{\overline{F}(T^*)}{\overline{F}(t)} - g\int_t^{T^*}\frac{\overline{F}(u)}{\overline{F}(t)}\mathrm{d}u\right\} \\
&= -(C_f - C_p)h(t)\frac{\overline{F}(T^*)}{\overline{F}(t)} - gh(t)\int_t^{T^*}\frac{\overline{F}(u)}{\overline{F}(t)}\mathrm{d}u + g \\
&= (C_f - C_p)\left[-h(t)\frac{\overline{F}(T^*)}{\overline{F}(t)} + h(T^*)\left\{-h(t)\int_t^{T^*}\frac{\overline{F}(u)}{\overline{F}(t)}\mathrm{d}u + 1\right\}\right] \\
&\geq (C_f - C_p)\frac{\overline{F}(T^*)}{\overline{F}(t)}\{-h(t) + h(T^*)\} \\
&\geq 0 \quad \text{for} \quad 0 < t < T^*,
\end{aligned}$$

where the second equality holds because

$$g = (C_f - C_p)h(T^*),$$

and the first inequality follows from

$$-h(t)\int_t^{T^*}\frac{\overline{F}(u)}{\overline{F}(t)}\mathrm{d}u + 1 \geq -\int_t^{T^*}h(u)\frac{\overline{F}(u)}{\overline{F}(t)}\mathrm{d}u + 1 = \frac{\overline{F}(T^*)}{\overline{F}(t)}$$

by Assumption 2.1 (A2).

We now consider two cases.

(i) $C_m - (C_f - C_p)\overline{F}(T^*) - g\int_0^{T^*}\overline{F}(u)\mathrm{d}u \geq 0$. Then

$$\frac{\mathrm{d}}{\mathrm{d}t}G((0,1),t,T^*) > 0 \quad \text{for} \quad 0 < t < T^*,$$

and $t^* = 0$ is the unique minimum solution of $G((0,1),t,T^*)$.

(ii) $C_m - (C_f - C_p)\overline{F}(T^*) - g\int_0^{T^*} \overline{F}(u)\mathrm{d}u < 0$. Then

$$\frac{\mathrm{d}}{\mathrm{d}t}G((0,1),t,T^*)$$

changes its sign exactly once from $-$ to $+$ as t moves from 0 to T^*, and the unique zero point t^* minimizes $G((0,1),t,T^*)$. From (4.21), such t^* satisfies (4.17). Therefore, from (4.21) again,

$$\frac{\mathrm{d}}{\mathrm{d}t}\left\{C_m\overline{F}(t) - (C_f - C_p)\overline{F}(T^*) - g\int_t^{T^*} \overline{F}(u)\mathrm{d}u\right\}\Bigg|_{t=t^*} = -\overline{F}(t^*)(C_m h(t^*) - g) \geq 0,$$

and this implies (4.18). Further, we have from (4.21) that

$$B((0,1),t^*,T^*) - gA((0,1),t^*,T^*) + (C_m - C_f) - C_m\int_0^{t^*} h(u)\mathrm{d}u + gt^* = 0.$$

This and relation

$$B((0,1),t^*,T^*) - gA((0,1),t^*,T^*) = 0$$

shown by Theorem 4.1 together prove (4.19).

In the remaining part we show that (t^*,T^*) is the minimum solution of $G((0,1),t,T)$ over the whole domain $0 \leq t \leq T$, that is,

$$G((0,1),t^*,T^*) \leq G((0,1),t,T) \quad \text{for} \quad 0 \leq t \leq T.$$

(a) If $0 \leq t \leq T^*$, then

$$G((0,1),t,T) \geq G((0,1),t,T^*) \geq G((0,1),t^*,T^*),$$

where the first inequality holds because

$$\frac{\partial}{\partial T}G((0,1),t,T) = \frac{\overline{F}(T)}{\overline{F}(t)}\{(C_f - C_p)h(T) - g\}$$

changes its sign from $-$ to $+$ exactly once at $T = T^*$ as T moves from t to $+\infty$, while the second inequality follows from the argument of the first part of this proof.

(b) If $T^* \leq t \leq T$, then

$$G((0,1),t,T) \geq G((0,1),t,t) \geq G((0,1),T^*,T^*) \geq G((0,1),t^*,T^*). \tag{4.22}$$

The first inequality holds because

$$\frac{\partial}{\partial T}G((0,1),t,T) = \frac{\overline{F}(T)}{\overline{F}(t)}\{(C_f - C_p)h(T) - g\}$$

is always positive for $T^* \leq t < T$. The second inequality of (4.22) follows from the fact that

$$G((0,1),t,t) = C_m\int_0^t h(u)\mathrm{d}u + C_p - gt$$

is increasing in t when $T^* < t$ because

$$\frac{\mathrm{d}}{\mathrm{d}t}G((0,1),t,t) = C_m h(t) - g > (C_m + C_p - C_f)h(T^*) > 0. \tag{4.23}$$

Finally, the third inequality of (4.22) was shown in the first part of this proof. □

Remark 4. 1

(1) The relations (2), (3), and (5) in Theorem 4.2 yield a simultaneous system of two nonlinear equations with unknown (t^*, T^*), which can be solved numerically by a certain standard computational method, e.g., Newton–Raphson method.

(2) Tahara and Nishida [11] also obtained some characterization similar to Theorem 4.2 by examining the optimality conditions of the minimization problem P1 in Theorem 4.1 instead of P2. However, the derivation process by our approach of investigating P2 includes a lot of information useful to our subsequent argument. □

5. Optimality of (t, T)–Policy

In this section, we show that a (t, T)–policy is optimal over the set Π of all allowable policies. Of course, among (t, T)–policies, (t^*, T^*)–policy of Section 4 is the only candidate for the optimum. Define

$$v(0,1) = \inf_{0 \le t \le T \le +\infty} G((0,1), t, T), \tag{5.1}$$

$$v(x,0) = \min\left\{C_f, \inf_{x \le t \le T \le +\infty} G((x,0), t, T)\right\} \quad \text{for } 0 < x < +\infty \tag{5.2}$$

In the following, we will show that function $v : S \to \mathcal{R}$ is the relative cost function of (t^*, T^*)–policy, and that the function $v(\cdot)$ and the constant g of (4.7) respectively serve as $v^*(\cdot)$ and g^* in Theorem 3.1.

First we have the following two lemmas.

Lemma 5. 1

(1)

$$\inf_{0 \le t \le T \le +\infty} G((0,1), t, T) = G((0,1), t^*, T^*) = 0, \tag{5.3}$$

(2)

$$\inf_{0 \le t \le T \le +\infty} G((x,0), t, T) = \begin{cases} G((x,0), t^*, T^*) & \text{if } 0 < x < t^*, \\ G((x,0), x, T^*) & \text{if } t^* \le x < T^*, \\ G((x,0), x, x) & \text{if } T^* \le x. \end{cases} \tag{5.4}$$

Proof. The relation (1) follows from Theorem 4.1. To show (2), note that

$$G((x,0), t, T) = G((0,1), t, T) + C_m - C_m \int_0^x h(u)\mathrm{d}u + gx \tag{5.5}$$

by the definitions of $A(s,t,T)$, $B(s,t,T)$, and $G(s,t,T)$. Therefore, similarly to (4.20) and (4.21),

$$\begin{aligned} \frac{\partial}{\partial T} G((x,0), t, T) &= \frac{1}{\overline{F}(t)}\left\{(C_f - C_p) f(T) - g\overline{F}(T)\right\} = \frac{\overline{F}(T)}{\overline{F}(t)}\left\{(C_f - C_p) h(T) - g\right\}, \\ \frac{\partial}{\partial t} G((x,0), t, T) &= h(t)\left\{C_m - (C_f - C_p)\frac{\overline{F}(T)}{\overline{F}(t)} - g\int_t^T \frac{\overline{F}(u)}{\overline{F}(t)}\mathrm{d}u\right\}. \end{aligned}$$

Now by an argument similar to the proof of Theorem 4.2, we obtain the following properties.

(i) $0 < x \le t^*$:

$$\inf_{x \le t \le T \le +\infty} G((x,0), t, T) = \inf_{x \le t \le T^*} G((x,0), t, T^*) = G((x,0), t^*, T^*).$$

(ii) $t^* \leq x \leq T^*$:

$$\inf_{x\leq t\leq T\leq +\infty} G((x,0),t,T) = \inf_{x\leq t\leq T^*} G((x,0),t,T^*) = G((x,0),x,T^*).$$

(iii) $T^* \leq x$:

$$\inf_{x\leq t\leq T\leq +\infty} G((x,0),t,T) = \inf_{x\leq t\leq +\infty} G((x,0),t,t) = G((x,0),x,x).$$

□

The next lemma implies that function $v : S \to \mathcal{R}$ is the relative cost function of (t^*,T^*)–policy.

Lemma 5. 2 Function $v(\cdot)$ of (5.1) and (5.2) satisfies

$$\begin{aligned} v(0,1) &= G((0,1),t^*,T^*) = 0, &(5.6)\\ v(x,0) &= \begin{cases} G((x,0),t^*,T^*) & \text{for } 0 < x \leq t^*, \\ C_f & \text{for } t^* \leq x. \end{cases} &(5.7) \end{aligned}$$

Furthermore $v(x,0)$ is bounded, continuous, and nondecreasing in x.

Proof. (5.6) was shown in Theorem 4.1. (5.7) is proved as follows.

(i) $0 < x \leq t^*$: By Lemma 5.1 (2), we have

$$\begin{aligned} \inf_{x\leq t\leq T\leq +\infty} G((x,0),t,T) &= G((x,0),t^*,T^*) \\ &= G((0,1),t^*,T^*) + C_m - C_m \int_0^x h(u)\mathrm{d}u + gx \quad \text{(by (5.5))} \\ &= C_m - C_m \int_0^x h(u)\mathrm{d}u + gx \quad \text{(by (5.6))} \\ &= C_m - C_m \int_0^x h(u)\mathrm{d}u + (C_f - C_p)h(T^*)x. \quad \text{(by (4.16))} \end{aligned}$$

Because

$$\frac{\mathrm{d}}{\mathrm{d}x} G((x,0),t^*,T^*) = -C_m h(x) + g$$

is decreasing in x, and

$$-C_m h(t^*) + g \geq 0$$

by Theorem 4.2 (4), $G((x,0),t^*,T^*)$ is increasing in x over the interval $(0,t^*]$. Furthermore

$$G((t^*,0),t^*,T^*) = C_m - C_m \int_0^{t^*} h(u)\mathrm{d}u + gt^* = C_f$$

by Theorem 4.2 (5). Thus

$$\begin{aligned} v(x,0) &= \min\left\{C_f, \inf_{x\leq t\leq T\leq +\infty} G((x,0),t,T)\right\} \\ &= G((x,0),t^*,T^*) \ \text{ for } \ 0 < x \leq t^*, \end{aligned}$$

and $v(x,0)$ is increasing in $x \in (0,t^*]$.

(ii) $t^* \leq x \leq T^*$: From Lemma 5.1 (2), we have

$$\begin{aligned} \inf_{x\leq t\leq T\leq +\infty} G((x,0),t,T) &= G((x,0),x,T^*) \\ &= C_m + C_f - (C_f - C_p)\frac{\overline{F}(T^*)}{\overline{F}(x)} - g\int_x^{T^*} \frac{\overline{F}(u)}{\overline{F}(x)}\mathrm{d}u, \end{aligned}$$

by the definition of function $G(\cdot)$. Noting that

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}x}G((x,0),x,T^*) &= -(C_f - C_p)h(x)\frac{\overline{F}(T^*)}{\overline{F}(x)} - gh(x)\int_x^{T^*}\frac{\overline{F}(u)}{\overline{F}(x)} + g \\
&= (C_f - C_p)\left[-h(x)\frac{\overline{F}(T^*)}{\overline{F}(x)} + h(T^*)\left\{-h(x)\int_x^{T^*}\frac{\overline{F}(u)}{\overline{F}(x)}\mathrm{d}u + 1\right\}\right] \\
&\geq (C_f - C_p)\frac{\overline{F}(T^*)}{\overline{F}(x)}\{-h(x) + h(T^*)\} \\
&\geq 0
\end{aligned}$$

and

$$G((t^*,0),t^*,T^*) = C_f,$$

we have

$$\begin{aligned}
v(x,0) &= \min\left\{C_f, \inf_{x\leq t\leq T\leq+\infty} G((x,0),t,T)\right\} \\
&= \min\{C_f, G((x,0),x,T^*)\} \\
&= C_f \quad \text{for } t^* \leq x \leq T^*.
\end{aligned}$$

(iii) $T^* \leq x$: From Lemma 5.1 (2), we have

$$\inf_{x\leq t\leq T\leq+\infty} G((x,0),t,T) = G((x,0),x,x) = C_m + C_p > C_f.$$

Thus

$$\begin{aligned}
v(x,0) &= \min\left\{C_f, \inf_{x\leq t\leq T\leq+\infty} G((x,0),t,T)\right\} \\
&= \min\{C_f, G((x,0),x,x)\} \\
&= C_f \quad \text{for } T^* \leq x.
\end{aligned}$$

□

Next we evaluate the values of the right hand side of the optimality equations (3.8) and (3.9) in which $v^*(\cdot)$ and g^* are replaced with $v(\cdot)$ and g, respectively. To this end, we prepare the following lemma. Since its proof is straightforwardly done, it is given in Appendices.

Lemma 5.3 For any $x \in [0, t^*)$,

$$\begin{aligned}
&\frac{1}{\overline{F}(x)}\int_x^{T^*} v(u,0)\mathrm{d}F(u) \\
&= C_m\int_x^{t^*} h(u)\mathrm{d}u - C_p\frac{\overline{F}(T^*)}{\overline{F}(x)} + g\int_x^{t^*}\frac{\overline{F}(u)}{\overline{F}(x)}\mathrm{d}u + \frac{C_f\left(\overline{F}(t^*) - \overline{F}(T^*)\right) + C_p\overline{F}(T^*)}{\overline{F}(t^*)} \\
&\quad - g\left(t^* - x + \int_0^{T^*}\frac{\overline{F}(u)}{\overline{F}(x)}\mathrm{d}u\right).
\end{aligned}$$

□

The next lemma evaluates the right hand side of the optimality equation (3.8) in which $v^*(\cdot)$ and g^* are replaced with $v(\cdot)$ and g, respectively.

Lemma 5.4 Define

$$D(T) := C_p\overline{F}(T) - g\int_0^T \overline{F}(u)\mathrm{d}u + \int_0^T v(u,0)\mathrm{d}F(u).$$

Then

$$\inf_{0<T\le+\infty} D(T) = D(T^*) = 0.$$

Proof. Since

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}T}D(T) &= -C_pf(T) - g\overline{F}(T) + v(T,0)f(T)\\ &= \overline{F}(T)\left\{(v(T,0)-C_p)h(T) - g\right\}\\ &= \overline{F}(T)\left\{(v(T,0)-C_p)h(T) - (C_f-C_p)h(T^*)\right\},\end{aligned}$$

and, by (5.7) of Lemma 5.2,

$$v(T,0)\begin{cases}\le C_f & \text{for } 0<T\le t^*,\\ = C_f & \text{for } t^*\le T,\end{cases}$$

we have

$$\frac{\mathrm{d}}{\mathrm{d}T}D(T)\begin{cases}\le 0 & \text{for } 0<T<t^*,\\ <0 & \text{for } t^*\le T<T^*,\\ =0 & \text{for } T=T^*,\\ >0 & \text{for } T^*<T.\end{cases}$$

Hence

$$\inf_{0<T\le+\infty} D(T) = D(T^*) = C_p\overline{F}(T^*) - g\int_0^{T^*}\overline{F}(u)\mathrm{d}u + \int_0^{T^*} v(u,0)\mathrm{d}F(u).$$

Further we have from Lemma 5.3 that

$$\begin{aligned}&C_p\overline{F}(T^*) - g\int_0^{T^*}\overline{F}(u)\mathrm{d}u + \int_0^{T^*} v(u,0)\mathrm{d}F(u)\\ =\;& C_m\int_0^{t^*}h(u)\mathrm{d}u + \frac{C_f\left(\overline{F}(t^*)-\overline{F}(T^*)\right)+C_p\overline{F}(T^*)}{\overline{F}(t^*)} - g\left(t^* + \int_0^{T^*}\frac{\overline{F}(u)}{\overline{F}(t^*)}\mathrm{d}u\right)\\ =\;& G((0,1),t^*,T^*) = 0.\end{aligned}$$

□

The next lemma evaluates the main part of the right hand side of the optimality equation (3.9) in which $v^*(\cdot)$ and g^* are replaced with $v(\cdot)$ and g, respectively.

Lemma 5.5 Define

$$E(x,T) := C_m + \frac{1}{\overline{F}(x)}\left\{C_p\overline{F}(T) - g\int_x^T\overline{F}(u)\mathrm{d}u + \int_x^T v(u,0)\mathrm{d}F(u)\right\} \quad \text{for } 0<x\le T\le+\infty.$$

Then the following properties hold.

(1) For $0<x\le t^*$ $(<T^*)$,

$$\inf_{x\le T\le+\infty} E(x,T) = E(x,T^*) = G((x,0),t^*,T^*) \le C_f.$$

(2) For $t^* \leq x < T^*$,

$$\inf_{x \leq T \leq +\infty} E(x,T) = E(x,T^*) = G((x,0),x,T^*) \geq C_f.$$

(3) For $T^* \leq x$,

$$\inf_{x \leq T \leq +\infty} E(x,T) = E(x,x) = G((x,0),x,x) = C_m + C_p > C_f.$$

Proof. Since

$$\begin{aligned}
\frac{\partial}{\partial T} E(x,T) &= \frac{1}{\overline{F}(x)} \left\{ -C_p f(T) - g\overline{F}(T) + v(T,0) f(T) \right\} \\
&= \frac{\overline{F}(T)}{\overline{F}(x)} \left\{ (v(T,0) - C_p) h(T) - g \right\} \\
&= \frac{\overline{F}(T)}{\overline{F}(x)} \left\{ (v(T,0) - C_p) h(T) - (C_f - C_p) h(T^*) \right\},
\end{aligned}$$

an argument similar to the proof of Lemma 5.1 yields the following.

(1) $0 < x \leq t^*$ $(< T^*)$:

$$\begin{aligned}
\inf_{x \leq T \leq +\infty} E(x,T) &= E(x,T^*) \\
&= C_m + \frac{1}{\overline{F}(x)} \left\{ C_p \overline{F}(T^*) - g \int_x^{T^*} \overline{F}(u) \mathrm{d}u + \int_x^{T^*} v(u,0) \mathrm{d}F(u) \right\} \\
&= C_m + C_m \int_x^{t^*} h(u) \mathrm{d}u + \frac{C_f \left(\overline{F}(t^*) - \overline{F}(T^*) \right) + C_p \overline{F}(T^*)}{\overline{F}(t^*)} \\
&\quad - g \left(t^* - x + \int_0^{t^*} \frac{\overline{F}(u)}{\overline{F}(x)} \mathrm{d}u \right) \\
&= G((x,0),t^*,T^*),
\end{aligned}$$

where the third equality holds by Lemma 5.3.

(2) $t^* \leq x \leq T^*$:

$$\begin{aligned}
\inf_{x \leq T \leq +\infty} E(x,T) &= E(x,T^*) \\
&= C_m + \frac{1}{\overline{F}(x)} \left\{ C_p \overline{F}(T^*) - g \int_x^{T^*} \overline{F}(u) \mathrm{d}u + \int_x^{T^*} v(u,0) \mathrm{d}F(u) \right\} \\
&= C_m + \frac{C_f \left(\overline{F}(x) - \overline{F}(T^*) \right) + C_p \overline{F}(T^*)}{\overline{F}(x)} - g \int_x^{T^*} \frac{\overline{F}(u)}{\overline{F}(x)} \mathrm{d}u \\
&= G((x,0),x,T^*).
\end{aligned}$$

(3) $T^* \leq x$:

$$\inf_{x \leq T \leq +\infty} E(x,T) = E(x,x) = C_m + C_p.$$

□

Thus we complete the proof of the following theorem.

Theorem 5.1 If we let

$$v^*(\cdot) = v(\cdot); \quad g^* = g,$$

then the optimality equations (3.8) and (3.9) hold. Moreover, (t^*, T^*)–policy is optimal over the set Π of all allowable policies. □

6. Concluding Remark

In this paper we discussed the optimal minimal–repair and replacement problem under the average cost criterion. We formulated the problem as a semi–Markov decision process, and showed that an optimal policy of all allowable policies is in the class of (t,T)–policies under the assumption the failure rate function monotonically increases to infinity. Further we characterized the optimal pair (t^*, T^*).

In this paper, we assumed that the costs incurred for minimal–repair and replacement are independent of the system age. The problems with age–dependent cost structures seem to be practically more important. Recently, Segawa, Ohnishi, and Ibaraki [10] discussed a such optimal minimal–repair and failure–replacement problem under an assumption that preventive replacements are not allowed. Optimal maintenance problems of reliability systems with age–dependent cost structures which consider minimal–repair, and both of failure and preventive replacements are left for future researches.

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A. Appendices

Proof of Lemma 4.1. The second inequality of (4.13) is obvious because

$$g_{(0,+\infty)} = C_f \mu,$$

where we note that $(0,+\infty)$-policy means to use only failure replacement. Since, by de l'Hôspital's rule,

$$\begin{aligned}\lim_{t,T\to+\infty;t\le T} g_{(t,T)} &= \lim_{t\to+\infty}\frac{f(t)\left(C_m - C_f - C_m\int_0^t h(u)\mathrm{d}u\right)}{-tf(t)}\\ &= \lim_{t\to+\infty}\left(\frac{C_f-C_m}{t}+\frac{C_m\int_0^t h(u)\mathrm{d}u}{t}\right)\\ &= +\infty,\end{aligned}$$

the first inequality of (4.13) can be proved by showing that, for a sufficiently large but finite t^+, there exists some δ (>0) such that

$$B((0,1),t,T)-(C_f-C_p)h(0)A((0,1),t,T)>\delta \tag{A.1}$$

for $0\le t\le t^+$ and $t\le T\le+\infty$, which implies

$$\begin{aligned}g_{(t,T)}=\frac{B((0,1),t,T)}{A((0,1),t,T)} &> (C_f-C_p)h(0)+\frac{\delta}{A((0,1),t,T)}\\ &\ge (C_f-C_p)h(0)+\frac{\delta}{t^+ + \frac{1}{\mu}}.\end{aligned}$$

The above second inequality holds because

$$A((0,1),t,T)=t+\int_t^T\frac{\overline{F}(u)}{\overline{F}(t)}\mathrm{d}u\le t+\frac{1}{\mu}\le t^+ +\frac{1}{\mu}$$

by using the well-known fact that IFR implies NBUE (New Better that Used in Expectation) (e.g., Chapter 6 of Barlow and Proschan [4]).

(A.1) is proved by multiplying the left hand side by $\overline{F}(t)$:

$$\begin{aligned}&\overline{F}(t)\left\{B((0,1),t,T)-(C_f-C_p)h(0)A((0,1),t,T)\right\}\\ &= C_m\overline{F}(t)\int_0^t h(u)\mathrm{d}u + C_f\left(\overline{F}(t)-\overline{F}(T)\right)+C_p\overline{F}(T)\\ &\quad -(C_f-C_p)h(0)t\overline{F}(t)-(C_f-C_p)h(0)\int_t^T\overline{F}(u)\mathrm{d}u\\ &> (C_f-C_p)\overline{F}(t)\left\{\int_0^T h(u)\mathrm{d}u - th(0)\right\}\\ &\quad +(C_f-C_p)\left\{\int_t^T f(u)\mathrm{d}u - h(0)\int_t^T\overline{F}(u)\mathrm{d}u\right\}+C_p\overline{F}(t)\\ &> C_p\overline{F}(t),\end{aligned}$$

where the last inequality holds because, by Assumption 2.1 (A2),

$$\int_0^t h(u)\mathrm{d}u - th(0)>0,$$

and

$$h(0)\int_t^T\overline{F}(u)\mathrm{d}u<\int_t^T h(u)\overline{F}(u)\mathrm{d}u=\int_t^T f(u)\mathrm{d}u.$$

Therefore, let $\delta = C_p$ to prove (A.1). □

Proof of Lemma 5.3. By Lemma 5.2 we have

$$\int_x^{T^*} v(u,0)\mathrm{d}F(u) = \int_x^{t^*} G((u,0),t^*,T^*)\mathrm{d}F(u) + C_f\left(\overline{F}(t^*) - \overline{F}(T^*)\right).$$

Further, for any $x \in (0,t^*)$,

$$\begin{aligned}
&\int_x^{t^*} G((u,0),t^*,T^*)\mathrm{d}F(u) \\
&= C_m \int_x^{t^*}\left(1+\int_u^{t^*} h(s)\mathrm{d}s\right)\mathrm{d}F(u) + g\int_x^{t^*} u\mathrm{d}F(u) \\
&\quad + \left(\overline{F}(x) - \overline{F}(t^*)\right)\left\{\frac{C_f\left(\overline{F}(t^*) - \overline{F}(T^*)\right) + C_p\overline{F}(T^*)}{\overline{F}(t^*)} - g\left(t^* + \int_x^{T^*}\frac{\overline{F}(u)}{\overline{F}(t^*)}\mathrm{d}u\right)\right\} \\
&= C_m\overline{F}(x)\int_x^{t^*} h(u)\mathrm{d}u + g\int_x^{t^*}\overline{F}(u)\mathrm{d}u - g\left(t^*\overline{F}(t^*) - x\overline{F}(x)\right) \\
&\quad + \left(\overline{F}(x) - \overline{F}(t^*)\right)\left\{\frac{C_f(\overline{F}(t^*) - \overline{F}(T^*)) + C_p\overline{F}(T^*)}{\overline{F}(t^*)} - g\left(t^* + \int_x^{T^*}\frac{\overline{F}(u)}{\overline{F}(t^*)}\mathrm{d}u\right)\right\} \\
&= C_m\overline{F}(x)\int_x^{t^*} h(u)\mathrm{d}u - C_f\left(\overline{F}(t^*) - \overline{F}(T^*)\right) - C_p\overline{F}(T^*) + g\int_x^{t^*}\overline{F}(u)\mathrm{d}u \\
&\quad + \overline{F}(x)\left\{\frac{C_f(\overline{F}(t^*) - \overline{F}(T^*)) + C_p\overline{F}(T^*)}{\overline{F}(t^*)} - g\left(t^* - x + \int_x^{T^*}\frac{\overline{F}(u)}{\overline{F}(t^*)}\mathrm{d}u\right)\right\},
\end{aligned}$$

where the first equality holds by the definition of $G(\cdot)$, and the second one follows from the following identities:

$$\begin{aligned}
&\frac{1}{\overline{F}(x)}\int_x^{t^*}\left(1+\int_u^{t^*} h(s)\mathrm{d}s\right)\mathrm{d}F(u) = \int_x^{t^*} h(u)\mathrm{d}u, \\
&\int_x^{t^*} u\mathrm{d}F(u) = \int_x^{t^*}\overline{F}(u)\mathrm{d}u - \left(t^*\overline{F}(t^*) - x\overline{F}(x)\right).
\end{aligned}$$

□

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