Journal of the Operations Research
Society of Japan
Vol. 40, No. 3, September 1997

OPTIMAL INVESTIGATING SEARCH MAXIMIZING THE DETECTION PROBABILITY

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(Received June 12, 1995; Revised August 27, 1996)

Abstract In this paper, we deal with a two-stage search consisting of the broad search and the investigating search, and derive conditions of an optimal investigating search plan maximizing the detection probability of a target. We consider a search for a target in an area under a restriction of search time. In this search, contacts of false signal caused by system noise are inevitable, and when a contact is gained, the searcher must investigate the contact to ascertain whether it is the true target or not. We derive conditions of the optimal investigating time and elucidate properties of the optimal plan. Several numerical examples are examined and interpretations of the optimal conditions are given.

1. Introduction

In this paper, we consider a search for a target in an area under a restriction of total search time T. Here, we assume that the detection device being used by the searcher is frequently disturbed by false signals that are similar to the true target. Hence, the searcher must investigate the obtained signal (called a contact) in detail by another sensors whether it comes from the true target searching for or not. Therefore, the search process is composed of two stages ; in the first stage, a broad search is carried out to gain contacts, and in the second stage (called an investigating search), the contact obtained in the first stage is examined in detail to confirm whether it is the true target or not. As for the broad search, the random search [4] characterized by a Poisson process is assumed and the searcher contacts the true target with a Poisson rate λ_0 if there is not any false contact. On the other hand, the false contacts consist of two types which cannot be distinguished by the detection device used in the broad search. One is the contact of a false signal emitted from a real object being similar to the true target and the false contact of this type can be distinguished from the true target by the investigating search. Another type of the false contact is caused by a false signal in background noise or system noise. In this case, since there is no object to be investigated, usually no positive information telling it to be false is obtained by the investigating search. Hence, if the investigating search is continued long without any information, the searcher must stop it at some appropriate time, otherwise he cannot avoid the risk of wasting the search time by the investigating search for the false contact. In this case, the searcher must abandon the unconfirmed contact and restart the broad search to get a new contact. Since we assume that the target stays in the search area during the search and the random search is carried out in the broad search, the contact rate of the true target λ_0 does not change even if the unconfirmed contact is the true target. It should be noted that the search-and-investigation process mentioned above is a sampling test with replacement.

In this paper, we consider a search involving the false contact of signal from the system noise. The searcher begins his search with the broad search and he gets a contact sooner or later. The searcher cannot know whether the contact is true or not by the detection device used in the broad search, but he observes the distinctive feature of the obtained signal and can know the probability of the contact being true. This probability is referred to the reliability of the contact. The investigating search for the contact is started immediately if it is worthy of investigation. The investigating search is stopped if the true target is confirmed or if it is continued so long without any information. Here, we assume that the searcher desires to maximize the detection probability of the target (the term "detection" is used only when the investigation of the true target is finished successfully). The purpose of this paper is to determine optimally the stopping time of the investigating search for the contact obtained at time t according to its reliability.

The optimal search with false contacts have been studied by several authors. As for search problems with the false contacts by real objects being similar to the true target, Stone and Stanshine [6], Stone et al. [7] and Dobbie [1] dealt with the optimal broad search including the investigating search uninterrupted until finishing the investigation. These studies are compiled in the text book by Stone [5]. Preceding studies of our problem, the optimal investigating search for the noise type false contacts, were studied by Kisi [3] and Iida [2]. Assuming a large number of targets and false contacts of noise type, Kisi studied the optimal investigating plan maximizing the expected number of targets gained during a limited search time. He derived a necessary condition of the optimal investigating time and showed numerical examples of the optimal solution. On the other hand, considering a single target and noise type false contacts and assuming the search to be continued until detection of the target, Iida studied the optimal investigating plan minimizing the expected time to detection of the true target. He showed the necessary and sufficient condition for the optimal investigating search plan and elucidated the structure of the optimal plan. In contrast with these studies, this paper deals with an unsolved investigating search problem maximizing the detection probability of the target under a limited search time and analyze the properties of the optimal investigating search plan.

In the next section, we describe the assumptions of the model and formulate the search process, and in Section 3, theorems elucidating the structure of the optimal investigating search plan are presented. Several numerical examples are analyzed to show the properties of the optimal plan in Section 4. Finally in Section 5, we examine meanings of the conditions for the optimal investigating plan and discuss the relations between our model and results studied by the previous authors.

2. Assumptions and Formulation of the Model

Assumptions of the model dealt with here are as follows.

(1). A searcher searches a target distributed uniformly in a known region (area A) and starts his search with the broad search. In the broad search, the random search is assumed; the searcher searches the area for the target randomly with speed v and by a detection device with sweep width W, and the search pattern is random in the sense that the path can be thought of as having its different (not too near) portions placed independently and randomly of one another in A. By this assumption, the searcher contacts the true target with a Poisson rate $\lambda_0 = vW/A$ [4,8].

(2). The search time is assumed to be continuous and be restricted by T. The time is counted in the reverse order and we call the time t when the search time t is remained until the end of the search, namely, $t = T$ at the beginning of the search and $t = 0$ at the end of the search.

(3). The searcher gains contacts randomly in the broad search with Poisson rate λ. When the searcher gets a contact, he observes its similarity to the signal of the true target and classifies it into n classes. The i-class contact is specified by its reliability p_i, $i = 1,2,\cdots,n$, (the probability of the contact being true), and without any loss of generality, $1>p_1>p_2>\cdots>p_n>0$ is assumed. The occurrence of i-class contact is assumed to be a Poisson process with rate λ_i which does not change in time and is not influenced by the past contacts. Let λ_{i0} be the Poisson rate of the false contact of i-class and p_{0i} be the probability that the true contact is classified in i-class. Then, we have $\lambda_i = \lambda_0 p_{0i} + \lambda_{i0}$, $p_i = \lambda_0 p_{0i}/\lambda_i$ and $\lambda = \Sigma_i \lambda_i = \lambda_0 + \Sigma_i \lambda_{i0}$. This assumption of Poisson arrival of contacts results from the assumptions of the random system noise and the random search in the broad search.

(4). When a contact is gained, the searcher decides whether or not to investigate it. If the contact is worth investigating, the searcher immediately stops the broad search and begins the investigation of the contact. It is assumed that the searcher does not gain any new contact during the investigating search.

(5). If the contact is true, the investigating time X until finishing the investigation of the i-class contact has a c.d.f. $H_i(x)$. ($H_i(x)$ is called the investigating function.) $H_i(x)$ is assumed to be continuous, differentiable and the p.d.f. of the investigating time X is denoted by $h_i(x)$; $h_i(x) = dH_i(x)/dx$.

(6). If the contact is false (with probability $(1-p_i)$ for the i-class contact), no positive information telling it to be false is obtained by the investigating search. This comes from our assumption of the false contacts caused by system noise. If the investigating search is continued long without detection, the contact turns out to be suspicious and the hope for detection of the target becomes dimmer. Therefore, the searcher should stop the investigating search at some appropriate time and return to the broad search to get a new contact. Let $z_i(t)$ be the stopping time of investigation for the i-class contact obtained at t (sometimes, $z_i(t)$ is abbreviated as z if any confusion is not expected). $\{z_i(t), i=1,2,\cdots,n, 0\leqq z_i(t)\leqq t, 0\leqq t\leqq T\}$ is called the investigating search plan. The optimal value of $z_i(t)$ is denoted by $z_i^*(t)$. When the investigating search is stopped unsuccessfully, the broad search is resumed immediately.

(7). Let $P(t)$ be the detection probability of the target when the optimal investigating plan $\{z_i^*(t)\}$ is used. The measure of effectiveness of the investigating search is assumed to be the detection probability of the target until T; $P(T)$.

Under the assumptions mentioned above, the problem is formulated as follows. Applying the dynamic programming formulation, we have the next relation by considering possible events in $[t, t+\Delta t]$.

$$P(t+\Delta t) = (1-\Sigma_i \lambda_i \Delta t)P(t) + \Sigma_i \lambda_i \Delta t\{\max_{0\leqq z_i(t)\leqq t} G_i(z_i(t))\}, \tag{1}$$

where $G_i(z_i(t))$ is the conditional detection probability when the i-class contact is gained at t and the investigating time $z_i(t)$ is adopted. $G_i(z_i(t))$ is given by

$$G_i(z_i(t)) = p_iH_i(z_i(t)) + (1-p_iH_i(z_i(t)))P(t-z_i(t)). \quad (2)$$

Rearranging Eq. (1) and considering the limit when $\Delta t \to 0$, we have the next differential equation.

$$\frac{dP(t)}{dt} = \Sigma_i \lambda_i \{G_i(z_i^*(t))-P(t)\}, \quad (3)$$

where

$$G_i(z_i^*(t)) = \max_{0 \leqq z_i(t) \leqq t} G_i(z_i(t)). \quad (4)$$

We define $dP(t)/dt$ at the boundary points as the right differential at $t = 0$ and the left differential at $t = T$.

The boundary condition of Eq. (3) is given by

$$P(0) = 0. \quad (5)$$

The formulation of our problem has been completed by Eqs. (2)～(5).

3. The Optimal Plan of the Investigating Search

In this section, the optimal stopping time of the investigating search is studied and the properties of the optimal investigating search plan are analyzed. First, to clarify the discussion, we consider a fundamental model with a single class of contacts, and later, we generalize it to the model with multiple classes.

3.1. Optimal plan for single contact class

As shown in Eqs. (2) and (4), maximizing $G_i(z_i(t))$ by $z_i(t)$, $0 \leqq z_i(t) \leqq t$, the optimal investigating time $z_i^*(t)$ is determined in every contact of i-class gained at t. Hence, first of all, we must examine properties of $G_i(z_i(t))$. In this paragraph, we deal with the case of single contact class, and therefore, we omit the suffix i in p, H, G, etc. since $i = 1$ always. The derivative $df(x)/dx$ is denoted by $f'(x)$. The fundamental properties of $G(z(t))$ are given by the next lemma.

[Lemma 1] *$G(z)$ given t has the following properties.*

(1). *The boundary values of $G(z)$ at $z = 0$ and $z = t$ are given by*

$$G(0) = P(t) \ (\geqq 0),$$
$$G(t) = pH(t) \ (\geqq 0).$$

(2). *The boundary values of $G'(z)$ at $z = 0$ and $z = t$ are given by*

$$G'(0) = ph(0)(1-P(t)) - P'(t) \geqq (1-P(t))(ph(0)-\lambda), \quad (6)$$
$$G'(t) = ph(t) \geqq 0. \quad \square$$

(Proof)

(1). Since $H(0) = P(0) = 0$, we have $G(0) = P(t)$ and $G(t) = pH(t)$ from Eq. (2). Therefore, obviously $G(0) \geqq 0$ and $G(t) \geqq 0$.

(2). Since $H(z)$ is continuous and differentiable, $G(z)$ given by Eq. (2) is also differentiable. Differentiating Eq. (2) and substituting Eq. (3), we have

$$\begin{aligned} G'(z) &= ph(z)(1-P(t-z))-(1-pH(z))P'(t-z) \quad (7)\\ &= ph(z)(1-P(t-z))-(1-pH(z))\lambda\{G(z^*(t-z))-P(t-z)\}\\ &\geqq (1-P(t-z))\{ph(z)-\lambda(1-pH(z))\}. \end{aligned}$$

Since $H(0) = 0$, $P(0) = 0$ and $P'(0) = 0$ by Eqs. (2), (3) and (5), $G'(0)$ and $G'(t)$ are obtained as Eq. (6). (q.e.d.)

The curve of $G(z(t))$ is complicated according to the function $H(z)$ and has several extreme points in general. Here, we define $z^0(t)$ as the point z that gives the largest local maximum of $G(z)$ in $0 < z < t$ if it exists, and $z^0(t) = 0$ if $G(z)$ has not any local maximum. The optimal investigating plan is given by the next theorem.

[Theorem 1]

(1). *If $z^0(t)$ exists, it is a solution of the equation:*

$$\frac{ph(z)}{1-pH(z)} = \frac{P'(t-z)}{1-P(t-z)} . \qquad (8)$$

(2). $z^*(t) \neq 0$, *that is, $z^*(t)$ is either $z^0(t)$ or t, and we have*

$$G(z^*(t)) = \max\{G(z^0(t)),\ pH(t)\} > P(t). \quad \square$$

(Proof)

(1). Setting $G'(z) = 0$ in Eq.(7), we obtain Eq.(8) easily and the solutions of Eq.(8) give extreme points of $G(z)$. Hence, if there exists local maximums, $z^0(t)$ is one of the solutions of Eq.(8).

(2). $z^*(t) \neq 0$ is proved by the reductive absurdity. Suppose $z^*(t) = 0$. Then $P(t) = 0$ for any t is deduced from Eqs.(2) and (3), and it contradicts the definition of the optimal $z^*(t)$. Therefore, $z^*(t)$ is either $z^0(t)$ or t and $G(z^*(t)) > G(0) = P(t)$. This implies that the contact should be investigated in any case and it is natural intuitively since we have only one class of contact. (q.e.d.)

In general, Eq.(8) may have plural solutions. In this case, since Eq.(8) means $G'(z) = 0$ and it is only a necessary condition for $z^0(t)$, we must check the signs of $G'(z \pm \Delta z)$ and examine which solutions give the local maximum. Then, evaluating $G(z)$ of the local maximums, we determine z^0 which gives the largest local maximum. Here, we define two time points t^{00} and t^0 as follows.

$$t^{00} = \min\{t \mid \exists z^0(t)\}, \qquad (9)$$

$$t^0 = \min\{t \mid \exists z^0(t) \text{ and } G(z^0(t)) \geqq G(t)\}. \qquad (10)$$

t^{00} is the minimum time point t such that $G(z(t))$ has a local maximum and t^0 is the minimum time point that the largest local maximum $G(z^0(t))$ becomes larger than the value of the upper boundary t; $G(t)$. Since the condition of t^0 given by Eq.(10) is more restrictive than that of t^{00} given by Eq.(9), $t^{00} \leqq t^0$ is concluded, and the next lemma is described.

[Lemma 2] *Suppose there exist the positive t^{00} and t^0 defined by Eqs.(9) and (10) respectively, then $t^{00} \leqq t^0$.* □

Using t^{00} and t^0 defined above, the optimal investigating search plan is given by the next corollary.

[Corollary 1-1]

$$\text{If } 0 < t \leqq t^0,\ z^*(t) = t, \quad \text{and if } t^0 < t,\ z^*(t) = z^0(t), \qquad (11)$$

where $z^0(t)$ is a solution of Eq.(8). □

We omit the proof of this corollary since it is obvious by Theorem 1-(2) and the definition of t^0 given by Eq.(10).

In many cases of real world applications, usually, the marginal investigation rate; $ph(z)/(1-pH(z))$ is a decreasing function of z and the contact rate λ in the broad search is smaller than the investigation rate. Hence, usually the maximum value of the marginal investigation rate; $ph(0)/(1-pH(0)) = ph(0) > \lambda$ is valid. Hereafter, we examine this case in detail. To illustrate the feature of $G(z(t))$, we calculate $G(z(t))$ by Eqs.(2)~(5) assuming $H(t) = 1-\exp(-t)$, $\lambda = 0.2$ and $p = 0.5$

(in this case, $G'(0) > 0$ by Eq.(6) since $ph(0) > \lambda$) varying $t = 1, 2, \cdots, 10$. Fig.1 shows the curves of $G(z)$ given t. As shown in Fig.1-A, if the remained search time t is short, $G(z)$ is a strictly increasing function of z in $0 \leqq z \leqq t$ (see the curves of $t = 1, 2$ and 3). In this case, z^* which maximizes $G(z)$ in $0 \leqq z \leqq t$ is obviously at the upper boundary; $z^* = t$. If t increases more, $G(z)$ has a local maximum at z which satisfies $G'(z) = 0$; Eq.(8). This time t is $t^{00} = 3.75$ for the case of Fig.1. Fig.1-B shows the magnified $G(z)$ at the neighborhood of t^{00}. As shown in Fig.1-B, $G(z^0)$ is not maximum globally in $3.75 < t < 3.85$; the global maximum is still given at the upper boundary; $z^* = t$. The search time increases more, the (largest) local maximum $G(z^0)$ grows up rapidly and z^* is shifted to the (largest) local maximum point z^0 from the upper boundary as seen in the case of $t \geqq 3.85$ in Fig.1-B. We have $t^0 = 3.85$ for the case of Fig.1.

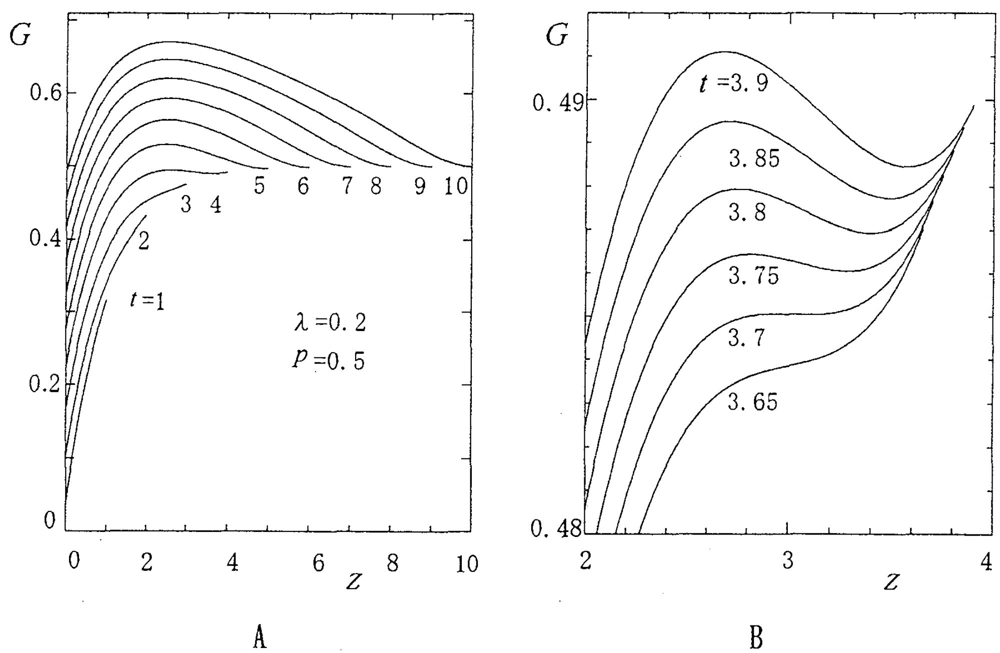


Fig.1. $G(z)$ given t

Properties of $G(z)$ illustrated in Fig.1 are summarized as follows.

[Corollary 1-2] *Suppose $ph(0) > \lambda$, then $G(z)$ has the next properties.*

(1). *$G(z)$ is increasing in the neighborhood of $z = 0$, i.e. $G'(0) > 0$ for any t.*

(2). *Suppose that there exist a positive t^{00} and t^0 defined by Eqs.(9) and (10). Then, if $0 \leqq t < t^{00}$, $G(z)$ is strictly increasing in $z \in [0, t]$, and if $t^{00} \leqq t < t^0$, $G(z)$ has at least a local maximum at z given by Eq.(8) (hence, there exists z^0). If $t^0 \leqq t$, the global maximum of $G(z)$ is obtained at z^0.*

(3). *If $G(z)$ has a local maximum, $G(z)$ also has a local minimum.* □

(Proof)

(1). $G'(0) > 0$ is derived from Eq.(6) by the assumption of Corollary; $ph(0) > \lambda$.

(2). From the assumption; $\exists t^{00} > 0$ and the definition of t^{00} given by Eq.(9), we can conclude $G'(z) > 0$ and $G(z)$ is strictly increasing in $z \in [0, t]$; $0 \leqq t < t^{00}$. If $t^{00} \leqq t < t^0$, there exists at least a local maximum $G(z)$ by the definition of t^{00} at $z(t)$ given by Eq.(8). However, $G(z^0)$ is not the global maximum from the definition of t^0 given by Eq (10). If $t > t^0$, $G(z^0) \geqq G(t)$ is obvious from the definition of t^0.

(3). If $t > t^{00}$, $G(z)$ has a local maximum and $G'(z) < 0$ in $z^0 < z < z^0+\Delta z$. However, since $G'(t) = ph(t) \geq 0$ for the boundary t from Lemma 1-(2), $G'(z)$ changes its sign from - to + in $(z^0, t]$ at least once, and hence, $G(z)$ has a local minimum in this interval. (q.e.d.)

The optimal investigating time $z^*(t)$ varying $t = 1, 2, \cdots, 20$ is shown in Fig.2 for the case of Fig.1: $H(z) = 1-\exp(-z)$, $\lambda = 0.2$ and $p = 0.5$. As stated before, $t^0 = 3.85$ in this case, and as shown in Fig.2, $z^*(t) = t$ if $t \leqq 3.85$ and $z^*(t) = z^0$ if $t > 3.85$.

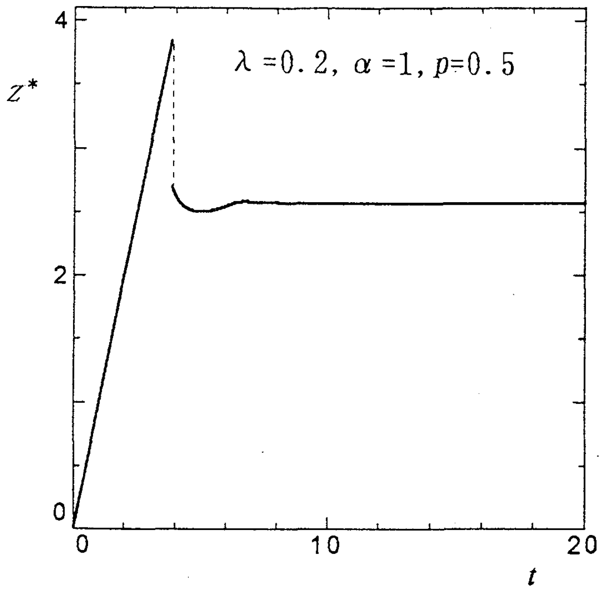


Fig.2. The optimal investigating time z^*

Table 1. z^* and $P(t)$

t	z^*	$P(t)$
2	2.00	0.098
4	2.63	0.222
5	2.51	0.275
6	2.55	0.324
8	2.58	0.413
10	2.57	0.491
20	2.57	0.749

In Fig.2, it should be noted that $z^*(t)$ is constant for large t. This property is presented by the next theorem.

[Theorem 2] *The optimal investigating time z^* converges to a constant value when the search time is sufficiently long*; $t \gg t^0$. □

(Proof) We assume $t \gg t^0$. Substituting Eqs.(2) and (3) into the r.h.s. of Eq.(8), we have the next equation for sufficiently large t and $z^0 \ll t$.

$$\frac{ph(z^0)}{1-pH(z^0)} = \frac{\lambda\{pH(z^0)+(1-pH(z^0))P(t-2z^0)-P(t-z^0)\}}{1-P(t-z^0)}$$
$$\fallingdotseq \lambda p\{H(z^0)-z^0h(z^0)\}.$$

The last equation is derived by substituting the next approximation for $t \gg t^0$ and $z^0 \ll t$ into the numerator of the r.h.s of the first equation.

$$P(t-2z^0) \fallingdotseq P(t-z^0) - P'(t-z^0)z^0$$
$$= P(t-z^0) - \frac{ph(z^0)(1-P(t-z^0))z^0}{1-pH(z^0)}.$$

Therefore, Eq.(8) becomes $ph(z^0)/(1-pH(z^0)) = \lambda p\{H(z^0)-z^0h(z^0)\}$ for large t. Since this equation does not depend on t, the solution z^0 of this equation also does not depend on t and becomes constant approximately if t is sufficiently large. (q.e.d.)

From the proof of Theorem 2, the next corollary is presented.

[Corollary 2-1] *If $H(z)$ is a strictly concave function of z with $h(0) > 0$ and the marginal investigation rate $ph(z)/(1-pH(z))$ is a strictly decreasing function of z, an approximation of $z^0(t)$ for $t \gg t^0$ is given by the unique solution z^0 of the next equation:*

$$\frac{h(z)}{1-pH(z)} = \lambda\{H(z)-zh(z)\}. \qquad (12)\ \square$$

(Proof) The l.h.s. of Eq.(12) is a strictly decreasing function of z from the assumption of the corollary and the r.h.s. is strictly increasing by the concavity of $H(z)$; $h'(z) < 0$. The value of the l.h.s. at $z = 0$ is larger than the r.h.s. since $h(0) > 0$. Therefore, Eq.(12) has always a unique solution z^0 at which $G'(\cdot)$ changes its sign + to −, and z^0 gives an approximation of the optimal. (q.e.d.)
As mentioned later in Section 4, the solution given by Eq.(12) is a fairly good approximation of $z^0(t)$ for $t \gg t^0$. As for $z^0(t)$, the next corollary is obtained.

[Corollary 2-2] *The approximate value $z^0(t)$ for $t \gg t^0$ given by Eq.(12) is a decreasing function of λ.* $\square$

(Proof) Eq.(12) is rearranged as $\lambda = [h(z^0)/(1-pH(z^0))][1/(H(z^0)-z^0h(z^0))]$. In the r.h.s. of this equation, the term in the first bracket is a decreasing function of z^0 from the assumption of Corollary 2-1 and the term in the second bracket is proved to be also a decreasing function by using the concavity of $H(z^0)$. Hence, λ is a decreasing function of z^0. Therefore, if λ increases, z^0 given by this equation is decreases. (q.e.d.)
The main properties of $z^*(t)$ have been elucidated by the theorems mentioned above. Here, we proceed to study the optimal investigating time $z_i^*(t)$ in environment of multiple contacts classes.

3.2. Optimal plan for multiple contacts classes

In this paragraph, the case of multiple contact classes is considered. The contacts are classified into n classes and the reliability of the i-class contact is denoted by p_i, $1>p_1>p_2>\cdots>p_n>0$. Here, we define $z_i^0(t)$ and t_i^0 as follows.

$z_i^0(t)$: the value of z which gives the largest local maximum of $G_i(z)$ in $0 < z < t$ for the i-class contact.

$t_i^0 = \min\{t \mid \exists z_i^0(t)$ and $G_i(z_i^0(t)) \geqq G_i(t) = p_iH_i(t)\}$.

We have the next theorem.

[Theorem 3]

(1). *$z_i^0(t)$ is a solution of the equation:*

$$\frac{p_ih_i(z)}{1-p_iH_i(z)} = \frac{P'(t-z)}{1-P(t-z)}. \qquad (13)$$

(2). *We have*

$$G_i(z_i^*(t)) = \max\{P(t),\ p_iH_i(t),\ G_i(z_i^0(t))\}, \qquad (14)$$

that is, $z_i^*(t)$ is one of $\{0,\ t,\ z_i^0(t)\}$.

(3). *If the search time t is shorter than t_i^0; $t < t_i^0$, then $z_i^*(t) = t$.*

(4). *A sufficient condition for $z_i^*(t) > 0$ is*

$$p_ih_i(0) > \frac{P'(t)}{1-P(t)}. \qquad (15)\ \square$$

(Proof)

(1). Eq.(13) is derived from $G_i'(z_i(t)) = 0$. Since $z_i^0(t)$ gives the largest local maximum by the definition, it satisfies this equation.

(2). The optimal $z_i^*(t)$ is one of the boundary points ($z_i^*(t) = 0$ or t) or the largest local maximum point $z_i^0(t)$, and therefore, Eq.(14) is obvious.

(3). Since $G_i(z_i^0(t)) < p_i H_i(t)$ for $t < t_i^0$ by the definition of t_i^0, $z_i^*(t)$ is either $z_i^*(t) = 0$ or $z_i^*(t) = t$ by Eq.(14). Here, we assume $z_i^*(t) = 0$ for some t_i^0, then there exists a time $t^0 = \mathrm{mim}_i\ t_i^0$ for which $z_i^*(t) = 0$ for all i in $t < t^0$. Hence, we have $P(t) = 0$ for all $t < t^0$ since $G_i(z_i^*(t)) = P(t)$ and $P'(t) = 0$ by Eqs.(2) and (3). This contradicts the definition of the optimal $z_i^*(t)$. Hence, $z_i^*(t) = t$ in $t < t_i^0$ is concluded.

(4). Differentiating Eq.(2) with respect to $z_i(t)$, we have

$$G_i'(z_i(t)) >(\leqq)\ 0 \rightleftarrows \frac{p_i h_i(z)}{1-p_i H_i(z)} >(\leqq) \frac{P'(t-z_i)}{1-P(t-z_i)}\ . \tag{16}$$

(The signs > and ≦ are applied in same order.)

Here, we assume Ineq.(15). We have $G_i'(0) > 0$ from Eq.(16), and to neglect the i-class contact is not optimal since the investigation is effective to increase the detection probability. Hence, $z_i^*(t) > 0$ if Ineq.(15) holds. (q.e.d.)

In the case of multiple contact classes, some ineffective contacts are neglected in the investigation in $t > t^0$. In Section 4, we shall show a numerical example in which the contact with low reliability is neglected. The next corollary is directly derived from Theorems 3-(2), (3) and the definition of t_i^0.

[Corollary 3-1] *Suppose there exists* t_i^0. *If* $0 < t \leqq t_i^0$, $z_i^*(t) = t$. *If* $t > t_i^0$ *and* $G_i(z_i^0(t)) > P(t)$, *then* $z_i^*(t) = z_i^0$, *and if* $t > t_i^0$ and $G_i(z_i^0(t)) \leqq P(t)$, *then* $z_i^*(t) = 0$. □

The meaning of this corollary will be discussed in Section 5.

The relations between the optimal investigating time of the i-class contact and the $(i+1)$-class contact are presented by the next theorem.

[Theorem 4]

(1). *If* $p_i h_i(z) \geqq p_{i+1} h_{i+1}(z)$ *in* $0 \leqq z \leqq t$, *we have the relation*; $t_i^0 \geqq t_{i+1}^0$.

(2). *If* $p_i h_i(z) \geqq p_{i+1} h_{i+1}(z)$ *in* $0 \leqq z \leqq t$, *the optimal investigating times for the contacts* i *and* $i+1$ *classes have the relation*; $z_i^*(t) \geqq z_{i+1}^*(t)$. □

(Proof)

(1). We define $\Delta G(z) = G_i(z) - G_{i+1}(z)$, then we can easily confirm that $\Delta G(z)$ is an increasing function of z if $p_i h_i(z) \geqq p_{i+1} h_{i+1}(z)$ in $z \in [0, t]$. Since $z_i^*(t) \leqq t$ for all t and any i, we have $\Delta G(t) \geqq \Delta G(z_i^*(t))$, and therefore,

$$G_i(t) - G_i(z_i^*(t)) \geqq G_{i+1}(t) - G_{i+1}(z_i^*(t)).$$

From the above, we have the next inequality since $G_{i+1}(z_{i+1}^*(t)) \geqq G_{i+1}(z_i^*(t))$.

$$G_i(t) - G_i(z_i^*(t)) \geqq G_{i+1}(t) - G_{i+1}(z_{i+1}^*(t)).$$

Here, to prove Theorem by the reductive absurdity, we suppose $t_i^0 < t_{i+1}^0$ and consider t such as $t_i^0 < t < t_{i+1}^0$. Then, the r.h.s. of the above inequality is zero since $z_i^*(t) < t$ and $z_{i+1}^*(t) = t$ from Corollary 3-1, and therefore, we have $G_i(t) > G_i(z_i^*(t))$. This contradicts the definition of the optimal $z_i^*(t)$. Therefore, $t_i^0 \geqq t_{i+1}^0$ is concluded.

(2). By the definition of $z_i^*(t)$, the next relations are obvious.

$$G_i(z_i^*(t)) \geqq G_i(z_{i+1}^*(t)) \quad \text{and} \quad G_{i+1}(z_{i+1}^*(t)) \geqq G_{i+1}(z_i^*(t)).$$

From the above, we have

$$G_i(z_i^*(t)) - G_{i+1}(z_i^*(t)) \geqq G_i(z_{i+1}^*(t)) - G_{i+1}(z_{i+1}^*(t)). \qquad (17)$$

On the other hand, $\Delta G(z) = G_i(z)-G_{i+1}(z)$ is a strictly increasing function of z if $p_i h_i(z) \geqq p_{i+1}h_{i+1}(z)$ as mentioned above. Here, we assume $z_i^*(t) < z_{i+1}^*(t)$, then since $\Delta G(z)$ is the increasing function of z, $\Delta G(z_i^*(t)) < \Delta G(z_{i+1}^*(t))$, we have the relation;

$$G_i(z_i^*(t)) - G_{i+1}(z_i^*(t)) < G_i(z_{i+1}^*(t)) - G_{i+1}(z_{i+1}^*(t)).$$

This inequality contradicts Ineq.(17), and therefore, $z_i^*(t) \geqq z_{i+1}^*(t)$ is concluded. (q.e.d.)

[Corollary 4-1] *If the function $H_i(z)$ is common to all contacts; $H_i(z) = H(z)$ for all i, we have relations; $t_i^0 \geqq t_{i+1}^0$ and $z_i^*(t) \geqq z_{i+1}^*(t)$.* □

(Proof) If $H_i(z)$ is common to all contact, $p_i h_i(z) \geqq p_{i+1}h_{i+1}(z)$ in all z, $0 \leqq z \leqq t$, is obvious from the definition of p_i, and therefore, Corollary is established by Theorem 4. (q.e.d.)

[Theorem 5] *Suppose the marginal investigation rate; $p_i h_i(z)/(1-p_i H_i(z))$ is a decreasing function of z. If λ $(=\Sigma_i \lambda_i)$ increases, $z_i^*(t)$ for $t \gg t_i^0$ decreases.* □

(Proof) Let $f_i = \lambda_i/\lambda$, where λ $(=\Sigma_i\lambda_i)$ is the rate of any contact and f_i is the conditional probability of the i-class contact when a contact is gained. Rewriting Eq.(3) by using λ and f_i, we have $P'(t) = \lambda\Sigma_i f_i\{G_i(z_i^*(t))-P(t)\}$, hence, $P'(t)$ (obviously $P'(t) \geqq 0$ and $G_i(z_i^*(t)) \geqq P(t)$) increases as λ increases. Since the r.h.s. of Eq.(13) increases as λ increases, and the solution $z_i^*(t)$ of Eq.(13) decreases because the l.h.s. of Eq.(13) is strictly decreasing function of $z_i^*(t)$ by the assumption of Theorem. (q.e.d.)

4. Numerical Examples

In this section, assuming the exponential investigating function; $H_i(z) = 1-\exp(-\alpha_i z)$, we analyze several numerical examples to see the sensitivity of the system parameters to the optimal investigating plan. Here, taking the mean investigating time of Contact 1 as the unit time, we set $\alpha_1 = 1$. For the sake of convenience for the numerical analysis of this section, we define the contact rate $\lambda = \Sigma_i\lambda_i$ and the conditional probability of the i-class contact $f_i = \lambda_i/\lambda$, then obviously $\lambda_i = f_i\lambda$. Hereafter, we set the primary case, Case 1, and then varying system parameters in Cases 2, 3 and 4, we evaluate the optimal investigating plan to see the sensitivities of each parameter. Here, we set the parameters of Case 1 as

$$\text{Case 1}: n = 3, \quad \{p_i\} = \{0.9,\ 0.5,\ 0.1\},$$
$$\alpha_i = 1 \text{ for all } i, \quad \lambda = \Sigma_i\lambda_i = 0.2,$$
$$\{f_i\} = \{\lambda_i/\lambda\} = \{1/3, 1/3, 1/3\}.$$

The optimal investigating plan of Case 1 is calculated discretely by every $\Delta t = 0.01$ using Eqs.(2),(3),(4) and (5). The solutions, $z_i^*(t)$ and $P(t)$ are presented in Table 2 and $z_i^*(t)$, i=1,2,3, are visualized in Fig.3. As seen in Fig.3, the properties described in Corollary 3-1; if $t \leqq t_i^0$, $z_i^*(t) = t$, and if $t > t_i^0$, $z_i^*(t) = z_i^0$, Corollary 4-1; $t_i^0 \geqq t_{i+1}^0$, and $z_i^*(t) \geqq z_{i+1}^*(t)$, are shown clearly.

Next, to examine the sensitivity of the rate of contacts $\lambda = \Sigma_i\lambda_i$, λ is varied from $\lambda = 0.2$ in Case 1 to $\lambda = 0.1$ in Case 2, whereas the other parameters

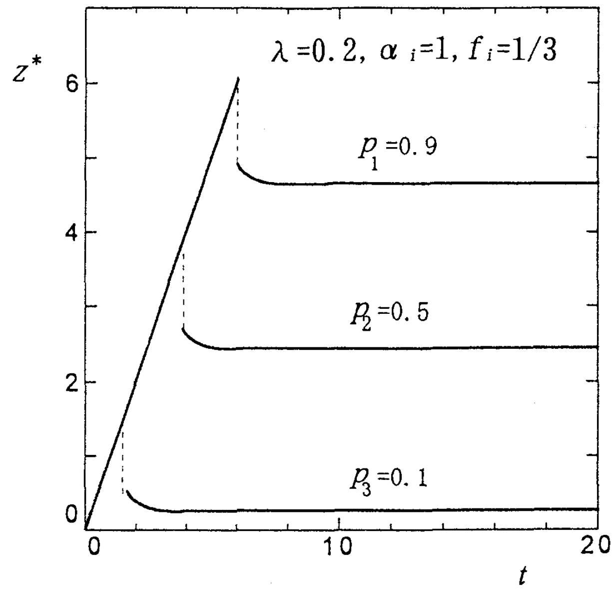


Fig.3. The optimal plan of Case 1

Table 2. $z_i{}^*(t)$ and $P(t)$ of Case 1

i	1	2	3	
$t_i{}^0$	6.05	3.85	1.65	
t	$z_i{}^*(t)$			$P(t)$
2	2.00	2.00	0.38	0.098
4	4.00	2.63	0.25	0.230
5	5.00	2.46	0.25	0.289
6	6.00	2.44	0.25	0.344
8	4.64	2.45	0.26	0.440
10	4.65	2.45	0.26	0.522
20	4.65	2.45	0.26	0.783

are kept the same as Case 1:

Case 2 : λ = 0.1.

Furthermore, to see the sensitivity of λ_i keeping constant λ= 0.2, we evaluate Case 3 varying f_i from $\{f_i\}$ = {1/3,1/3,1/3} in Case 1 to

Case 3 : $\{f_i\}$ = {0.8, 0.1, 0.1}.

The optimal investigating plans of Cases 2 and 3 are evaluated and shown in Table 3 and are visualized in Figs.4 and 5, respectively. By comparing Fig.4 (Case 2) with Fig.3 (Case 1), we can see that the optimal investigating time increases when λ decreases as stated in Theorem 5. In Fig.5 (Case 3), it should be noted that Contact 3 is neglected in the optimal investigating plan for $t > t_3{}^0$. This corresponds to the case: $\{t > t_i{}^0$ and $G_i(z_i{}^0(t)) \leqq P(t)\}$ in Corollary 3-1, and then $z_i{}^*(t)$ = 0.

Table 3 $z_i{}^*(t)$ and $P(t)$ of Cases 2 and 3

Case	Case 2				Case 3			
i	1	2	3		1	2	3	
$t_i{}^0$	6.73	4.53	2.34		5.56	3.37	1.17	
t	$z_i{}^*(t)$			$P(t)$	$z_i{}^*(t)$			$P(t)$
1	1.00	1.00	1.00	0.025	1.00	1.00	1.00	0.075
4	4.00	4.00	0.91	0.130	4.00	1.94	0.00	0.352
5	5.00	3.20	0.90	0.167	5.00	1.86	0.00	0.434
6	6.00	3.11	0.90	0.202	4.18	1.85	0.00	0.505
8	5.32	3.10	0.90	0.268	4.04	1.87	0.00	0.621
10	5.29	3.10	0.90	0.329	4.07	1.87	0.00	0.710
20	5.30	3.10	0.90	0.564	4.07	1.87	0.00	0.924

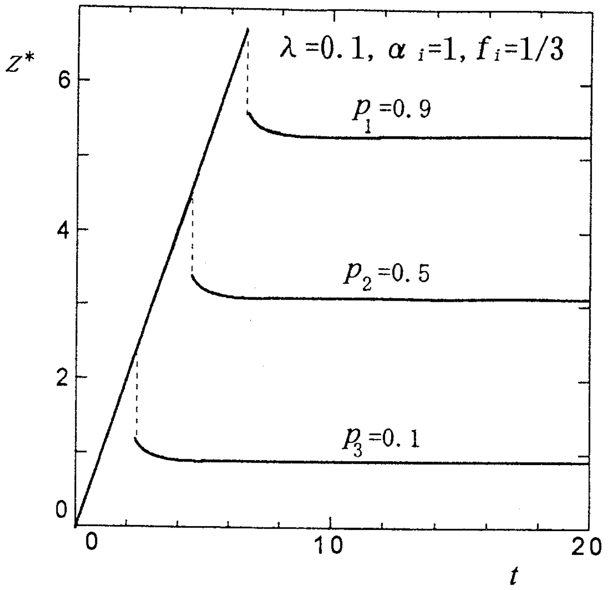


Fig.4. The optimal plan of Case 2

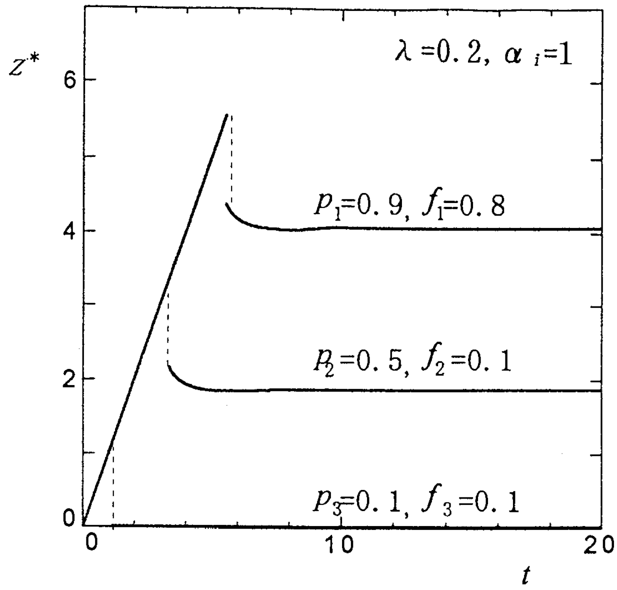
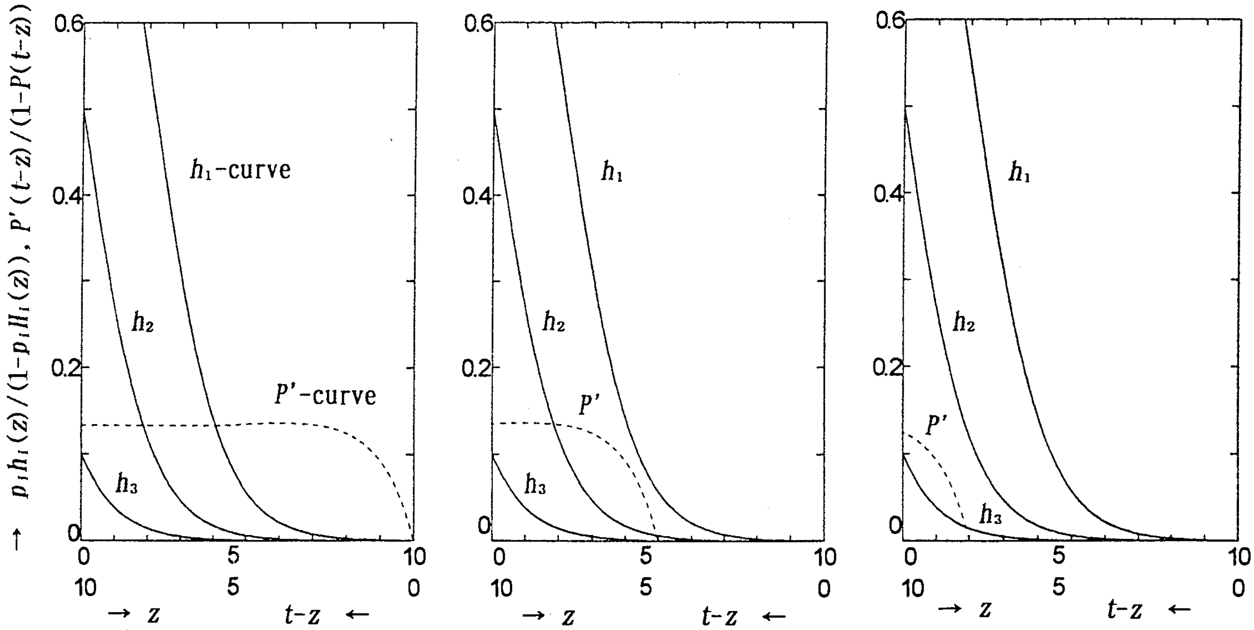


Fig.5. The optimal plan of Case 3

To examine the extreme value of $G_i(z_i(t))$ in detail for Case 3, we evaluate the marginal investigation rate ; $p_i h_i(z_i(t))/\{1-p_i H_i(z_i(t))\}$ (referred to h_i-curve) and the marginal detection rate ; $P'(t-z_i(t))/\{1-P(t-z_i(t))\}$ (referred to P'-curve) at t = 2,5,10 and show them in Fig.6. From Eq.(16), we can see that $G_i'(z) >(\leq) 0 \rightleftarrows h_i$-curve $>(\leq)$ P'-curve. As shown in Fig.6-A, if t = 10, h_i-curves, i = 1,2, have two intersections with P'-curve. We can easily confirm by Eq.(16) that the smaller z_i of the intersection corresponds to the local maximum (the optimal z_i^*) and the larger one does the local minimum. On the other hand, the h_3-curve has



—— : h_i-curve, ········ : P'-curve

A : t = 10 B : t = 5 C : t = 2

Fig.6. The marginal investigating rate and the marginal detection rate

only one intersection with P'-curve at $z \doteqdot 10$ and it gives the local minimum. Hence if $G_3(0) > G_3(t)$, the global maximum is given by $G_i(0)$. In this case, we have $G_3(0) = P(t) = 0.710$ and $G_3(t) = p_3H_3(t) = 0.1$ at $t = 10$ and $z_3{}^*(t) = 0$ is confirmed. Next let's consider the case of $t = 5$ in Fig.6-B. In this case, h_1-curve is always larger than P'-curve in $z \in [0, t]$ and $G_1{}'(z) > 0$ in this interval, hence, $\max_{0 \leq z \leq t} G_1(z^*) = G_1(t)$, namely, $z_1{}^*(5) = 5$. As for h_i-curve, $i = 2, 3$, the intersections of h_i-curve with P'-curve are same as the case of $t = 10$ and $\max_{0 \leq z \leq t} G_2(z)$ is the local maximum and $\max_{0 \leq z \leq t} G_3(z) = G_3(0)$ as stated above. Next, we examine the case of $t = 2$ (Fig.6-C). In this case, h_i-curves, $i = 1, 2$, are always larger than P'-curve in $z \in [0, t]$ and then $\max_{0 \leq z \leq t} G_i(z) = G_i(t), i = 1, 2$, and h_3-curve is the same as Fig.6-A.

Next, we examine Case 4 in which the investigating rates α_i are changed from $\alpha_i = 1$, $i = 1, 2, 3$, in Case 1 to

$$\text{Case 4 : } \alpha_1 = 1 \text{ and } \alpha_2 = \alpha_3 = 0.5.$$

In this case, the Contacts 2 and 3 are time-consuming to investigate compared with Contact 1 (it takes twice longer investigating time of Contacts 2, 3 than that of Contact 1 on the average). The optimal investigating plan is shown in Table 4 and Fig.7. By Comparing Fig.7 (Case 4) with Fig.3 (Case 1), we can see that Contact 3 is neglected in $t > t_3{}^0$ in Case 4, and Contacts 1 and 2 are investigated thoroughly, namely, the investigating time for these contacts are prolonged in Case 4.

The results of Cases 3 and 4 seem to be instructive. The assertions of these Cases are stated as follows. The contacts with poor reliability or low efficiency should be neglected and the investigating effort should be concentrated to the contact with high reliability and efficiency if there is plenty of search time remained, and if the remained time is short, all contacts should be investigated until the end of the search. This assertion may be reasonable intuitively and Theorems are very useful to give a explicit quantitative bases for the decision of the optimal investigating search plan.

Finally, assuming single contact class and the exponential investigating function $H(z) = 1\text{-}\exp(-\alpha z)$, we examine the accuracy of the approximation z^0 given

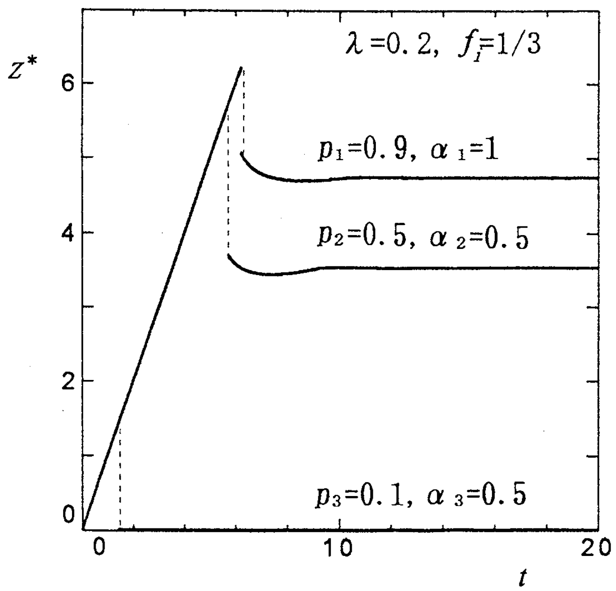


Fig.7. The optimal plan of Case 4

Table 4. $z_i{}^*(t)$ and $P(t)$ of Case 4

i	1	2	3	
$t_i{}^0$	6.23	5.73	1.43	
t	$z_i{}^*(t)$			$P(t)$
1	1.00	1.00	1.00	0.041
4	4.00	4.00	0.00	0.210
5	5.00	5.00	0.00	0.267
6	6.00	3.59	0.00	0.319
8	4.71	3.46	0.00	0.411
10	4.73	3.55	0.00	0.491
20	4.74	3.54	0.00	0.754

by Eq.(12) for $t \gg t_i^0$. Substituting $H(z)$ into Eq.(12), we have the next equation of z^0.

$$\exp(\alpha z^0)-\alpha z^0-1 = \frac{\alpha}{\lambda\{1-p-p\exp(-\alpha z^0)\}} . \quad (18)$$

Applying the Newton method to the above equation, we obtain the solution $z^0 = 2.55$ for the example given in Section 3.1; $\lambda = 0.2$, $\alpha = 1$, $p = 0.5$. In this case, the optimal investigating time $z^*(20) = 2.57$ is given as shown in Table 1. The absolute error of z^0 is 0.02 and the relative error is only 0.8 %. This accuracy of the approximation may be satisfactory for the use of practical applications. However, as mentioned before, the approximation z^0 given by Eq.(12) is derived by assuming the single contact class. If we apply Eq.(12) to the case of the multiple contacts classes, the approximation is not sufficient; we must consider other approximations than this for the multiple contacts classes.

5. Discussions

In this section, we elucidate the meaning of the conditions of the optimal investigating plan and discuss the differences between the optimal plan obtained above and that of the previous studies.

5.1. Interpretation of the conditions for the optimal plan

(1). The assertions of Corollary 1-1 or 3-1 is stated as follows; if the remained search time is short, it should be used exhaustively to investigate the contact on his hand, and if plenty of search time is remained, the investigation should be stopped at some appropriate time. This assertion seems to be reasonable since the searcher cannot expect to gain a new contact in the broad search if the remained time is short, and then he should not stop the investigation to the contact on his hand. On the contrary, if the remained time is long enough to detect a new contact, he should stop the investigation of the suspicious contact and return to the broad search for new contacts. When the optimal investigating search is stopped at $z_i^0(t)$, the optimal condition of z^* given by Eq.(13) is elucidated as follows. The l.h.s. of Eq.(13) is the conditional investigating probability of the contact being examined in unit time at z^* given that it has been investigated until z^* unsuccessfully (called the marginal investigation rate). On the other hand, the r.h.s. of Eq.(13) is the conditional detection probability in unit time when the searcher stops the investigation at z^* and return to the broad search (called the marginal detection rate of the broad search). The optimal stopping time z^* is the balance point of the both marginal values mentioned above. Such property is reasonable for the optimal conditions of the investigating search.

(2). As stated in Corollary 2-2 and Theorem 5, if λ increases, $z^*(t)$ for $t \gg t^0$ decreases. This property seems to be natural since we can expect to contact frequently when λ is large, and then the searcher does not necessary to stick to a contact, and therefore, he should stop the investigation early.

(3). In the case of multiple contacts classes, if the search time $t < t_i^0$, $z_i^*(t) = t$ for any i as stated in Theorem 3-(3), and if p_i is small enough, $z_i^*(t) = 0$ for $t > t_i^0$ as shown in the examples; Cases 3 and 4 in Section 4. This property is deduced from the similar reason discussed above in (1), namely, if the remained time is short, since the searcher cannot expect to gain a new contact in the broad search, he should continue his investigation for any contact even if its reliability is low, on the contrary, if the remained time is long enough to detect a new contact, he should neglect the contact with low reliability and return to the broad search for new contacts.

From the view point of real world applications, we may be able to approximate the optimal plan by such a plan controlled by two values t_i^0 and z_i that $z_i^*(t) = t$ for $t < t_i^0$ and $z_i^*(t) = z_i$ for any $t \geqq t_i^0$ if it is investigated. The validity of this approximation is expected from the numerical examples shown in Section 3.

(4). As for the time t_i^0, if $p_ih_i(t) > p_{i+1}h_{i+1}(t)$, we have the relation; $t_i^0 \geqq t_{i+1}^0$ as stated in Theorem 4-(1), and the optimal investigating time has the relation; $z_i^*(t) \geqq z_{i+1}^*(t)$ by Theorem 4-(2). These properties are the same as mentioned above in (1) and (3); the contact with high reliability should be investigated thoroughly.

5.2. Relations to the optimal plans studied previously

As stated in Section 1, Kisi [3] presented a model of the similar investigating search to our model. Assuming single contact class, a large number of targets and Poisson arrival of the contact in the broad search, he formulated the model with the criterion of the expected number of the target detected in time T. He gave the equation of the optimal investigating time and showed several numerical examples. Let $E(t)$ be the maximum expected number of detected target when time t remains. The necessary condition of the optimal investigating time is given by

$$\frac{ph(z^*(t))}{1-pH(z^*(t))} = E'(t-z^*(t)). \tag{19}$$

Assuming $H(z) = 1-\exp(-\alpha z)$ in Eq.(19), Kisi derived the next equation of $z^*(t)$ for a sufficiently large t.

$$\exp(\alpha z^*(t)) - \alpha z^*(t) - 1 = \frac{\alpha}{\lambda(1-p)}. \tag{20}$$

Eqs.(19) and (20) correspond to Eqs.(8) and (18) in our model. Comparing the r.h.s. of Eq.(18) with Eq.(20), we can see that the optimal investigating time z^0 of our model given by Eq.(18) is larger than that of Kisi's model given by Eq.(20).

Iida [2] studied the optimal investigating search under assumptions of multiple contact classes and unrestricted search time with the criterion of expected time to detection. Let ET be the expected time to detection by the optimal plan. The equation of the optimal investigating time z_i^* is given by

$$p_ih_i(z_i^*) = \frac{1}{ET}.$$

In this case, the optimal investigating plan such as $z_i^*(t) = t$ stated in Corollaries 1-1 and 3-1 does not exist since the search time is not restricted.

6. Concluding Remarks

In this paper, we consider a two-stage search consisting of the broad search and the investigating search for false contacts caused by system noise. We study the optimal stopping time of the investigating search maximizing the detection probability of the target until the end of the search. We derive the conditions of the optimal investigating search plan and give clear interpretations for the conditions. Furthermore, by using the conditions, we elucidate the structure of the optimal plan and explain its behavior when the system parameters are varied.

Acknowledgments

The authors would like to express their sincere thanks to anonymous referees for their useful comments and suggestions to refine this paper.

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