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STRUCTURES OF SUBLATTICES RELATED TO VEINOTT RELATION

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Abstract Let E be a nonempty finite set. H. Narayanan showed a theorem describing that the family $\{\Pi' \mid \Pi' \in P_E, \sum_{X\in\Pi'} f(X) = \min_{\Pi\in P_E} \sum_{X\in\Pi} f(X)\}$ forms a lattice, where f is a submodular function on 2^E and P_E is the set of all partitions of E. On the other hand, L. S. Shapley gave a theorem on a necessary and sufficient condition for a convex game to be decomposable. We give a theorem which is a generalization of these two theorems.

1. Introduction

Let E be a nonempty finite set and $f : 2^E \to \mathbf{R}$ submodular. For the purpose of solving problems in electrical network theory, H. Narayanan [3] studied structure of the set of the partitions $P_E^* = \{\Pi' \mid \Pi' \in P_E, \overline{f}(\Pi') = \min_{\Pi\in P_E} \overline{f}(\Pi)\}$, where P_E is the set of all partitions of E and $\overline{f}(\Pi) = \sum_{X\in\Pi} f(X)$. He showed that P_E^* forms a lattice. On the other hand, L. S. Shapley [4] introduced convex games and studied their properties. One of his results is a necessary and sufficient condition for a convex game to be decomposable.

Our purpose is aimed at generalizing these two results. We pay attention to two following concepts: first, Veinott relation ([5]) which is an order on a set of sublattices, and secondly, properties of sum of a submodular function $f : 2^E \to \mathbf{R}$ over the subpartitions of E. In section 2 we introduce a concept of $\preceq^V$-chain related to Veinott relation and that of V-function as a generalization of such a function $\overline{f}$. In section 3 we give our main result with an additional property obtained from it, and show that the result is a generalization of Narayanan's and Shapley's results.

2. Definitions and preliminaries

Suppose a finite lattice $\mathbf{L} = (L, \vee, \wedge)$ with order $\preceq$ is given. For $X, Y \in 2^L$, if $x \in X$ and $y \in Y$ imply that $x \wedge y \in X$ and $x \vee y \in Y$, then we denote $X \preceq^V Y$.

Theorem 2.1 (Topkis [5]): *The set of all nonempty sublattices of a lattice* $\mathbf{L}$ *is a poset with* $\preceq^V$. □

We call the order $\preceq^V$ *Veinott relation*, which is named after a person introducing the order. Let $\mathbf{L}_1 = (L_1, \vee, \wedge), \mathbf{L}_2 = (L_2, \vee, \wedge), \cdots, \mathbf{L}_n = (L_n, \vee, \wedge)$ be sublattices of $\mathbf{L}$. If $\{L_1, L_2, \cdots, L_n\}$ satisfies

(a.1) $i, j \in \{1, 2, \cdots, n\},\ i \neq j \Longrightarrow L_i \cap L_j = \emptyset$,
(a.2) $i, j \in \{1, 2, \cdots, n\} \Longrightarrow L_i \preceq^V L_j$ or $L_j \preceq^V L_i$,

then we call it a $\preceq^V$*-chain* of L. By the definition, $(\cup_{i=1}^n L_i, \vee, \wedge)$ is a sublattice of $\mathbf{L}$.

Example 1: Let $(L, \vee, \wedge)$ be a finite lattice with order $\preceq$ and $x, y \in L$. Let $X = \{z \mid z \preceq x\}$ and $Y = \{z \mid y \preceq z\}$. If $y \notin X$ then $\{X, Y\}$ is a $\preceq^V$-chain of L. □

Example 2: Let $\mathbf{A} = (A, \vee_A, \wedge_A)$ and $\mathbf{C} = (C, \vee_C, \wedge_C)$ be finite lattices with order $\preceq_A$ and $\preceq_C$ respectively, where C is a chain (i.e., for all distinct $x, y \in C$, either $x \preceq_C y$ or $y \preceq_C x$). We denote the direct product of A and C by $A \times C$. We define $\vee$ and $\wedge$ on $A \times C$ as follows: for any two elements $(a_i, c_i) \in A \times C$ $(i = 1, 2)$, $(a_1, c_1) \vee (a_2, c_2) = (a_1 \vee_A a_2, c_1 \vee_C c_2)$, $(a_1, c_1) \wedge (a_2, c_2) = (a_1 \wedge_A a_2, c_1 \wedge_C c_2)$. Then, for each nonempty subset $\{c_{i_1}, c_{i_2}, \cdots, c_{i_k}\}$ of C and sublattice $\mathbf{B} = (B, \vee_A, \wedge_A)$ of $\mathbf{A}$, $\{B \times \{c_{i_1}\}, B \times \{c_{i_2}\}, \cdots, B \times \{c_{i_k}\}\}$ is a $\preceq^V$-chain of $A \times C$. ◻

Example 3: A *partition* Π' of a nonempty finite set E is a set of nonempty disjoint subsets of E whose union is E. A *subpartition* Π of a set E is a set of nonempty disjoint subsets of E. Thus if $E_1 \subseteq E$ and Π_1 is a partition of E_1, then Π_1 is a subpartition of E. We refer to an element N_i of a subpartition Π as a *block* of Π. The collection of all subpartitions (partitions) of E is denoted by SP_E (P_E). We define a partial order $\preceq^{SP}$ on SP_E by defining $\Pi_2 \preceq^{SP} \Pi_1$ if and only if each block of Π_2 is contained in some block of Π_1. The least (greatest) element of SP_E above (below) Π_1 and Π_2 in the partially ordered set SP_E is denoted by $\Pi_1 \vee \Pi_2$ $(\Pi_1 \wedge \Pi_2)$. We should notice that $\Pi_1 \wedge \Pi_2$ does not always exist for two arbitrary subpartitions Π_1, Π_2. However, by defining $\{\emptyset\} \preceq^{SP} \Pi \in SP_E$ and $\{\emptyset\} \preceq^{SP} \{\emptyset\}$, $SP_E \cup \{\{\emptyset\}\}$ forms a lattice with $\preceq^{SP}$.

Let $\emptyset \neq E_1 \subset E_2 \subset \cdots \subset E_n \subseteq E$. Then $\{P_{E_1}, P_{E_2}, \cdots, P_{E_n}\}$ is a $\preceq^V$-chain of $SP_E \cup \{\{\emptyset\}\}$. ◻

We denote by AL_C the set of all $\preceq^V$-chain of L. Let x_* be the minimal element of L and $L_C \in AL_C$. For a given $L_p \in L_C$ and $x \in L_p$, if $x \neq x_*$ and $Z_x = \{z_1, z_2, \cdots, z_k\} \subseteq L - \{x_*\}$ satisfies

(b.1) $z_1 \vee z_2 \vee \cdots \vee z_k = x$,
(b.2) $i, j \in \{1, 2, \cdots, k\}$, $i \neq j \Longrightarrow z_i \wedge z_j = x_*$,
(b.3) For each z_j, there exists sublattice L^j of L such that $z_j \in L^j \preceq^V L_p$
(Note that L^j need not be contained in L_C),

then we call it a *decomposition* of x. One may notice that $\{x\}$ is a decomposition of x. We define that $\{x_*\}$ is the decomposition of x_*. We denote by D_x the set of all decompositions of x. If $Z_x \in D_x$ and $|Z| \leq |Z_x|$ for all $Z \in D_x$, we call Z_x a *finest decomposition of* x. We denote by FD_x the set of all finest decompositions of x.

Let $V : L \rightarrow \mathbf{R}$ be a function. For any $L_i, L_j \in L_C$, suppose

(c.1) x_*^i is the minimal element of L_i (i.e., $x_*^i \preceq x$ $(x \in L_i)$),
(c.2) $\hat{x} \in L_i$ and $V(\hat{x}) = \min_{x \in L_i} V(x)$,
(c.3) $Z_{\hat{x}} \in FD_{\hat{x}}$,
(c.4) $\hat{y} \in L_j$ and $V(\hat{y}) = \min_{y \in L_j} V(y)$,
(c.5) $L_i \preceq^V L_j$.

The following conditions characterize a special class of functions on L.

Condition 1: If $\hat{z}_h \in Z_{\hat{x}}$, then $V(\hat{y}) + V(\hat{z}_h \vee x_*^i) \geq V(\hat{y} \vee (\hat{z}_h \vee x_*^i)) + V(\hat{y} \wedge (\hat{z}_h \vee x_*^i))$. (Note that $\hat{z}_h \vee x_*^i \in L_i$.)

Condition 2: For a given $\hat{z}_h \in Z_{\hat{x}}$, if $x \in L_i$ and $x \preceq \hat{z}_h \vee x_*^i$, then $V(\hat{z}_h \vee x_*^i) \leq V(x)$.

Condition 3: $V(\hat{y} \wedge \hat{x}) - V(\hat{x}) = \sum_{\hat{z}_h \in Z_{\hat{x}}} \{V(\hat{y} \wedge (\hat{z}_h \vee x_*^i)) - V(\hat{z}_h \vee x_*^i)\}$.

If a function $V : L \rightarrow \mathbf{R}$ satisfies above three conditions for every $Z_{\hat{x}} \in FD_{\hat{x}}$, then we call V a *V-function* with respect to L_C.

Let E be a nonempty finite set. A function $f: 2^E \to \mathbf{R}$ is called a *submodular function* on 2^E if

$$f(X)+f(Y) \geq f(X \cup Y)+f(X \cap Y) \tag{2.1}$$

for all $X, Y \subseteq E$. (If $-f$ is a submodular function, we call f a *supermodular function*.) For a subpartition Π of E, we define $\overline{f}(\Pi) \equiv \sum_{X\in\Pi} f(X)$.

An example of V-function: Let us reconsider Example 3. A subpartition whose blocks are all singletons of $N \subseteq E$ is denoted by $\Pi_0(N)$, i.e., $\Pi_0(N)=\{\{e\} \mid e \in N\}$. We show that $\overline{f}: SP_E \cup \{\{\emptyset\}\} \to \mathbf{R}$ is a V-function with respect to $\{P_{E_1}, P_{E_2}, \cdots, P_{E_n}\}$ satisfying $\emptyset \neq E_1 \subset E_2 \subset \cdots \subset E_n \subseteq E$ when f is a submodular function on 2^E.

Let $1 \leq p, q \leq n$. Note that $\{\emptyset\}$ is the minimal element of $SP_E \cup \{\{\emptyset\}\}$ and $\{\{e\} \mid e \in E_p\}$ is the minimal element of P_{E_p}. Let $\Pi_p=\{N_1, N_2, \cdots, N_r\}$ and Π_q be partitions in P_{E_p}, P_{E_q} such that $\overline{f}(\Pi_p)=\min_{\Pi\in P_{E_p}} \overline{f}(\Pi)$ and $\overline{f}(\Pi_q)=\min_{\Pi\in P_{E_q}} \overline{f}(\Pi)$. We may assume that $\emptyset \neq E_p \subseteq E_q$ without loss of generality. Note here that L_i, L_j in the definition of V-function correspond to P_{E_p}, P_{E_q} in this $\preceq^V$-chain $\{P_{E_1}, P_{E_2}, \cdots, P_{E_n}\}$ and Π_p, Π_q correspond to $\hat{x}$, $\hat{y}$. Then $\{\{N_1\},\{N_2\}, \cdots,\{N_r\}\}$ is the finest decomposition of Π_p. For $i=1,2, \cdots, r$, define $\Pi_{N_i}=\{N_i\} \vee \{\{e\} \mid e \in E_p\}=\{N_i, \Pi_0(E_p-N_i)\}$. (More precisely, if $r=1$, then define $\Pi_{N_1}=\{N_1\}$.) Since f is submodular, an elementary calculation yields the following inequality (cf. Corollary 3.3 of [3]).

$$\overline{f}(\Pi_q)+\overline{f}(\Pi_{N_i}) \geq \overline{f}(\Pi_q \vee \Pi_{N_i})+\overline{f}(\Pi_q \wedge \Pi_{N_i}). \tag{2.2}$$

Hence, $\overline{f}$ satisfies Condition 1.

Secondly, we show that $\overline{f}$ satisfies Condition 2. Let $\Pi' \in P_{E_p}$. For $N_i \in \Pi_p$, $\Pi_{N_i} \succeq \Pi'$ implies that Π' can be described as $\{M_1, M_2, \cdots, M_s, \Pi_0(E_p-N_i)\}$ where $\{M_1, M_2, \cdots, M_s\}$ is a partition of N_i. Let $\widehat{\Pi}=\{M_1, M_2, \cdots, M_s\}$. Since $\overline{f}(\Pi_p)=\min_{\Pi\in P_{E_p}} \overline{f}(\Pi)$, from Lemma 3.1 of [3] we have

$$\overline{f}(\Pi')-\overline{f}(\Pi_{N_i})=\sum_{M_j\in\widehat{\Pi}} f(M_j)-f(N) \geq 0. \tag{2.3}$$

Finally, we show that $\overline{f}$ satisfies Condition 3. From the definition of Π_{N_i} we derive

$$\begin{aligned} \sum_{N_i\in\Pi_p}\left\{\overline{f}(\Pi_q \wedge \Pi_{N_i})-\overline{f}(\Pi_{N_i})\right\} &= \sum_{N_i\in\Pi_p}\left\{\overline{f}(\Pi_q \wedge \{N_i\})-f(N_i)\right\} \\ &= \overline{f}(\Pi_q \wedge \Pi_p)-\overline{f}(\Pi_p). \end{aligned} \tag{2.4}$$

3. Structure of $\preceq^V$-chain related to V-function minimization

In this section $\mathbf{L}=(L, \vee, \wedge)$ is a finite lattice. A function $V: L \to \mathbf{R}$ is a V-function with respect to L_C, where L_C is a $\preceq^V$-chain of L. The main purpose of this section is to give the following theorem.

Theorem 3.1: *Let $L_i^*=\{\hat{x} \mid \hat{x} \in L_i, V(\hat{x})=\min_{x\in L_i} V(x)\}$ for $L_i \in L_C$. Then $L_C^*=\{L_i^* \mid L_i \in L_C\}$ is also a $\preceq^V$-chain of L.*

Proof: It is simple to show that L_C^* satisfies condition (a.1). We establish that L_C^* satisfies condition (a.2) by using the similar argument what Narayanan proved Theorem 3.5 of [3]. (The fact that L_i^* is a sublattice of $\mathbf{L}$ is obtained simultaneously.) Let x_*^i be the minimal element of L_i and $L_i, L_j \in L_C$. We assume $L_i \preceq^V L_j$ without loss of generality. First, we state the following claim for emphasis.

Claim: Suppose $\hat{x} \in L_i^*$ and $\hat{y} \in L_j^*$. For $\hat{z}_h \in Z_{\hat{x}} \in FD_{\hat{x}}$, we have

$$V(\hat{y} \vee (\hat{z}_h \vee x_*^i)) = \min_{y \in L_j} V(y), \tag{3.1}$$

$$V(\hat{y} \wedge (\hat{z}_h \vee x_*^i)) = V(\hat{z}_h \vee x_*^i). \tag{3.2}$$

(Proof of Claim) From $L_i \preceq^V L_j$ and $V(\hat{y}) = \min_{y \in L_j} V(y)$, we have

$$V(\hat{y}) \leq V(\hat{y} \vee (\hat{z}_h \vee x_*^i)). \tag{3.3}$$

Moreover, from $L_i \preceq^V L_j$ and Condition 2, we obtain

$$V(\hat{z}_h \vee x_*^i) \leq V(\hat{y} \wedge (\hat{z}_h \vee x_*^i)). \tag{3.4}$$

Therefore, from Condition 1, (3.3) and (3.4) we have (3.1) and (3.2).
(The end of the proof of Claim)

Let $Z_{\hat{x}} = \{\hat{z}_1, \hat{z}_2, \cdots, \hat{z}_k\}$. Repeating the application of (3.1) with $(\hat{y} \vee (\hat{z}_1 \vee x_*^i) \vee \cdots \vee (\hat{z}_h \vee x_*^i))$ and $\hat{z}_{h+1} \vee x_*^i$ for $h = 1$ to $k-1$, we have

$$V(\hat{y} \vee \hat{x}) = V(\hat{y} \vee (\hat{z}_1 \vee x_*^i) \vee \cdots \vee (\hat{z}_k \vee x_*^i)) = \min_{y \in L_j} V(y). \tag{3.5}$$

Moreover, from (3.2) for $\hat{z}_h \in Z_{\hat{x}}$ and Condition 3, we have $V(\hat{y} \wedge \hat{x}) = V(\hat{x})$. These imply that $\hat{y} \vee \hat{x} \in L_j^*$ and $\hat{y} \wedge \hat{x} \in L_i^*$. □

Let E be a nonempty finite set and $f : 2^E \to \mathbf{R}$ submodular. From Theorem 3.1 and the example of V-function in the previous section we have the following theorem.

Theorem 3.2 (cf. [2]): *Let $\phi \neq E_1 \subseteq E_2 \subseteq \cdots \subseteq E_n \subseteq E$. Let Π_i be a partition of E_i and Π_j a partition of E_j for $1 \leq i \leq j \leq n$. If $\overline{f}(\Pi_i) = \min_{\Pi \in P_{E_i}} \overline{f}(\Pi)$ and $\overline{f}(\Pi_j) = \min_{\Pi \in P_{E_j}} \overline{f}(\Pi)$, then $\overline{f}(\Pi_i \vee \Pi_j) = \min_{\Pi \in P_{E_j}} \overline{f}(\Pi)$ and $\overline{f}(\Pi_i \wedge \Pi_j) = \min_{\Pi \in P_{E_i}} \overline{f}(\Pi)$.* □

By setting $E_i = E_j = E$ in Theorem 3.2 we obtain the following theorem.

Theorem 3.3 (Narayanan [3]): *Let Π_1 and Π_2 be two partitions of E. If $\overline{f}(\Pi_1) = \overline{f}(\Pi_2) = \min_{\Pi \in P_E} \overline{f}(\Pi)$, then $\overline{f}(\Pi_1 \vee \Pi_2) = \overline{f}(\Pi_1 \wedge \Pi_2) = \min_{\Pi \in P_E} \overline{f}(\Pi)$.* □

We consider the case of $f(\emptyset) = 0$. In the remaining part of this section, we define $P_\emptyset = \{\{\emptyset\}\}$ and $\overline{f}(\{\emptyset\}) = f(\emptyset)$. Since f is submodular, for each $S \subseteq E$, we have $\overline{f}(\{S\}) = \min_{\Pi \in P_S} \overline{f}(\Pi)$. Hence, from Theorem 3.2, we have the following corollary.

Corollary 3.4: *If $\Pi \in P_{E_2}$ satisfies $\overline{f}(\Pi) = f(E_2)$ then $\overline{f}(\Pi \wedge \{E_1\}) = \overline{f}(\{E_1\})$ for each $E_1 \subseteq E_2$.* □

The following property immediately follows from the above corollary.

Property 3.5 (Girlich, Schneidereit and Zaporozhets [1]): *Let Y be a nonempty subset of E such that $f(Y) = f(Y_1) + f(Y_2)$, where Y_1 and Y_2 are nonempty disjoint sets, $Y = Y_1 \cup Y_2$. Then $f(X) = f(X \cap Y_1) + f(X \cap Y_2)$ for each $X \subseteq Y$.* □

Finally, we refer to a decomposable convex game. Let $E = \{1, 2, \cdots, n\}$ be a set of players and (E, v) be an n-person game in characteristic function form where v is a real-valued function on 2^E and $v(\emptyset) = 0$. A game (E, v) is convex if v is a supermodular function. Let $\Pi = \{X_1, X_2, \cdots, X_p\} \in P_E$ and $p \geq 2$. (E, v) is said to be *decomposable* (with respect to Π) if for each $S \subseteq E$, $v(S) = \sum_{X_i \in \Pi} v(X_i \cap S)$. Let $f = -v$. From Corollary 3.4 we derive the following Shapley's Theorem.

Theorem 3.6 (Shapley [4]): *A convex game (E, v) is decomposable if and only if $f(E) = f(X_1) + \cdots + f(X_p)$ holds for some partition $\{X_1, X_2, \cdots, X_p\}$ of E into $p \geq 2$ nonempty subsets, where $f = -v$.* □

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