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STOCHASTIC PROPERTIES OF FORK/JOIN MULTI-STAGE PRODUCTION SYSTEMS WITH GENERAL BLOCKING

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Abstract For tandem queues and fork/join queueing networks with communication blocking, the stochastic comparison, reversibility and other equivalence properties have been studied. In this paper, we consider a fork/join multi-stage production system with general blocking which includes the above models as special cases. Under the weak conditions of the initial numbers of items, we formulate this system into a generalized semi-Markov process (GSMP). Then we show the convex property of the GSMP, the stochastic comparison with respect to stochastic and convex ordering, the reversibility and the structural equivalence.

1. Introduction

Many practical models as production systems and communication networks are classified as a discrete event system, which has a discrete state space and a finite event space. The clocks for multiple events may run simultaneously, and the state changes only when an event occurs. As a model of the discrete event system, a generalized semi-Markov process (GSMP) is well-known. The framework of the GSMP is proposed by Mathes [11] and it is applied to show the performance insensitivity of queueing systems by Schassberger[14] and Whitt[18]. Recently, Glasserman and Yao [8]-[10] gave attention to the structural properties of the GSMP, and investigated the stochastic comparison, reversibility and equivalence properties of (a, b, k) tandem production lines with blocking.

There are many studies which show the reversibility, equivalence properties and stochastic comparisons in various production models including tandem queues. Avi-itzhak and Yadin [2] considered a tandem queue with two single-server stations, no intermediate buffer and Poisson arrivals. They showed that when both service stations have either exponential or constant services, the steady state distribution of the sojourn times of customers does not depend on arrangements of stations. Yamazaki and Sakasegawa [19] dealt with tandem queues with multiple single-server stations and finite intermediate buffers. They showed that when there are n batch input jobs and no jobs at machines initially, the nth departure time has the same distribution as that in the reversed system, where stations and buffers are arranged in a reversed order. Muth [12] considered a production line with unlimited raw material and no intermediate buffers. He showed that when a sequence of service times at each station forms independent and identically distributed random variables, the throughput is the same as that in the reversed system. Yamazaki et al. [20] considered a blocking system with two multi-server stations, and showed that the system has the same throughput as its reversed system. For a kanban controlled system, Tayur[17] showed its reversibility and discussed the performance comparison with respect to kanban allocations. Cheng and Yao [5] dealt with tandem queues with general blocking, called an (a, b, k) model, and investigated the stochastic comparison and convexity. Glasserman and Yao [9] and [10] formulated the (a, b, k) model into a GSMP, and derived the reversibility using the convex property of GSMP. Cheng[6] showed that in the (a, b, k) model the completion times of the nth processing are the same among several types of lines. Baccelli and Makowski[3] considered one-stage fork/join queues and showed their stochastic convex property. Ammer and Gershwin [1] considered fork/join queueing networks with finite buffers, exponential servers and communication blocking mechanism, and showed that two systems are stochastically equivalent when upstream servers of several buffers are exchanged with downstream servers and the initial number of items in each buffer is exchanged with the number of its empty space. Paik and Tcha [13] generalized this result to the case in which the service times are generally distributed. Dallery and Towsley [7] showed the similar equivalence properties for a closed tandem queue. Buzacott and Shanthikumar[4] discussed the reversibility, the stochastic comparison and optimal sequences of stations for tandem queues with finite buffers.

In a production system, some stations receive material from different sources and produce multiple parts which are sent to different stations. Their blocking mechanisms are production or kanban blocking, although only communication blocking is dealt with in [1] and [13]. In this paper, we consider a general fork/join multi-stage system with general blocking which includes (a, b, k) tandem queues and all the systems discussed in the abovementioned papers. The purpose of this paper is to show the stochastic comparison, the reversibility and the structural equivalence property of the general fork/join systems with general blocking. We first give conditions under which each station in the system can process the items and then we formulate it into a GSMP. Then we show that when the processing, operation and walking times are comparable in the stochastic or convex order, the nth completion times of processing at each station are comparable in the same order. This implies that the less the moments of processing times are, the larger the throughput of completed items at each station is. Therefore, reducing the variance of processing times of items is essential to increase the throughput. Then we show the reversibility that the distribution of the maximum of the nth completion times of all stations is the same as that in the reversed system in which the items are processed in the reversed order. We also show the structural equivalence property that two different systems under a structural condition have the same nth completion times at each station. When we design a production system, it is important to arrange the ordering of stations. The reversibility and the structural equivalence imply that the throughput is the same among several arrangements of stations, which reduces the number of ordering of stations to be considered.

This paper is organized as follows. In section 2, we describe the general fork/join multistage production system, and give some conditions on the initial numbers of items. Then we derive necessary and sufficient conditions for each station to be in process. In section 3, we briefly review the GSMPs and their properties, and formulate the system into a GSMP. In section 4, we show the convex property of the GSMP and the stochastic comparison, the reversibility and the structural equivalence property of the general fork/join system. Concluding remarks are given in section 5, and the notations used throughout this paper are listed in Appendix.

2. General Fork/Join Multi-Stage Production System

We consider a fork/join multi-stage production system with $M+2$ stations $\{0, 1, 2, \ldots, M+1\}$, shown in Figure 1. Each station $i \in \mathcal{M} = \{1, 2, \ldots, M\}$ has a set of immediate upstream stations $U(i)$ and that of immediate downstream stations $D(i)$. Station 0 represents the input station and station $M+1$ represents the output station. That is, $0 \in U(i)$ if station i receives raw material from infinite resource and $M+1 \in D(i)$ if an item processed at station i leaves the system. Station i receives one item from each station $j \in U(i)$, processes the items and produces $|D(i)|$ finished products, one of which is sent to each station $k \in D(i)$, where $|D(i)|$ is the number of elements of $D(i)$. There are two buffers between stations i and $k \in D(i)$. The buffer in rear of station i and the buffer in front of station k are denoted by B_{ik} and A_{ik}, respectively, where A_{ik} includes the item which is in process at station k. The process at station i proceeds as follows:

1. Each item processed at station i which should be sent to station $k \in D(i)$ remains at buffer B_{ik} if one of the following conditions hold:

a) there are a_{ik} items in buffer A_{ik}, or

b) for some $k' \in D(k)$, the sum of the numbers of items in buffers A_{ik} and $B_{kk'}$ is $c_{ikk'}$.

If neither of these conditions holds, the processed item is sent to A_{ik}.

2. Station i can process items only if there is at least one item in buffer A_{ji} for every $j \in U(i)$ and the number of items in buffer B_{ik} is less than b_{ik} for every $k \in D(i)$. We say that station i is blocked when the number of items in buffer B_{ik} is b_{ik} for some $k \in D(i)$. In particular, if $b_{ik} = 0$ for some $k \in D(i)$, we say that station i is blocked when the item produced at station i cannot be sent to buffer A_{ik}.

The parameters a_{ji} and b_{ik} represent the capacities of buffers A_{ji} and B_{ik}, respectively, and $c_{ikk'}$ is a production control parameter. For each $i \in \mathcal{M}$, $j \in U(i)/\{0\}$ and $k \in D(i)/\{M+1\}$, we assume that

$$c_{jik} \geq a_{ji} \geq 1, \quad c_{jik} \geq b_{ik} \geq 0 \text{ and } a_{ji} + b_{ik} \geq c_{jik}.$$

We also set $c_{0ik} = b_{ik}$ and $c_{jiM+1} = a_{ji}$.

The various well-known production systems with blocking can be regarded as the special cases of this fork/join multi-stage production system. When $U(i) = \{i-1\}$ and $D(i) = \{i+1\}$, the above model becomes a well-known (a, b, k) system, which is analyzed in Cheng and Yao[5], Glasserman and Yao[9],[10] and Cheng[6]. In particular,

a) if the system has the communication blocking mechanism, that is, station i is blocked if buffer $A_{i\ i+1}$ is full, then $a_{ii+1} = c_{ii+1i+2}$ and $b_{i,i+1} = 0$,

b) if the system has the production blocking mechanism, that is, station i is blocked if the processed item at station i cannot be sent to station $i+1$, then $a_{ii+1} = c_{ii+1i+2}$ and $b_{i,i+1} = 1$, and

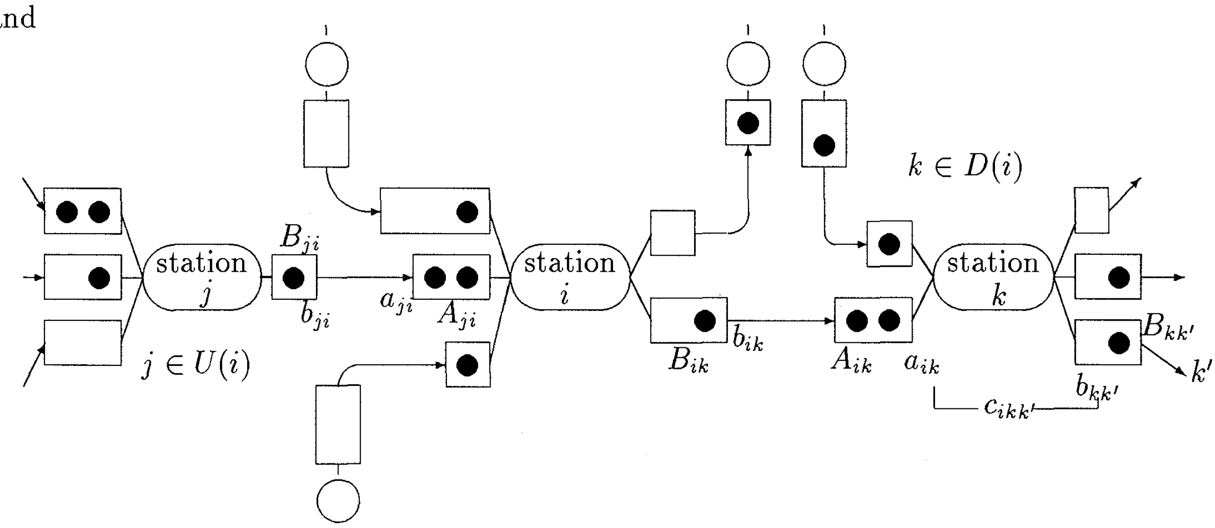


Fig. 1 Blocking Mechanism of the Model

c) if the system has the kanban blocking mechanism, where each station i can have at most k_i items which are waiting for processing, being processed or have already been processed, then $a_{ii+1} = c_{ii+1i+2} = k_{i+1}$ and $b_{i,i+1} = k_i$.

When it holds that $b_{ji} = 0$ and $c_{jik} = a_{ji}$, the system is a fork /join queueing network with communication blocking, which is discussed by Ammer and Gershwin [1] and Paik and Tcha [13].

In this paper, we assume that at time 0, there are m_{ji} items between stations j and i for $j \in U(i)/\{0\}$ and $i \in \mathcal{M}$. Let $L(i)$ be the set consisting of the station sequences in which one of items processed at station i will be processed later, that is,

$$L(i) = \{(i_0, i_1, \ldots, i_t); i_0 = i, i_1 \in D(i_0), i_2 \in D(i_1), \ldots, i_t \in D(i_{t-1}), (i_j, i_{j+1}) \neq (i_k, i_{k+1}) \\ \text{for } j \neq k,\ i_t \neq M+1, t \geq 1\}$$

and for any sequence of stations $I = (i_0, i_1, \ldots, i_t) \in L(i)$, its capacity u_I is defined by

$$u_I = b_{i_0,i_1} + \sum_{j=1}^{t-1} c_{i_{j-1},i_j,i_{j+1}} + a_{i_{t-1},i_t}.$$

Under the above blocking mechanism, the maximal numbers of items in buffers A_{ji} and B_{ik} are a_{ji} and b_{ik}, respectively, and the sum of them must be no more than c_{jik}. Therefore, we assume the following condition on the initial numbers:

(A1) $\sum_{(i_t,i_{t+1}) \in I} m_{i_t,i_{t+1}} \leq u_I$ for all $I \in L(i)$.

In the same way, the number of items on the closed sequence of stations $I = (i_0, i_1, \ldots, i_t, i_0)$ must be no more than $c_I = c_{i_0,i_1,i_2} + c_{i_1,i_2,i_3} + \cdots + c_{i_k,i_0,i_1}$. If it is equal to c_I, however, then the items in buffer $B_{i_t i_{t+1}}$ cannot go to the next buffer $A_{i_t i_{t+1}}$ by the blocking mechanism. Thus the processing at any station i_t will stop after buffer $B_{i_t i_{t+1}}$ is occupied. On the other hand, if there is no item on this sequence of stations, then all stations cannot start processing. The next condition prevents these situations.

(A2) For any closed sequence of stations $I = (i_0, i_1, \ldots, i_k, i_0)$ such that $i_t \in D(i_{t-1}), t = 1, \ldots, k$ and $i_0 \in D(i_k)$,

$$1 \leq \sum_{(i,i') \in I} m_{i,i'} < c_{i_0,i_1,i_2} + c_{i_1,i_2,i_3} + \cdots + c_{i_k,i_0,i_1}.$$

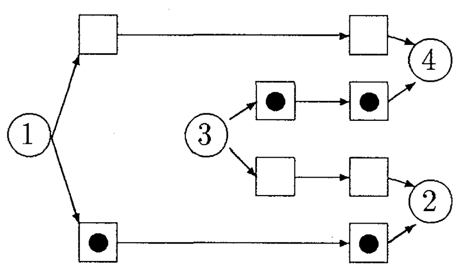


Fig. 2 An example model

Even if conditions (A1) and (A2) are satisfied, the system can not always start processing items at any machine. To illustrate this, we consider the fork/join multi-stage production system in Figure 2, in which $b_{ji} = a_{ji} = 1$ and $c_{jik} = 2$ for all $i \in \mathcal{M} = \{1,2,3,4\}$, $j \in U(i)/\{0\}$ and $k \in D(i)/\{5\}$, and $m_{12} = m_{34} = 2$ and $m_{32} = m_{14} = 0$. Then all stations cannot start processing items because stations 1 and 3 are blocked while stations 2 and 4 are starved. The following condition (A3) prohibits this type of deadlock.

(A3) For any sequence of stations $I = (i_0, i_1, \ldots, i_k, i_0)$ $(i_j \neq i_l, j \neq l)$ such that there is a sequence $(k_0, k_1, k_2, \ldots, k_m)$ where $0 = k_0 < k_1 \cdots < k_{m-1} \leq k$, $k_m = 0$, $i_t \in D(i_{t-1})$ for $(i_{k_{j-1}}, i_{k_{j-1}+1}, \ldots, i_{k_j})$, $j = 1, 3, \ldots$ and $i_t \in U(i_{t-1})$ for $(i_{k_{j-1}}, i_{k_{j-1}+1}, \ldots, i_{k_j})$, $j = 2, 4, \ldots$, it holds that

i) there is a sequence $I' = (i_{k_m}, i_{k_m+1}, \ldots, i_{k_{m+1}})$ with $i_t \in D(i_{t-1})$ which satisfies $\sum_{(i,i') \in I'} m_{i,i'} < u_I$, or

ii) there is a sequence $I' = (i_{k_m}, i_{k_m+1}, \ldots, i_{k_{m+1}})$ with $i_t \in U(i_{t-1})$ which satisfies $\sum_{(i,i') \in I'} m_{i',i} > 0$.

In the above system, for $I = (i_0, i_1, i_2, i_3, i_0) = (1,2,3,4,1)$ and $(k_0, k_1, k_2, k_3, k_4) = (0,1,2,3,0)$, there is no sequence I' that satisfies i) or ii). In this paper, we assume that the initial parameters satisfy (A1) through (A3).

Before formulating the model into a GSMP, we have to determine the set of events which may occur when the current numbers of items in all buffers are given. Let s_{ik} denote the sum of numbers of items in buffers B_{ik} and A_{ik}.

Lemma 2.1 Suppose that conditions (A1) through (A3) are satisfied. Station i is processing items if and only if

1) $s_{ji} > 0$ for all $j \in U(i)/\{0\}$, and

2) $\sum_{(i_t, i_{t+1}) \in I} s_{i_t, i_{t+1}} < u_I$ for all $I \in L(i)$.

Proof: We can show this lemma in the same way as in the (a, b, k) lines [9]. Station i is processing items if and only if there is an item in buffer A_{ji} for every $j \in U(i)/\{0\}$ and station i is not blocked. When station i is not blocked, there is an item between stations j and i if and only if there is an item in buffer A_{ji}. Hence station i is processing items if and only if 1) holds and station i is not blocked. Therefore, it suffices to show that station i is not blocked if and only if 2) holds.

We assume that station $i = i_0$ is blocked because there are b_{ii_1} items in buffer B_{ii_1} for some $i_1 \in D(i)$. If this blocking occurs by a_{ii_1} items in A_{ii_1}, then $s_{ii_1} = b_{ii_1} + a_{ii_1} = u_{ii_1}$, which violates 2). Otherwise, the sum of the numbers of items in buffers A_{ii_1} and $B_{i_1 i_2}$ is $c_{ii_1 i_2}$ for some $i_2 \in D(i_1)$, and then there is an item in $B_{i_1 i_2}$ because $c_{ii_1 i_2} \geq a_{ii_1}$. Therefore, there are $a_{i_1 i_2}$ items in buffer $A_{i_1 i_2}$ or there are $c_{i_1 i_2 i_3}$ items in buffers $A_{i_1 i_2}$ and $B_{i_2 i_3}$ all together for some $i_3 \in D(i_2)$. In the same way, we obtain that if station $i_0 = i$ is blocked then one of the following events occurs:

a) There is some sequence of stations $I_a = (i_0, i_1, \ldots, i_k)$ where $(i_j i_{j+1}) \neq (i_{j'}, i_{j'+1})$ for $j \neq j'$ such that the number of items in buffer $B_{i_0 i_1}$ is $b_{i_0 i_1}$, the sum of numbers of items in buffers $A_{i_{j-1} i_j}$ and $B_{i_j, i_{j+1}}$ is $c_{i_{j-1}, i_j, i_{j+1}}$ for $j = 1, 2, \ldots, k-1$, and the number of items in buffer A_{i_{k-1}, i_k} is a_{i_{k-1}, i_k}, or

b) there is a sequence of stations $I_b = (i_0, i_1, \ldots, i_k, i_{k+1}, \ldots, i_{k'}, i_k)$ where $(i_j i_{j+1}) \neq (i_{j'}, i_{j'+1})$ for $j \neq j'$, $j, j' \leq k'$ such that the number of items in buffer $B_{i_0 i_1}$ is $b_{i_0 i_1}$, the sum of numbers of items in buffers $A_{i_{j-1} i_j}$ and $B_{i_j, i_{j+1}}$ is $c_{i_{j-1}, i_j, i_{j+1}}$ for $j = 1, 2, \ldots, k' - 1$,

the one in buffers $A_{i_{k'-1}i_{k'}}$ and $B_{i_{k'},i_k}$ is $c_{i_{k'-1},i_{k'},i_k}$ and the one in buffers $A_{i_{k'}i_k}$ and $B_{i_k,i_{k+1}}$ is $c_{i_{k'},i_k,i_{k+1}}$.
In case b), however, we have for $I' = (i_k, i_{k+1}, \ldots, i_{k'}, i_k)$

$$\sum_{(i,i')\in I'} s_{i,i'} = c_{i_{k'}i_k i_{k+1}} + c_{i_k i_{k+1} i_{k+2}} + \cdots + c_{i_{k'-1}i_{k'}i_k}.$$

Since the sum of numbers of items in buffers on the closed sequence I' is constant, we have

$$\sum_{(i,i')\in I'} s_{i,i'} = \sum_{(i,i')\in I'} m_{i,i'},$$

which contradicts (A2). Therefore if station i is blocked then case a) occurs, and 2) does not hold for $I_a = (i_0, i_1, \ldots, i_k) \in L(i)$.

Inversely, assume that 2) does not hold for some $I = (i_0, i_1, \ldots, i_k) \in L(i)$. From the blocking mechanism, the sum of numbers of items in buffers A_{ji} and B_{ik} is no more than c_{jik}, and the numbers of A_{ji} and B_{ik} are no more than a_{ji} and b_{ik}, respectively. Since $\sum_{(i_t,i_{t+1})\in I} m_{i_t,i_{t+1}} \le u_I$ and $\sum_{(i_t,i_{t+1})\in I} s_{i_t,i_{t+1}}$ shows the sum of the numbers of items in buffers B_{i_0,i_1}, A_{i_0,i_1}, B_{i_1,i_2}, $\cdots$, B_{i_{k-1},i_k} and A_{i_{k-1},i_k} for all $I = \{i_0, i_1, \ldots, i_k\} \in L(i)$, it holds that $0 \le \sum_{(i_t,i_{t+1})\in I} s_{i_t,i_{t+1}} \le u_I$ for all $I \in L(i)$. Therefore, we have

$$\sum_{(i_t,i_{t+1})\in I} s_{i_t,i_{t+1}} = u_I = b_{i_0 i_1} + \sum_{j=1}^{k-1} c_{i_{j-1}i_j i_{j+1}} + a_{i_{k-1}i_k},$$

which implies that there are $b_{i_0 i_1}$ items in buffer B_{i_0,i_1} and station i is blocked. ■

3. Formulation into a Generalized Semi-Markov Process(GSMP)

3.1. A Generalized Semi-Markov Process

For this paper to be self-contained, we explain a generalized semi-Markov process (s_0, S, A, $\mathcal{E}, \phi$, ω) in the following and show its properties related to our model (See [9] and [10]).

Let S be a countable state space, s_0 denote an initial state, A be a finite event set $\{\alpha_1, \alpha_2, \ldots, \alpha_M\}$, $\mathcal{E}(s)$ be the set of events which may occur in state s, where $A = \cup_s \mathcal{E}(s)$. The transition function $\phi(s, \alpha)$ denotes the state just after event α occurs in state s and the sequence of clock samples ω denotes $\{(\omega_{\alpha_1}(n), \ldots, \omega_{\alpha_M}(n)), n \in \mathcal{Z}\}$ where $\omega_\alpha(n)$ is the nth clock of event α and $\mathcal{Z} = \{1, 2, \ldots\}$.

The GSMP $(s_0, S, A, \mathcal{E}, \phi, \omega)$ behaves as follows: At time $\tau_0 = 0$, we set clock $c(\alpha_i)$ $= \omega_{\alpha_i}(1)$ and $n_i = 1$ for each $\alpha_i \in \mathcal{E}(s_0)$, and clock $c(\alpha_i)$=0 and $n_i = 0$ for each $\alpha_i \notin \mathcal{E}(s_0)$. For each $n \in \mathcal{Z}$, the nth event occurs at time $\tau_n = \tau_{n-1} + \min_{\alpha_i \in \mathcal{E}(s_{n-1})} \{c(\alpha_i)\} = \tau_{n-1} + c(\beta_n)$, and the state is moved to $s_n = \phi(s_{n-1}, \beta_n)$. At time τ_n, the clock of event α_i, $c(\alpha_i)$, is set as follows:
If $\alpha_i \in \mathcal{E}(s_n)/(\mathcal{E}(s_{n-1})/\{\beta_n\})$ then set $c(\alpha_i) = \omega_{\alpha_i}(n_{\alpha_i} + 1)$ and let n_{α_i} increase by 1,
If $\alpha_i \in \mathcal{E}(s_n) \cap (\mathcal{E}(s_{n-1})/\{\beta_n\})$ then let clock $c(\alpha_i)$ decrease by $\tau_n - \tau_{n-1}$, and
If $\alpha_i \notin \mathcal{E}(s_n)$ then set $c(\alpha_i) = 0$.

We define a generalized semi-Markov scheme (GSMS) by $\mathcal{G} = (s_0, S, A, \mathcal{E}, \phi)$. We call an event sequence $\sigma = \beta_0\beta_1 \cdots \beta_n$ a string. When there are some event $\beta_0 \in \mathcal{E}(s_0)$ and some state sequence $(s_1, \ldots, s_{k+1})$ such that $\beta_i \in \mathcal{E}(s_i)$ and $s_{i+1} = \phi(s_i, \beta_i)$, $i = 0, 1, \ldots, n$, then string $\sigma = \beta_0\beta_1 \ldots \beta_n$ is said to be feasible in s_0. For string σ, $[\sigma]_i$ denotes the number of occurrences of event α_i. $[\sigma] = \{[\sigma]_1, \ldots, [\sigma]_M\}$ is called a score of σ.

We say that $\mathcal{G}$ is noninterruptive if $\beta \in \mathcal{E}(\phi(s, \alpha))$ whenever $\alpha, \beta \in \mathcal{E}(s)$ for $\alpha \neq \beta$ and $s \in S$, and $\mathcal{G}$ is permutable if $\mathcal{E}(\phi(s_0, \sigma_1)) = \mathcal{E}(\phi(s_0, \sigma_2))$ whenever σ_1 and σ_2 are feasible in s_0 and $[\sigma_1] = [\sigma_2]$, where $\phi(s,\sigma) = \phi(\cdots\phi(\phi(s_0,\beta_0),\beta_1),\cdots,\beta_n)$ for feasible string $\sigma = \beta_0\beta_1\cdots\beta_n$. We also say that $\mathcal{G}$ satisfies convexity condition (CX) if it holds that $\{\mathcal{E}(\sigma_1)\cap\mathcal{E}(\sigma_2)\} - A_{\sigma_1,\sigma_2,\sigma_3} \subseteq \mathcal{E}(\sigma_3)$, whenever strings σ_1, σ_2 and σ_3 are feasible in s_0 and $[\sigma_3] \geq [\sigma_1]\wedge[\sigma_2]$, where $A_{\sigma_1,\sigma_2,\sigma_3} = \{\alpha_i; [\sigma_3]_i > [\sigma_1]_i\wedge[\sigma_2]_i\}$, $x\wedge y = (\min\{x_1,y_1\}, \min\{x_2,y_2\}, \ldots, \min\{x_M,y_M\})$ for $x = \{x_1,\ldots,x_M\}$ and $y = \{y_1,\ldots,y_M\}$ and $\mathcal{E}(\sigma) = \mathcal{E}(\phi(s_0,\sigma))$. It is known that the GSMS $\mathcal{G}$ is noninterruptive and permutable if it satisfies (CX).

For each string σ and event $\alpha_i \in A$, we define

$$\chi_{\alpha_i}(\sigma) = [\sigma]_i + 1\{\alpha_i \in \mathcal{E}(\sigma)\},$$

where $1\{\cdot\}$ is an indicator function. $\chi(\sigma) = \{\chi_{\alpha_1}(\sigma),\ldots,\chi_{\alpha_M}(\sigma)\}$ is called a characteristic function of GSMS $\mathcal{G}$. When GSMS $\mathcal{G}$ is permutable, then $\mathcal{E}(\sigma)$ and $\chi(\sigma)$ can be represented as $\mathcal{E}(x)$ and $\chi(x) = \{\chi_{\alpha_1}(x),\ldots,\chi_{\alpha_M}(x)\}$ respectively, where $x = [\sigma]$ and $\chi_{\alpha_i}(x) = x_{\alpha_i} + 1\{\alpha_i \in \mathcal{E}(x)\}$. $\mathcal{N} = \{x \in Z_+^m : x = [\sigma]$ for some σ being feasible in s_0 $\}$ is called a score space of $\mathcal{G}$. We say that χ is increasing if $\chi(\sigma) \leq \chi(\sigma')$ for feasible strings σ, σ' with $x = [\sigma]$, $y = [\sigma']$ and $x \leq y$, and that χ is increasing and supermodular if χ is increasing and

$$\chi(x) + \chi(y) \leq \chi(x\wedge y) + \chi(x\vee y). \tag{1}$$

for any feasible $x, y, x\wedge y, x\vee y \in \mathcal{N}$, where $x\vee y = (\max\{x_1,y_1\},\ \ldots,\ \max\{x_M,y_M\})$.

When $\mathcal{G}$ is permutable, set $\mathcal{N}_{\alpha,n} = \{x \in \mathcal{N} : x_\alpha = n-1, \alpha \in \mathcal{E}(x)\}$ for $x = [\sigma]$. We say that $y \in \mathcal{N}_{\alpha,n}$ is a minimal element of $\mathcal{N}_{\alpha,n}$ if it holds that $x \leq y$ implies $x = y$ for all $x \in \mathcal{N}_{\alpha,n}$.

We give the well-known lemmas in the following (see [10]).

Lemma 3.1 a) If GSMS $\mathcal{G}$ is noninterruptive and permutable, then χ is increasing.
b) GSMS $\mathcal{G}$ satisfies (CX) if and only if χ is increasing and supermodular. ■

Lemma 3.2 If $\mathcal{G}$ satisfies Property (CX), then $x\vee y \in \mathcal{N}$ and $x\wedge y \in \mathcal{N}$ whenever $x, y \in \mathcal{N}$. There is a unique minimal element $x(\alpha,n) = \{x_\beta(\alpha,n); \beta \in A\}$ of $\mathcal{N}_{\alpha,n}$ and

$$T_\alpha(n) = \omega_\alpha(n) + \max_{\beta\in A}\{T_\beta(x_\beta(\alpha,n))\},$$

where $T_\alpha(n)$ denotes the nth epoch when event α occurs. ■

Lemma 3.3 We suppose that $\mathcal{G}$ satisfies (CX). We also assume that for each $\alpha \in A$ $\omega_\alpha = \{\omega_\alpha(n), n \in \mathcal{Z}\}$ is a sequence of mutually independent random variables and these sequences are mutually independent between different events. Then for such two sequences $\{\omega_\alpha(n)\}$ and $\{\omega'_\alpha(n)\}$, it follows that
if $\omega_{\alpha_i}(n) \leq_{st} \omega'_{\alpha_i}(n)$ for all $i \in \mathcal{M}$, $n \in \mathcal{Z}$, then $T_{\alpha_i}(n) \leq_{st} T'_{\alpha_i}(n)$ for all $i \in \mathcal{M}$, $n \in \mathcal{Z}$, and if $\omega_{\alpha_i}(n) \leq_{icx} \omega'_{\alpha_i}(n)$ for all $i \in \mathcal{M}$, $n \in \mathcal{Z}$, then $T_{\alpha_i}(n) \leq_{icx} T'_{\alpha_i}(n)$ for all $i \in \mathcal{M}$, $n \in \mathcal{Z}$, where $\leq_{st}$ and $\leq_{icx}$ denote stochastic ordering and nondecreasing convex ordering, respectively (See Stoyan[15]). ■

Lemma 3.2 shows that if the GSMP satisfies (CX) condition then the nth occurrence epoch of any event α is increasing and convex in ω.

3.2. GSMP Formulation

From Lemma 2.1, we can formulate the fork/join multi-stage production systems into the following GSMP $(s_0, S, A, \mathcal{E}, \phi, \omega)$:

$S = \{(s_1, \ldots, s_M);\ s_i = (s_{ik}; k \in D(i)/\{M+1\}),\ s_{ji} \geq 0$ for all $j \in U(i)/\{0\}$,
$\sum_{(i_t,i_{t+1}) \in I} s_{i_t,i_{t+1}} \leq u_I$ for all $I \in L(i),\ i \in \mathcal{M}\}$,
$s_0 = \{(m_{ik}); k \in D(i)/\{M+1\}, i \in \mathcal{M}\}$
$A = \{\alpha_i; i \in \mathcal{M}\}$, where the event α_i denotes the end of processing at station i.
$\mathcal{E}(s) = \{\alpha_i; s_{ji} > 0$ for all $j \in U(i)/\{0\}, \sum_{(i_t,i_{t+1}) \in I} s_{i_t,i_{t+1}} < u_I$ for all $I \in L(i)\}$, for each $s \in S$,
$\phi(s, \alpha_i) = s - \sum_{j \in U(i)/\{0\}} e_{ji} + \sum_{k \in D(i)/\{M+1\}} e_{ik}$ for $s \in S$ and $\alpha_i \in A$,
where e_{ij} denotes a unit vector with $\prod_{i \in \mathcal{M}} |D(i)/\{M+1\}|$ dimensions whose the jth element of station i is one, and
$\omega_{\alpha_i}(n) = S_i(n)$ for $i \in \mathcal{M}$ and $n \in \mathcal{Z}$,
where $S_i(n)$ is a random variable which represents the nth processing time at station i for $i \in \mathcal{M}$ and $n \in \mathcal{Z}$.

4. Stochastic Properties of the General Fork/Join System

4.1. Convex Property

Let x_i be the number of occurrences of event α_i for $i \in \mathcal{M}$. Then we have

$$s_{ik} = m_{ik} + x_i - x_k \quad \text{for } i \in \mathcal{M} \text{ and } \quad k \in D(i)/\{M+1\}.$$

Theorem 4.1 The GSMS $\hat{\mathcal{G}} = (s_0, S, A, \mathcal{E}, \phi)$ defined in section 3 satisfies Property (CX).
Proof: Since the system $\hat{\mathcal{G}}$ is noninterruptive and permutable, Lemma 3.1 implies that the characteristic function is increasing. To show the supermodularity of characteristic function χ, it suffices to prove that

$$1\{\alpha_i \in \mathcal{E}(x)\} + 1\{\alpha_i \in \mathcal{E}(y)\} \leq 1\{\alpha_i \in \mathcal{E}(x \vee y)\} + 1\{\alpha_i \in \mathcal{E}(x \wedge y)\}.$$

If $\alpha_i \notin \mathcal{E}(x)$ and $\alpha_i \notin \mathcal{E}(y)$ then it obviously holds. Assume that $\alpha_i \in \mathcal{E}(x)$ and $\alpha_i \in \mathcal{E}(y)$, and let a set of stations for which there exists a station sequence $(i_0, i_1, \ldots, k)$ in $L(i)$ be denoted by $\tilde{D}(i)$. Then for all $j \in U(i)$,

$$\begin{aligned}(x \wedge y)_j + m_{ji} - (x \wedge y)_i &\geq (x_j - x_i) \wedge (y_j - y_i) + m_{ji} \geq 1 \\ (x \vee y)_j + m_{ji} - (x \vee y)_i &\geq (x_j - x_i) \wedge (y_j - y_i) + m_{ji} \geq 1\end{aligned}$$

and for all $k \in \tilde{D}(i)$ and $I = (i, \ldots, k) \in L(i)$,

$$\begin{aligned}\sum_{(i_t,i_{t+1}) \in I} s_{i_t,i_{t+1}} + (x \wedge y)_i - (x \wedge y)_k &\leq \sum_{(i_t,i_{t+1}) \in I} s_{i_t,i_{t+1}} + (x_i - x_k) \vee (y_i - y_k) \leq u_I - 1, \\ \sum_{(i_t,i_{t+1}) \in I} s_{i_t,i_{t+1}} + (x \vee y)_i - (x \vee y)_k &\leq \sum_{(i_t,i_{t+1}) \in I} s_{i_t,i_{t+1}} + (x_i - x_k) \vee (y_i - y_k) \leq u_I - 1.\end{aligned}$$

Therefore, it holds that $\alpha_i \in \mathcal{E}(x \vee y)$ and $\alpha_i \in \mathcal{E}(x \wedge y)$. If $\alpha_i \in \mathcal{E}(x)$ and $\alpha_i \notin \mathcal{E}(y)$, we can show that $\alpha_i \in \mathcal{E}(x \vee y)$ if $x_i \geq y_i$ and $\alpha_i \in \mathcal{E}(x \wedge y)$ otherwise. ■

Let $T_i(n)$ be the nth completion epoch of processing at station i. Then we have the following theorem.

Theorem 4.2 For each $i \in \mathcal{M}$ and $n \in \mathcal{Z}$,

$$T_i(n) = S_i(n) + \max\left\{T_i(n-1), \max_{j\in U(i)}\{T_j(n-m_{ji})\}, \max_{k\in\tilde{D}(i)}\left\{T_k\left(n - \min_{(i_0,\ldots,i_t)\in L(i), i_t=k}(u_{(i_0,\ldots,i_t)} - \sum_{j=0}^{t-1} m_{i_j,i_{j+1}})\right)\right\}\right\}, \quad (2)$$

where $T_i(n) = 0$ for $n \leq 0$.

Proof: Since the GSMS $\hat{\mathcal{G}}$ has the property (CX), there is a unique minimal element $x(\alpha_i, n)$. It is obvious that $x_{\alpha_i}(\alpha_i, n) = n-1$. If $j \in U(i)$, then $s_{ji} = m_{ji} + x_j - x_i$ where x_j is the number of the completions of processing at station j, and hence by Lemma 2.1 $m_{ji} + x_{\alpha_j}(\alpha_i, n) - (n-1) = 1$, that is,

$$x_{\alpha_j}(\alpha_i, n) = n - m_{ji}.$$

If $k \in \tilde{D}(i)$, then for each $I = (i_0, \ldots, i_t) \in L(i)$ with $i_t = k$, we have $\sum_{(i_{t-1}i_t)\in I} s_{i_{t-1}i_t} = \sum_{(i_{t-1}i_t)\in I} m_{i_{t-1}i_t} + x_i - x_k$, and Lemma 2.1 implies that $\sum_{(i_{t-1}i_t)\in I} m_{i_{t-1}i_t} + n - 1 - x_{\alpha_k}(\alpha_i, n) \leq u_I - 1$. Hence

$$x_{\alpha_k}(\alpha_i, n) = \max_{I=(i_0,\ldots,i_t)\in L(i), i_t=k}\{\sum_{(i_{t-1}i_t)\in I} m_{i_{t-1}i_t} + n - u_I\}.$$

Therefore, we have equation (2) from Lemma 3.2. Note that if $j \in U(i)$ and $j \in \tilde{D}(i)$, then $x_{\alpha_j}(\alpha_i, n)$ becomes the maximum of the above two values, and if $j \notin U(i) \cup \tilde{D}(i)$, then the number of the occurrences of event α_j does not affect the maximum of (2). ∎

Using Theorem 4.2 and initial conditions (A1) to (A3), we show that station $i \in A$ finishes the nth service in a finite period with probability one.

Theorem 4.3 For each $n \in \mathcal{Z}$, $T_i(n)$ is finite with probability one for all $i \in \mathcal{M}$, when $P(\sum_{m=1}^{n} S_i(m) < \infty,\ i \in A) = 1$.

Proof: We define the notation $(j, m') \to (i, m)$, which represents $x_{\alpha_j}(\alpha_i, m) = m'$ for $i, j \in \mathcal{M}$ and $m, m' \in \mathcal{Z}$. From equation (2) $(j, m') \to (i, m)$ implies that $m' \leq m$ and $(j, m'+1) \to (i, m+1)$. Hence it is sufficient to show that the first event α_i occurs for all $i \in \mathcal{M}$ with probability one. Theorem 4.2 implies that we have that $(j, 1) \to (i, 1)$ if and only if either $j \in U(i)$ and $m_{ji} = 0$ or $j \in \tilde{D}(i)$ and $u_I = \sum_{p=1}^{t} m_{i_{p-1}i_p}$ for some $I = (i_0, \ldots, i_t) \in L(i)$ with $i_t = j$. Under initial assumptions (A2) and (A3) there is no station sequence $(i_0, i_1, \ldots, i_k, i_0)$ such that $(i_0, 1) \to (i_1, 1) \to \ldots \to (i_k, 1) \to (i_0, 1)$. Then there exist station i such that $U(i) = \{0\}$, which is in $\mathcal{E}(s_0)$. Since the system is noninterruptive and the event set is finite, the first event α_i occurs for all $i \in \mathcal{M}$ in a finite period. ∎

4.2. Stochastic Comparison

The following theorem shows that two systems with different clock distributions of events are comparable with respect to the event epochs.

Theorem 4.4 For two sequences of processing times $\omega = \{S_i(n); i \in \mathcal{M}, n \in \mathcal{Z}\}$ and $\omega' = \{S_i'(n); i \in \mathcal{M}, n \in \mathcal{Z}\}$, we denote the epochs when the nth processing is finished at station i by $T_i(n)$ and $T_i'(n)$, respectively. If each of ω and ω' forms sequences of independent

random variables, then
$S_i(n) \leq_{st} S_i'(n)$ implies that $T_i(n) \leq_{st} T_i'(n)$, and
$S_i(n) \leq_{icx} S_i'(n)$ implies that $T_i(n) \leq_{icx} T_i'(n)$.
Proof: We obtain the result from Theorem 4.1 and Lemma 3.3. ∎

This theorem shows that the less the moments of processing times are, the larger the throughput of the completed products at each station is. To increase the throughput requires reducing the variance of processing times of items.

4.3. Reversibility

For the fork/join multi-stage production system, called the original system in the following, we define the reversed system with parameters m_{ij}^r, $a_{ij}^r, b_{ij}^r, c_{kij}^r$, the sets $U^r(i)$, $D^r(i)$ and processing times $S_i^r(m)$ such that
1) for each $i \in \mathcal{M}$,

$$U^r(i) = D(i), \quad D^r(i) = U(i), \quad m_{ij}^r = m_{ji} \quad \text{for } j \in D^r(i),$$

where $0 \in U^r(i)$ if $M+1 \in D(i)$, and $M+1 \in D^r(i)$ if $0 \in U(i)$,
2) for any sequence $(i_0, i_1, \ldots, i_t) \in L(i)$,

$$u^r_{i_t, i_{t-1}, \ldots, i_1, i_0} = u_{i_0, i_1, \ldots, i_{t-1}, i_t}, \text{ and} \tag{3}$$

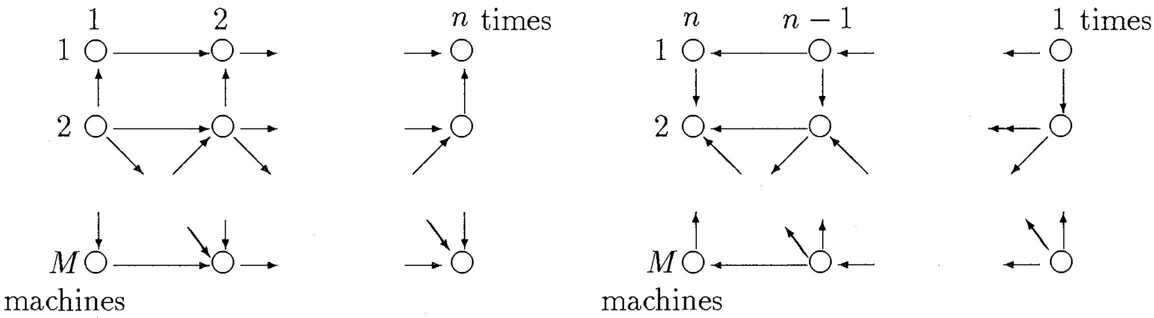
3) $S_i^r(m) = S_i(n+1-m)$ for $i \in \mathcal{M}$ and $m = 1, 2, \ldots, n$.
We also define $L^r(i)$ and $\tilde{D}^r(i)$ in the same way as in $L(i)$ and $\tilde{D}(i)$, respectively.
Condition 2) is satisfied if one of the following conditions holds. For any $i \in \mathcal{M}$, $j \in U(i)$ and $k \in D(i)$,
a) $a_{ij}^r = b_{ji}$, $b_{ki}^r = a_{ik}$ and $c_{kij}^r = c_{jik}$,
b) $b_{ik} = b$, $a_{ji} + b = c_{jik}$, $a_{ij}^r = a_{ji}$, $b_{ki}^r = b$ and $c_{kij}^r = a_{ki}^r + b$ for some constant b,
c) $a_{ji} = c_{jik} = d_i$ for all j and k, $b_{ik} = b$, $a_{ij}^r = a_{ji} = d_i, b_{ki}^r = b$ and $c_{kij}^r = a_{ki}^r = d_k$ for some constants b and d_i, or
d) $a_{ji} = c_{jik} = b_{ik} = d_i$ for all j and k, $a_{ij}^r = d_j, b_{ki}^r = d_k, c_{kij}^r = d_i$ for some constants d_i, $i \in \mathcal{M}$.

To show the equivalence property between the original and reversed systems, we derive the relation between the maximum of the nth completion times of processing at all stations, $\max_{i\in\mathcal{M}} T_i(n)$, and the processing times $\{S_i(m); i \in \mathcal{M}, m = 1, 2, \ldots, n\}$. Since the original system satisfies Property (CX), a unique minimal element $x(\alpha, m)$ exists for each $\alpha \in A$ and $m \in \mathcal{Z}$. Consider an activity network $V = (N, E)$ with a node set $N = \{(i, m); i \in \mathcal{M},\ m = 1, \ldots, n\}$ and an arc set $E = \{(j, m') \to (i, m); x_{\alpha_j}(\alpha_i, m) = m', (j, m'), (i, m) \in N\}$, which is shown in Figure 3 a). Let $\Pi(n)$ be the set of all paths from $(i, 1)$ to (j, n) for all pairs of stations (i, j) in V. Then Theorem 4.2 implies that

$$\max_{i\in\mathcal{M}} T_i(n) = \max_{\pi\in\Pi(n)} \sum_{(i,m)\in\pi} S_i(m).$$

Since the reversed system can be formulated into a GSMP $\hat{\mathcal{G}}^r$, Theorem 4.1 leads to the property (CX) of $\hat{\mathcal{G}}^r$. Consider an activity network $V^r = (N^r, E^r)$ with a node set $N^r = \{(i, m); i \in \mathcal{M},\ m = 1, \ldots, n\}$ and an arc set $E^r = \{(j, m') \to (i, m);\ x^r_{\alpha_j}(\alpha_i, m) = m',\ (j, m'), (i, m) \in N^r\}$, shown in Figure 3 b), where $x^r(\alpha, m)$ is a minimal element in the reversed system. Let $\Pi^r(n)$ denote the set of all paths from $(i, 1)$ to (j, n) for any pair of stations (i, j) in V^r. Then

$$\max_{i\in\mathcal{M}} T_i^r(n) = \max_{\pi^r\in\Pi^r(n)} \sum_{(i,m)\in\pi^r} S_i^r(m).$$



(a) original system (b) reversed system

Fig. 3 Activity Networks

Theorem 4.5

$$\max_{i\in\mathcal{M}} T_i^r(n) = \max_{i\in\mathcal{M}} T_i(n).$$

Proof: If $j \in U(i)$ then $i \in U^r(j)$ and

$$x^r_{\alpha_i}(\alpha_j, m) = m - m^r_{ij} = m - m_{ji} = x_{\alpha_j}(\alpha_i, m).$$

If $k \in \tilde{D}(i)$ then $i \in \tilde{D}^r(k)$ and

$$\begin{aligned} x^r_{\alpha_i}(\alpha_k, m) &= m - \min_{I=(i_0,\ldots,i_j)\in L^r(k), i_j=i} \{u^r_I - \sum_{(i,i')\in I} m^r_{i,i'}\} \\ &= m - \min_{I=(i_0,\ldots,i_j)\in L(i), i_j=k} \{u_I - \sum_{(i,i')\in I} m_{i,i'}\} = x_{\alpha_k}(\alpha_i, m). \end{aligned}$$

Therefore, $(j,m') \to (i,m)$ in E if and only if $(i,m') \to (j,m)$ in E^r, which implies that $(j,m') \to (i,m)$ in E if and only if $(i, n+1-m) \to (j, n+1-m')$ in E^r. Since $S_i^r(m) = S_i(n+1-m)$ for $i \in \mathcal{M}$ and $m = 1, \ldots, n$,

$$\max_{\pi^r\in\Pi^r(n)} \sum_{(i,m)\in\pi^r} S_i^r(m) = \max_{\pi\in\Pi(n)} \sum_{(i,m)\in\pi} S_i(m). \quad \blacksquare$$

Instead of 3), we assume that in the original and reversed systems,
3)' $\{S_i(m); m = 1, 2, \ldots, n\}$ is a sequence of independent and identically distributed random variables for each $i \in \mathcal{M}$ and $S_i(m)$ and $S_j(m')$ are mutually independent for all $i, j \in \mathcal{M}$, $i \neq j$ and $m, m' = 1, 2, \ldots, n$.
Then the following corollary is obtained immediately.

Corollary 4.1 If $\{S_i(m)\}$ satisfies 3)', then we have

$$\max_{i\in\mathcal{M}} T_i^r(n) =_{st} \max_{i\in\mathcal{M}} T_i(n). \quad \blacksquare$$

For the (a, b, k) system such that $m_{ij} = 0$, it is clear that $T_M(n) = \max_{i\in\mathcal{M}} T_i(n)$ and $T_1^r(n) = \max_{i\in\mathcal{M}} T_i^r(n)$. Hence Theorem 4.5 implies that $T_M(n) = T_1^r(n)$. This stochastic equivalence property has been shown in [9]. For the fork/join queueing system with communication blocking, this property is discussed in [13].

4.4. Structural Equivalence

In the original system, suppose that $a_{ji} + b_{ik} = c_{jik}$ for any $i \in \mathcal{M}$, $j \in U(i)$ and $k \in D(i)$. Then blocking at station i occurs only if there are a_{ik} items in buffer A_{ik} for some $k \in D(i)$. Therefore, in the same way as in the proof of Theorem 4.2, we have

$$T_i(n) = S_i(n) + \max\left\{T_i(n-1), \max_{j\in U(i)}\{T_j(n - m_{ji})\}, \max_{k\in D(i)}\{T_k(n - (u_{(i,k)} - m_{ik}))\}\right\}, \quad (4)$$

where $u_{(i,k)} = b_{ik} + a_{ik}$.

For this original system, we consider a locally reversed system $\hat{\mathcal{G}}^l$ in which the precedence relations between some pairs of stations are reversed and the numbers of the initial inventory and the initial empty spaces in buffers between these stations are exchanged. That is, parameters m_{ik}^l, D^l, a^l, b^l and c^l of $\hat{\mathcal{G}}^l$ satisfy that for some pairs of stations $(i_1, j_1), \ldots, (i_p, j_p)$ such that $j_q \in D(i_q)$, $q = 1, 2, \ldots, p$,

$$i_q \in D^l(j_q), \quad a^l_{j_q,i_q} = a_{i_q,j_q}, \quad b^l_{j_q,i_q} = b_{i_q,j_q}, \quad m^l_{j_q i_q} = u_{i_q j_q} - m_{i_q j_q}, \quad u^l_{(j_q,i_q)} = u_{(i_q,j_q)},$$

and if $k \in U^l(j_q)$ then $c^l_{kj_q i_q} = a^l_{kj_q} + b^l_{j_q i_q}$. Other parameters and processing times are the same as those in the original system, and if $U^l(i)$ becomes empty then put $U^l(i) = \{0\}$, and if $D^l(i)$ becomes empty then put $D^l(i) = \{M+1\}$. Then we obtain the following theorem.

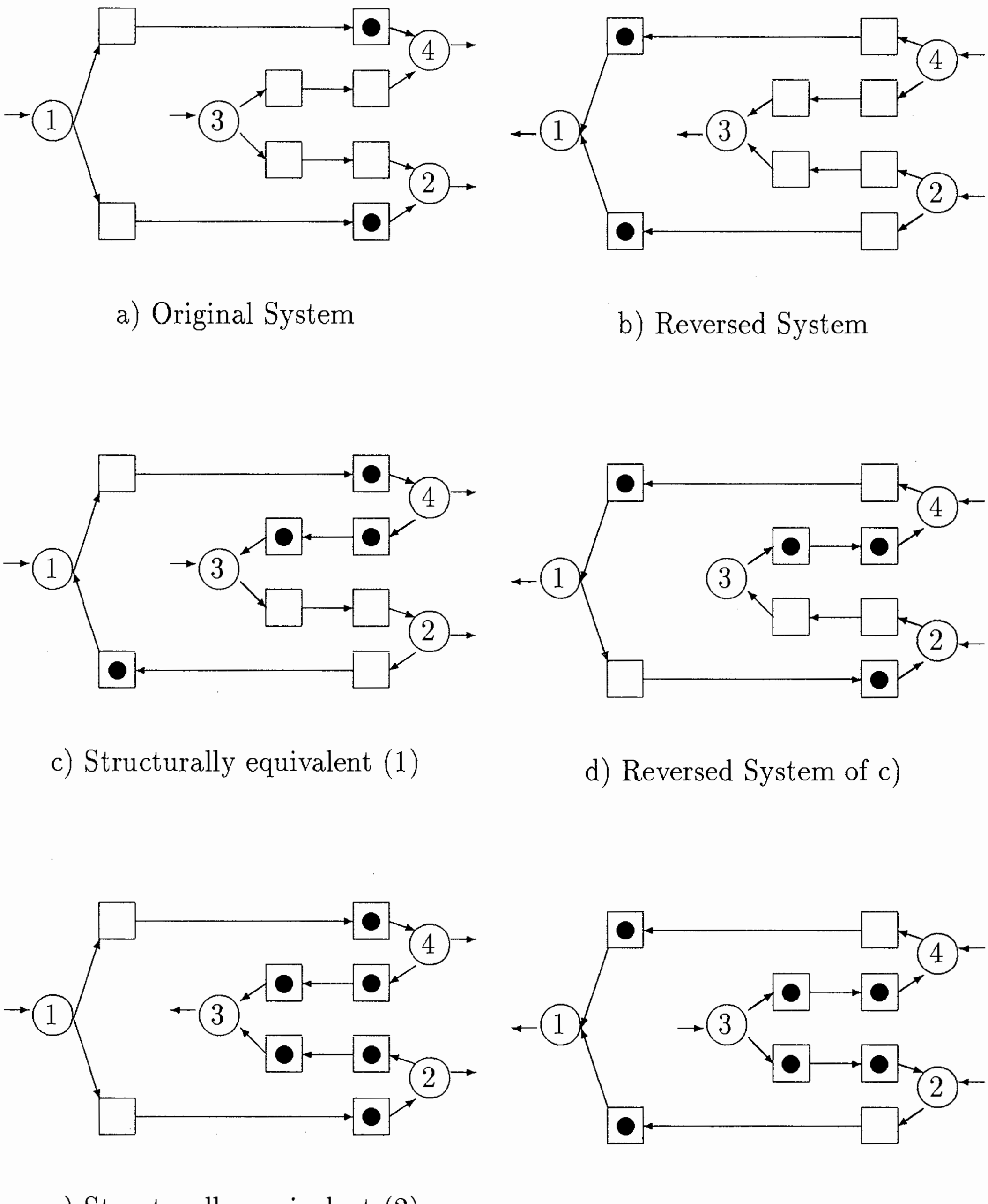
Theorem 4.6 Let $T_i^l(n)$ be the nth completion epoch of processing at station i of the locally reversed system. Then it holds that $T_i^l(n) = T_i(n)$ for $i \in \mathcal{M}$ and $n \in \mathcal{Z}$.
Proof: The recursive equations in the locally reversed system coincide with (4) in the original system. ■

This structural equivalence property has been shown for the (a, b, k) system in [9] and for the fork/join queueing network with communication blocking in [13].

Corollary 4.1 and Theorem 4.6 imply that if two different systems have the reversibility or structural equivalence relations, then the throughput of the completed products in one system is the same as that in another system.

4.5. An Example

To illustrate the reversibility or structural equivalence, we consider the fork/join multi-stage production system in Figure 4 a), which is the same as in Figure 2 except the initial numbers of items. For this system we have several different systems in b) through f) of Figure 4, whose parameters are $a_{ik} = b_{ik} = 1$ and $c_{jik} = 2$ for all $i \in M$, $j \in U(i)$ and $k \in D(i)$. Figure 4 b) is the reversed system of Figure 4 a), and c) and e) Figrue 4 are structurally equivalent to Figure 4 a). Figure 4 d) and Figure 4 f) are the reversed systems of c) and e), respectively. Note that c) and d) become equivalent to cyclic closed queueing networks with blocking by removing the input and output stations. Corollary 4.1 and Theorem 4.6 show that all systems have the same throughput of completed products.



a) Original System
b) Reversed System
c) Structurally equivalent (1)
d) Reversed System of c)
e) Structurally equivalent (2)
f) Reversed System of e)

Fig. 4 Equivalent Systems

5. Concluding Remarks

In this paper, we formulate the general fork/join multi-stage production system with general blocking into the GSMP, which includes open and closed (a, b, k) systems. We derive the convex property of the GSMP and show the stochastic comparison, the reversibility and the structural equivalence property of the general fork/join production system.

If a GSMP is a subscheme of another GSMP, and both GSMPs are irreducible, that is, for each GSMP any state in its state space is reachable from another state, then we can compare the performance of these systems. For example, Tayur[16] considered the irreducible kanban systems and discussed optimal allocations of kanbans. For the closed (a, b, k) system, Glasserman and Yao [10] discussed the relation between the nth completion time of processing and parameters $\{a, b, k\}$. It is not easy, however, to find the relations between the set of the reachable states and its initial inventory in the general fork/join system. Therefore, it seems difficult to compare two systems with different parameter set $\{a, b, c\}$ except special cases such as the closed (a, b, k) model.

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Appendix

For the convenience of the readers the notations used in this paper are listed in the following:

<u>Fork/Join Production Systems</u>

$\mathcal{M} = \{1, 2, \ldots, M\}$: the set of actual stations,
stations 0, $M+1$: the input and output stations, respectively,
$U(i)$: a set of immediate upstream stations $U(i)$,
$D(i)$: a set of immediate downstream stations $D(i)$,
$|D(i)|$: the number of elements of $D(i)$,
$B_{ik}(A_{ik})$: the buffer in rear of station i (the buffer in front of station k)
b_{ik} (a_{ik}): the capacities of buffers B_{ik} (A_{ik}),
c_{jik}: the capacity of the total numbers of items in buffers B_{ji} and A_{ik},
$L(i) = \{(i_0, i_1, \ldots, i_t); i_0 = i, i_1 \in D(i_0), i_2 \in D(i_1), \ldots, i_t \in D(i_{t-1}), (i_j, i_{j+1}) \neq (i_k, i_{k+1}),$
$\text{for } j \neq k,\ i_t \neq M+1, t \geq 1\}$,
$u_I = b_{i_0,i_1} + \sum_{j=1}^{t-1} c_{i_{j-1},i_j,i_{j+1}} + a_{i_{t-1},i_t}$ for $I = \{i_0, \ldots, i_t\} \in L(i)$,
m_{ik}: the initial total number of the items in buffers B_{ik} and A_{ik},
s_{ik}: the variable representing the total number of the items in buffers B_{ik} and A_{ik},
$S_k(n)$: the random variable denoting the nth processing time at machine k,
$T_k(n)$: the random variable denoting the nth completion epoch of processing at machine k,
$\tilde{D}(i)$: a set of stations k for which there exists a station sequence $(i_0, i_1, \ldots, k)$ in $L(i)$.

<u>GSMP:$(s_0, S, A, \mathcal{E}, \phi, \omega)$</u>

S: a countable state space,
s_0: an initial state,
$A = \{\alpha_1, \alpha_2, \ldots, \alpha_M\}$: a finite event set,
$\mathcal{E}(s)$: the set of events which may occur in state s, $A = \cup_s \mathcal{E}(s)$.
$\phi(s, \alpha)$: the transition function representing the state just after event α occurs in state s,
$\omega = \{(\omega_{\alpha_1}(n), \ldots, \omega_{\alpha_M}(n)), n \in \mathcal{Z}\}$: $\omega_\alpha(n)$ is the nth clock of event α,
$\sigma = \beta_0 \beta_1 \cdots \beta_n$: a string (the event sequence),
$\chi_{\alpha_i}(\sigma) = [\sigma]_i + 1\{\alpha_i \in \mathcal{E}(\sigma)\}$, a characteristic function of GSMS $\mathcal{G}$,
$\mathcal{N} = \{x \in Z_+^m : x = [\sigma] \text{ for some } \sigma \text{ being feasible in } s_0\}$: a score space of $\mathcal{G}$,
$\mathcal{N}_{\alpha,n} = \{x \in \mathcal{N} : x_\alpha = n-1, \alpha \in \mathcal{E}(x)\}$ for $x = [\sigma]$,
$x \vee y = (\max\{x_1, y_1\}, \ldots, \max\{x_M, y_M\})$,
$x \wedge y = (\min\{x_1, y_1\}, \ldots, \min\{x_M, y_M\})$,
$x(\alpha, n) = \{x_\beta(\alpha, n)\}$: a minimal element,
$T_\alpha(n)$: the nth epoch when event α occurs in GSMP.

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