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EIGENVALUE EXPRESSION FOR A BATCH MARKOVIAN ARRIVAL PROCESS

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Abstract Consider a batch Markovian arrival process (*BMAP*) as the counting process of an underlying Markov process representing the state of environment. Such a process is useful for representing correlated inputs for example. They are used both as a modeling tool and as a theoretical device to represent and approximate superposition of input processes and complex large systems. Our objective is to consider the first and second moments of the counting process depending on time and state. Assuming that the probability generating functions of batch size are analytic, and that eigenvalues of the infinitesimal generator are simple, we derive an analytic diagonalization for the matrix generating function of the counting process. Our main result gives the time-dependent form of the first and second factorial moments of the counting process, which is represented by eigenvalues and eigenvectors of the matrix generating function of the batch size.

1. Introduction

Consider a versatile Markovian point process, introduced by Neuts [6] and [7]. It is assumed that the change of an environment is formulated as a finite state Markov process and that the batch input rate of the point process depends on the state or the transition of the state of the underlying Markov process. Such processes are useful for representing correlated inputs, for example, the phase-type (PH) renewal process, the Markov modulated Poisson process ($MMPP$) and the phase transition arrival process. They are used both as a modeling tool and as a theoretical device to represent and approximate superposition of input processes and complex large systems. Under similar assumptions, the asymptotic normality of the counting process was discussed in [2]. Ramaswami [9] analyzed the single server queue with this process as input. Later, Lucantoni [4] reformulated the same process and called it a batch Markovian arrival process ($BMAP$). We follow in this paper his formulation and notation.

Our main result gives time-dependent forms of the first and second factorial moments of the $BMAP$ represented by eigenvalues and eigenvectors of the matrix generating function of batch size, only using ordinary exponential functions. In [6] and [7] the time-dependent mean arrivals and the equilibrium variance were derived. Narayana and Neuts in [5] studied the asymptotic and algorithmic properties of the first two moment matrices. In this paper, however, we show the time-dependent second factorial moment in non-equilibrium case. This expression can be used for the comparison of mean queue lengths between an $M/G/1$ queue and a $BMAP/G/1$ queue in [8].

In Section 2 we define the underlying Markov process and the $BMAP$ matrix generating function of the counting process. Our first assumption is the analyticity of the matrix generating function of batch size. In Section 3 we introduce our second assumption that the infinitesimal generator of the Markov process has simple eigenvalues. Under these assumptions the eigenvalues of the matrix generating function are analytic. We prove that each eigenvalue has an associated analytic eigenvector. In Section 4 we discuss properties of these analytic eigenvalues and eigenvectors. Then we obtain the algorithm to calculate the explicit form of coefficients of eigenvalues and eigenvectors up to the second order terms in a power series expansion. These coefficients are derived by a generalized form of the fundamental matrix in [3]. In Section 5 we obtain the first two factorial moments of the counting process depending on time.

2. Batch Markovian arrival process

Let D be the $m \times m$ irreducible transition rate matrix of an underlying Markov process of the batch Markovian arrival process. We adopt basic notations and definitions with respect to $BMAP$ given by [4]. Let the generating function of D_k be

$$D(z) = \sum_{k=0}^{\infty} D_k z^k. \tag{2.1}$$

We see also that

$$D(1) = \sum_{k=0}^{\infty} D_k = D. \tag{2.2}$$

As we need, later in this paper, to use the power series expansion of $D(z)$ at $z = 1$, we put the following assumption.

Assumption 1 *$D(z)$ is analytic in a neighbourhood of $z = 1$.*

This assumption is trivially satisfied if the batch sizes have a finite maximum.

Let $J(t)$ be the state of the Markov process D at time t and $N(t)$ be the total number of arrivals in $(0, t]$. We consider the probability

$$P_{ij}(\nu, t) = P\{N(t) = \nu, J(t) = j \mid N(0) = 0, J(0) = i\}.$$

Let $P(\nu, t)$ be the matrix form of $P_{ij}(\nu, t)$. It is then immediate that the matrix generating function

$$\tilde{P}(z, t) = \sum_{\nu=0}^{\infty} P(\nu, t) z^{\nu} \quad for |z| \leq 1,$$

satisfies the differential equation

$$\frac{\partial}{\partial t} \tilde{P}(z, t) = \tilde{P}(z, t) D(z), \quad \tilde{P}(z, 0) = I.$$

Thus it follows that

$$\tilde{P}(z, t) = exp\{D(z)t\}. \tag{2.3}$$

Now, by definition, the probability distribution of $N(t)$, under the initial condition $J(0) = i$, is the sequence of the i-th element of $\{P(k, t)\boldsymbol{e}\}_{k=0,\cdots}$ and its generating function is the i-th element of $\tilde{P}(z, t)\boldsymbol{e}$. Therefore, in order to obtain a concrete representation of the moments of the counting process $N(t)$, we need the power series expansion of $exp\{D(z)t\}$ in a neighbourhood of $z = 1$. This will be done in Section 3 and 4 by diagonalizing the generating function $D(z)$.

3. Eigenvector

We adopt in this paper the following notation: for any column vector $\boldsymbol{x} = (x_1, \cdots, x_m)^t$ we denote by $\Delta(\boldsymbol{x})$ the diagonal matrix whose diagonal elements are $(x_1, \cdots, x_m)$. The notation $A \circ B$ stands for the entrywise product of matrices A and B. Moreover we write:

$$\mathbf{o} = (0, \cdots, 0)^t, \qquad \boldsymbol{e} = (1, \cdots, 1)^t, \quad \mathbf{1} = (1, 0, \cdots, 0)^t$$

and the identity matrix is denoted by I.

In view of the diagonalization of $D(z)$, we put the following assumption on the transition rate matrix D of the underlying Markov process.

Assumption 2 *All eigenvalues of D are simple.*

There are m distinct eigenvalues and we define its first eigenvalue as $\lambda_1 = 0$ and other $m-1$ eigenvalues as λ_i $(i = 2, \cdots, m)$ with $Re(\lambda_i) < 0$. We put $\boldsymbol{\lambda} = (\lambda_1, \cdots, \lambda_m)^t$ as a column vector and $\Lambda = \Delta(\boldsymbol{\lambda})$ as a diagonal matrix. Then D is diagonalized as follows. Let $\boldsymbol{a}_i$ $(i = 1, \cdots, m)$ be a row eigenvector associated with λ_i such that $\boldsymbol{a}_i D = \lambda_i \boldsymbol{a}_i$. Especially we choose $\boldsymbol{a}_1$ to be the stationary probability vector $\boldsymbol{\pi}$ of D satisfying $\boldsymbol{\pi} > \mathbf{o}, \boldsymbol{\pi} D = \mathbf{o}$ and $\boldsymbol{\pi e} = 1$. Then

$$A = \begin{pmatrix} \boldsymbol{a}_1 \\ \vdots \\ \boldsymbol{a}_m \end{pmatrix} \tag{3.1}$$

is a nonsingular $m \times m$ matrix. If we put $A^{-1} = H = (\boldsymbol{h}_1, \cdots, \boldsymbol{h}_m)$, $\boldsymbol{h}_i$ $(i = 1, \cdots, m)$ is a column eigenvector associated with λ_i such that $D\boldsymbol{h}_i = \lambda_i \boldsymbol{h}_i$ satisfying

$$\boldsymbol{a}_i \boldsymbol{h}_j = \begin{cases} 0 & (i \neq j) \\ 1 & (i = j). \end{cases}$$

Especially $\boldsymbol{h}_1 = \boldsymbol{e}$ and

$$A\boldsymbol{e} = \mathbf{1}. \tag{3.2}$$

The diagonalization of D is given by

$$ADA^{-1} = \Lambda. \tag{3.3}$$

From Assumption 1, the characteristic equation $| D(z) - \lambda I | = 0$ is polynomial in λ of degree m and all its coefficients are analytic in a neighbourhood of $z = 1$. From the Assumption 2, this equation has m simple roots $\lambda = \lambda_1, \cdots, \lambda_m$ at $z = 1$. It follows from the theorem on implicit functions ([1] for example) that we have, for each $i (i = 1, \cdots, m)$ and in a neighborhood of $z = 1$, a unique eigenvalue $\omega_i(z)$ of $D(z)$ which is analytic in a neighborhood of $z = 1$ such that $\omega_i(1) = \lambda_i$. We may therefore write it in the following form:

$$\omega_i(z) = \lambda_i + (z-1)\mu_i + \frac{1}{2}(z-1)^2 \rho_i + o\left((z-1)^2\right). \tag{3.4}$$

For each eigenvalue $\omega_i(z)$ of $D(z)$, there exists a row eigenvector $\boldsymbol{u}_i(z)$, analytic at $z = 1$ and satisfying $\boldsymbol{u}_i(1) = \boldsymbol{a}_i$, so that it may be written in the form

$$\boldsymbol{u}_i(z) = \boldsymbol{a}_i + (z-1)\boldsymbol{b}_i + \frac{1}{2}(z-1)^2 \boldsymbol{c}_i + o_v\left((z-1)^2\right), \tag{3.5}$$

where $o_v((z-1)^2)$ is an m dimensional row vector of $o((z-1)^2)$. The existence of such an eigenvector results directly from the following lemma, the proof of which will be given in the Appendix.

Lemma 1 *Let $M(z)$ be an $m \times m$ matrix whose entries are analytic at $z = 0$. If an analytic function $\lambda(z)$ is an eigenvalue of $M(z)$ in a neighborhood of $z = 0$ and if $\lambda(0)$ is a simple eigenvalue of $M(0)$ and $\boldsymbol{x}_0$ its column eigenvector, then the followings hold:*

(i) There exists an eigenvector $\boldsymbol{x}(z)$ of $M(z)$, associated with the eigenvalue $\lambda(z)$, analytic at $z = 0$ and satisfying $\boldsymbol{x}(0) = \boldsymbol{x}_0$.

(ii) A vector $\boldsymbol{y}(z)$, analytic at $z = 0$ and satisfying $\boldsymbol{y}(0) = \boldsymbol{x}_0$, is an eigenvector of $M(z)$ associated with $\lambda(z)$ if and only if $\boldsymbol{y}(z) = c(z)\boldsymbol{x}(z)$ with an analytic function $c(z)$ at $z = 0$ satisfying $c(0) = 1$.

This lemma shows not only the existence of an analytic eigenvector $\boldsymbol{u}_i(z)$ of $D(z)$ associated with $\omega_i(z)$, but also the general form of analytic eigenvectors satisfying the same condition as $\boldsymbol{u}_i(z)$, i.e. $\boldsymbol{u}_i(1) = \boldsymbol{a}_i$. In fact, the condition $\boldsymbol{u}_i(1) = \boldsymbol{a}_i$ does not determine $\boldsymbol{u}_i(z)$ uniquely. When a particular choice of $\boldsymbol{u}_i(z)$, denoted by $\boldsymbol{u}_i^0(z)$, is given, then the general form of $\boldsymbol{u}_i(z)$ is given by

(3.6) $$\boldsymbol{u}_i(z) = f_i(z)\boldsymbol{u}_i^0(z),$$

where $f_i(z)$ is an arbitrary analytic function at $z = 1$ satisfying $f_i(1) = 1$.

4. Coefficients of expansion

In this section we will derive an algorithm to calculate coefficients at least up to second order terms in (3.4) and (3.5). As A was defined in (3.1), let $U(z)$, B and C be matrices of $\boldsymbol{u}_i(z), \boldsymbol{b}_i$ and $\boldsymbol{c}_i$, respectively. A matrix form of (3.5) is

(4.1) $$U(z) = A + (z-1)B + \frac{1}{2}(z-1)^2 C + o_M((z-1)^2).$$

Since $\boldsymbol{u}_i(z)$ is the eigenvector associated with $\omega_i(z)$, we get

(4.2) $$U(z)D(z) = \Omega(z)U(z),$$

where $\Omega(z)$ is the diagonal matrix with $\omega_i(z)$ as diagonal elements.

From (2.1), (3.4) and (3.5) it follows that comparing coefficients of $(z-1)^k$ ($k = 0, 1$ and 2) we have

(4.3) $$\boldsymbol{a}_i(D - \lambda_i I) = \mathbf{o}^t,$$

(4.4) $$\boldsymbol{b}_i(D - \lambda_i I) = -\boldsymbol{a}_i(D'(1) - \mu_i I),$$

(4.5) $$\boldsymbol{c}_i(D - \lambda_i I) = -\boldsymbol{a}_i(D''(1) - \rho_i I) - 2\boldsymbol{b}_i(D'(1) - \mu_i I).$$

Equation (4.3) implies that $\boldsymbol{a}_i$ is an eigenvector of D associated with λ_i. In (4.4) and (4.5) $(D - \lambda_i I)$ is a singular matrix of rank $m-1$. In order to solve the linear equations we prepare the following matrices.

Lemma 2 *There exists $Z_i \stackrel{\text{def}}{=} (D - \lambda_i I - \boldsymbol{h}_i \boldsymbol{a}_i)^{-1}$.*

Proof. Let $\boldsymbol{x}_i$ be an unknown vector of a homogeneous linear equation

(4.6) $$\boldsymbol{x}_i(D - \lambda_i I - \boldsymbol{h}_i \boldsymbol{a}_i) = \mathbf{o}^t.$$

Multiplying the equation by $\boldsymbol{h}_i$ and using the fact that $\lambda_i \boldsymbol{h}_i = D\boldsymbol{h}_i$ and $\boldsymbol{a}_i \boldsymbol{h}_i = 1$, we get $\boldsymbol{x}_i \boldsymbol{h}_i = 0$. We conclude $\lambda_i \boldsymbol{x}_i = \boldsymbol{x}_i D$, which implies that $\boldsymbol{x}_i$ is an eigenvector of D associated with λ_i. Then the general solution of $\boldsymbol{x}_i$ is given by $\boldsymbol{x}_i = c\boldsymbol{a}_i$, where c is a constant. Substituting it into (4.6)

$$\mathbf{o}^t = c\boldsymbol{a}_i \boldsymbol{h}_i \boldsymbol{a}_i = c\boldsymbol{a}_i.$$

It follows from $c = 0$ that $\boldsymbol{x}_i = 0$. We conclude that $(D - \lambda_i I - \boldsymbol{h}_i \boldsymbol{a}_i)$ is nonsingular. □

Lemma 3 *We have*

$$Z_i \boldsymbol{h}_j = \begin{cases} -\boldsymbol{h}_j & (i = j) \\ \frac{1}{\lambda_j - \lambda_i} \boldsymbol{h}_j & (i \neq j). \end{cases}$$

$$\boldsymbol{a}_j Z_i = \begin{cases} -\boldsymbol{a}_j & (i = j) \\ \frac{1}{\lambda_j - \lambda_i} \boldsymbol{a}_j & (i \neq j). \end{cases}$$

Proof. For the first equation if $i \neq j$, then

$$\begin{aligned} Z_i^{-1} \boldsymbol{h}_j &= (D - \lambda_i I - \boldsymbol{h}_i \boldsymbol{a}_i) \boldsymbol{h}_j \\ &= (\lambda_j - \lambda_i) \boldsymbol{h}_j. \end{aligned}$$

If $i = j$, then $Z_i \boldsymbol{h}_i = -\boldsymbol{h}_i$. Similarly, we get the second equation. □

Since $\boldsymbol{\pi} = \boldsymbol{a}_1$ and $\boldsymbol{e} = \boldsymbol{h}_1$, we define a matrix $\Pi = \boldsymbol{e}\boldsymbol{\pi} = \boldsymbol{h}_1 \boldsymbol{a}_1$. It follows from $(\Pi - D)^{-1} = -Z_1$ that

$$\boldsymbol{a}_i (\Pi - D)^{-1} = \begin{cases} \boldsymbol{a}_1 & (i = 1) \\ -\frac{\boldsymbol{a}_i}{\lambda_i} & (i \neq 1). \end{cases} \tag{4.7}$$

Lemma 4 *Let $\boldsymbol{f}_i$ be an m-dimensional row vector. The linear equation $\boldsymbol{x}_i(D - \lambda_i I) = \boldsymbol{f}_i$ has a solution, if and only if $\boldsymbol{f}_i \boldsymbol{h}_i = 0$. And if the linear equation has a solution, its general solution is given by*

$$\boldsymbol{x}_i = \boldsymbol{f}_i Z_i + s_i \boldsymbol{a}_i,$$

where $s_i = \boldsymbol{x}_i \boldsymbol{h}_i$ is an arbitrary constant.

Proof. Multiplying the equation by $\boldsymbol{h}_i$, it is necessary that $\boldsymbol{f}_i \boldsymbol{h}_i = 0$. Since the rank of $(D - \lambda_i I)$ is $m - 1$, the first assertion of this lemma is obvious. From the definition of Z_i, we have

$$D - \lambda_i I = (I - \boldsymbol{h}_i \boldsymbol{a}_i) Z_i^{-1},$$

which implies immediately that

$$\boldsymbol{f}_i Z_i = \boldsymbol{x}_i (D - \lambda_i I) Z_i = \boldsymbol{x}_i (I - \boldsymbol{h}_i \boldsymbol{a}_i)$$

and

$$\boldsymbol{x}_i = \boldsymbol{f}_i Z_i + \boldsymbol{x}_i \boldsymbol{h}_i \boldsymbol{a}_i.$$

□

In the previous section the existence of analytic eigenvalues and eigenvectors are proved, so we will need an algorithm to calculate them explicitly, at least up to the second order terms in (3.4) and (3.5). For the leading terms λ_i and $\boldsymbol{a}_i$ it is already done in Section 3. Thus our next objective is to obtain an algebraic procedure to determine μ_i and ρ_i in (3.4) on the one hand, $\boldsymbol{b}_i$ and $\boldsymbol{c}_i$ in (3.5) on the other.

Lemma 5 *Let $\boldsymbol{u}_i(z) = \boldsymbol{a}_i + (z-1)\boldsymbol{b}_i + \frac{1}{2}(z-1)^2 \boldsymbol{c}_i + o_v((z-1)^2)$ be an eigenvector of $D(z)$, analytic at $z = 1$ and associated with the eigenvalue*

$$\omega_i(z) = \lambda_i + (z-1)\mu_i + \frac{(z-1)^2}{2} \rho_i + o\left((z-1)^2\right).$$

Then μ_i and ρ_i are determined by

$$\mu_i = \boldsymbol{a}_i D'(1) \boldsymbol{h}_i \tag{4.8}$$

$$\rho_i = \boldsymbol{a}_i D''(1) \boldsymbol{h}_i + 2\boldsymbol{b}_i (D'(1) - \mu_i I) \boldsymbol{h}_i. \tag{4.9}$$

and $\boldsymbol{b}_i$ *and* $\boldsymbol{c}_i$ *satisfy*

$$\begin{aligned} (4.10) \qquad \boldsymbol{b}_i &= -\boldsymbol{a}_i(D'(1) - \mu_i I)Z_i + s_i^{(1)}\boldsymbol{a}_i \\ (4.11) \qquad \boldsymbol{c}_i &= -\boldsymbol{a}_i(D''(1) - \rho_i I)Z_i - 2\boldsymbol{b}_i(D'(1) - \mu_i I)Z_i + s_i^{(2)}\boldsymbol{a}_i, \end{aligned}$$

where $s_i^{(1)}$ *and* $s_i^{(2)}$ *are certain constants.*

Conversely, for any choice of constants $s_i^{(1)}$ *and* $s_i^{(2)}$, $\boldsymbol{b}_i$ *and* $\boldsymbol{c}_i$ *obtained by (4.10) and (4.11) are coefficients of the first and second order terms respectively of an analytic eigenvector* $\boldsymbol{u}_i(z)$ *at* $z = 1$, *associated with* $\omega_i(z)$ *such that (3.5) holds.*

Proof. Putting (3.4) and (3.5) into the equation $\boldsymbol{u}_i(z)D(z) = \omega_i(z)\boldsymbol{u}_i(z)$ and comparing the coefficients of the first and the second order terms on both sides, we get readily

$$\begin{aligned} (4.12) \qquad \boldsymbol{b}_i(D - \lambda_i I) &= -\boldsymbol{a}_i(D'(1) - \mu_i I), \\ (4.13) \qquad \boldsymbol{c}_i(D - \lambda_i I) &= -\boldsymbol{a}_i(D''(1) - \rho_i I) - 2\boldsymbol{b}_i(D'(1) - \mu_i I). \end{aligned}$$

Applying Lemma 4 to (4.12), we get $\boldsymbol{a}_i(D'(1) - \mu_i I)\boldsymbol{h}_i = 0$, which leads to (4.8) and (4.10). Using the same discussion for (4.13) we see:

$$\boldsymbol{a}_i(D''(1) - \rho_i I)\boldsymbol{h}_i = -2\boldsymbol{b}_i(D'(1) - \mu_i I)\boldsymbol{h}_i,$$

which implies (4.9), independently on the constant $s_i^{(1)}$. We may then apply Lemma 4 once more to (4.13) and obtain (4.11).

To prove the converse part of this lemma, let

$$\boldsymbol{u}_i^0 = \boldsymbol{a}_i + (z-1)\boldsymbol{b}_i^0 + \frac{1}{2}(z-1)^2\boldsymbol{c}_i^0 + o_v\left((z-1)^2\right)$$

be an analytic eigenvector of $D(z)$ associated with $\omega_i(z)$. By what we have just proved, we can write

$$\begin{aligned} \boldsymbol{b}_i^0 &= -\boldsymbol{a}_i(D'(1) - \mu_i I)Z_i + s_{0i}^{(1)}\boldsymbol{a}_i, \\ \boldsymbol{c}_i^0 &= -\boldsymbol{a}_i(D''(1) - \rho_i I)Z_i - 2\boldsymbol{b}_i^0(D'(1) - \mu_i I)Z_i + s_{0i}^{(2)}\boldsymbol{a}_i, \end{aligned}$$

where $s_{0i}^{(1)}$ and $s_{0i}^{(2)}$ are appropriate constants. Let $\boldsymbol{b}_i$ and $\boldsymbol{c}_i$ be defined by (4.10) and (4.11) with an arbitrary choice of constants $s_i^{(1)}$ and $s_i^{(2)}$ respectively. Then we have

$$\begin{aligned} (4.14) \qquad \boldsymbol{b}_i &= \boldsymbol{b}_i^0 \ + (s_i^{(1)} - s_{0i}^{(1)})\boldsymbol{a}_i, \\ \boldsymbol{c}_i &= \boldsymbol{c}_i^0 \ + 2(s_i^{(1)} - s_{0i}^{(1)})\boldsymbol{b}_i^0 \\ (4.15) \qquad & \qquad + [(s_i^{(2)} - s_{0i}^{(2)}) - 2s_{0i}^{(1)}(s_i^{(1)} - s_{0i}^{(1)})]\boldsymbol{a}_i. \end{aligned}$$

Let $\boldsymbol{u}_i(z)$ be defined by $\boldsymbol{u}_i(z) = f_i(z)\boldsymbol{u}_i^0(z)$, where

$$\begin{aligned} f_i(z) &= 1 + (z-1)(s_i^{(1)} - s_{0i}^{(1)}) \\ & \quad + \frac{(z-1)^2}{2}[(s_i^{(2)} - s_{0i}^{(2)}) - 2s_{0i}^{(1)}(s_i^{(1)} - s_{0i}^{(1)})], \end{aligned}$$

is analytic at $z = 1$ satisfying $f_i(1) = 1$. Then by what is remarked right after Lemma 1, we see readily that $\boldsymbol{u}_i(z)$, too, is an analytic eigenvector of $D(z)$ at $z = 1$ associated with $\omega_i(z)$. Direct calculus using (4.14) and (4.15) reveals that $\boldsymbol{u}_i(z)$ satisfies (3.5) with $\boldsymbol{b}_i$ and $\boldsymbol{c}_i$ given above as coefficients of the first and the second order terms respectively. □

Without loss of generality, we may and will suppose $s_i^{(1)} = s_i^{(2)} = 0$ in (4.10) and (4.11). We summarize as

Lemma 6 *The following satisfy (4.4) and (4.5):*

$$\mu_i = \boldsymbol{a}_i D'(1)\boldsymbol{h}_i,$$

$$\boldsymbol{b}_i = -\boldsymbol{a}_i(D'(1) - \mu_i I)Z_i,$$

$$\rho_i = \boldsymbol{a}_i D''(1)\boldsymbol{h}_i + 2\boldsymbol{b}_i(D'(1) - \mu_i I)\boldsymbol{h}_i,$$

$$\boldsymbol{c}_i = -\boldsymbol{a}_i(D''(1) - \rho_i I)Z_i - 2\boldsymbol{b}_i(D'(1) - \mu_i I)Z_i.$$

In the next section it will be discussed that under the stationary condition, the average arrival rate per unit time is $\mu^* = \mu_1 = \boldsymbol{\pi} D'(1)\boldsymbol{e}$. And we have

$$\begin{aligned} \rho_1 &= \boldsymbol{\pi} D''(1)\boldsymbol{e} + 2\boldsymbol{b}_1(D'(1) - \mu_1 I)\boldsymbol{e} \\ &= \boldsymbol{\pi} D''(1)\boldsymbol{e} - 2\boldsymbol{\pi} D'(1)Z_1 D'(1)\boldsymbol{e} - 2(\mu^*)^2. \end{aligned} \tag{4.16}$$

If we put $\beta_i = \boldsymbol{b}_i\boldsymbol{e}$ and $\gamma_i = \boldsymbol{c}_i\boldsymbol{e}$, from Lemma 3, we have

$$\begin{aligned} \beta_1 &= -\boldsymbol{\pi}(D'(1) - \mu_1 I)Z_1\boldsymbol{e} = 0, \\ \beta_i &= \boldsymbol{a}_i D'(1)\boldsymbol{e}/\lambda_i, \quad (i \neq 1) \\ \gamma_1 &= \boldsymbol{\pi} D''(1)\boldsymbol{e} - \rho_1 + 2\boldsymbol{b}_1(D'(1) - \mu_1)\boldsymbol{e} = 0, \\ \gamma_i &= \{\boldsymbol{a}_i D''(1)\boldsymbol{e} + 2\boldsymbol{b}_i D'(1)\boldsymbol{e} - 2\mu_i\beta_i\}/\lambda_i \quad (i \neq 1). \end{aligned} \tag{4.17}$$

We will denote them by $\boldsymbol{\beta} = (\beta_1, \cdots, \beta_m)^t$ and $\boldsymbol{\gamma} = (\gamma_1, \cdots, \gamma_m)^t$.

Lemma 7 *We have*

$$\begin{aligned} [I - exp(Dt)][\Pi - D]^{-1}D'(1)\boldsymbol{e} &= -A^{-1}(I - \Delta(e^{\boldsymbol{\lambda}t}))B\boldsymbol{e} \\ &= -\sum_{i=2}^{m} \boldsymbol{h}_i\boldsymbol{a}_i D'(1)\boldsymbol{e}(1 - e^{\lambda_i t})/\lambda_i. \end{aligned}$$

Proof. Since $D = A^{-1}\Delta(\boldsymbol{\lambda})A$, we easily have

$$[I - exp(Dt)][\Pi - D]^{-1}D'(1)\boldsymbol{e} = A^{-1}(I - \Delta(e^{\boldsymbol{\lambda}t}))A[\Pi - D]^{-1}D'(1)\boldsymbol{e}.$$

Using (4.7), the i^{th} element of $A[\Pi - D]^{-1}D'(1)\boldsymbol{e}$ yields

$$\boldsymbol{a}_i[\Pi - D]^{-1}D'(1)\boldsymbol{e} = \begin{cases} \mu_1 & (i = 1) \\ -\boldsymbol{a}_i D'(1)\boldsymbol{e}/\lambda_i & (i \neq 1). \end{cases}$$

Thus $A[\Pi - D]^{-1}D'(1)\boldsymbol{e}$ is equal to $-\boldsymbol{\beta}$ except the first element. On the other hand from $\lambda_1 = 0, I - \Delta(e^{\boldsymbol{\lambda}t})$ is a diagonal matrix whose $(1,1)$ element is 0, thus proving the lemma. □

5. First and second factorial moments

In this section we derive the first and the second factorial moments of *BMAP*. It follows from (3.4) that

$$e^{\omega_i(z)t} = e^{\lambda_i t}(1 + (z-1)\mu_i t + \frac{1}{2}(z-1)^2(\rho_i t + \mu_i^2 t^2) + o((z-1)^2)). \tag{5.1}$$

We define as $\boldsymbol{\mu} = (\mu_1, \cdots, \mu_m)^t$, $\boldsymbol{\mu}^{(2)} = (\mu_1^2, \cdots, \mu_m^2)^t$ and $\boldsymbol{\rho} = (\rho_1, \cdots, \rho_m)^t$. Since $\Omega(z)$ is a diagonal matrix, the matrix form of (5.1) is

$$\begin{aligned} \exp\{\Omega(z)t\} &= \Delta(e^{\boldsymbol{\lambda}t}) \times \Delta(\boldsymbol{e} + (z-1)\boldsymbol{\mu}t \\ &+ \frac{1}{2}(z-1)^2(\boldsymbol{\rho}t + \boldsymbol{\mu}^{(2)}t^2)) + o_V((z-1)^2). \end{aligned} \tag{5.2}$$

Since $U(z)$ is expanded as (4.1) and $U(1) = A$ is nonsingular, $U(z)^{-1}$ is also analytic and can be expanded as

$$\begin{aligned} U(z)^{-1} &= A^{-1} - (z-1)A^{-1}BA^{-1} \\ &+ (z-1)^2\{A^{-1}BA^{-1}BA^{-1} - \frac{1}{2}A^{-1}CA^{-1}\} + o_M((z-1)^2). \end{aligned} \tag{5.3}$$

Since $\exp\{D(z)t\}$ is an analytic function of z, each element of $\tilde{P}(z,t)\boldsymbol{e}$ is also analytic in z. The i th element of $\tilde{P}(z,t)\boldsymbol{e}$ is the generating function of the counting process $N(t)$ when the initial state is i.

From (2.3), we have

$$\tilde{P}(z,t)\boldsymbol{e} = U(z)^{-1}\exp\{\Omega(z)t\}U(z)\boldsymbol{e}. \tag{5.4}$$

Using (4.1), (5.2) and (5.3) , we can expand (5.4) into

$$\tilde{P}(z,t)\boldsymbol{e} = \boldsymbol{x}_0(t) + (z-1)\boldsymbol{x}_1(t) + \frac{1}{2}(z-1)^2\boldsymbol{x}_2(t) + o_V((z-1)^2). \tag{5.5}$$

The coefficients $\boldsymbol{x}_0(t)$, $\boldsymbol{x}_1(t)$ and $\boldsymbol{x}_2(t)$ are the 0^{th}, the first and the second factorial moment vectors of the counting process $N(t)$ during $(0,t)$, respectively. For $\boldsymbol{x}_0(t)$ it follows from (3.2) and $\lambda_1 = 0$ that we have

$$\begin{aligned} \boldsymbol{x}_0(t) &= A^{-1}\Delta(e^{\boldsymbol{\lambda}t})A\boldsymbol{e} \\ &= A^{-1}\Delta(e^{\boldsymbol{\lambda}t})\mathbf{1} \\ &= A^{-1}\mathbf{1} \\ &= \boldsymbol{e}. \end{aligned} \tag{5.6}$$

For $\boldsymbol{x}_1(t)$, we have the following result.

Theorem 1 *The time-dependent form of the mean vector is*

$$\boldsymbol{x}_1(t) = t\mu^*\boldsymbol{e} - \sum_{i=2}^{m}\boldsymbol{h}_i\boldsymbol{a}_iD'(1)\boldsymbol{e}(1-e^{\lambda_i t})/\lambda_i. \tag{5.7}$$

Proof. Comparing the coefficient of $(z-1)$ in (5.4) we have

$$\begin{aligned} \boldsymbol{x}_1(t) = &-A^{-1}BA^{-1}\Delta(e^{\boldsymbol{\lambda}t})A\boldsymbol{e} \\ &+ tA^{-1}\Delta(e^{\boldsymbol{\lambda}t})\Delta(\boldsymbol{\mu})A\boldsymbol{e} \\ &+ A^{-1}\Delta(e^{\boldsymbol{\lambda}t})B\boldsymbol{e}. \end{aligned}$$

Just as in (5.7), it follows from Lemma 7 that

$$\begin{aligned} \boldsymbol{x}_1(t) &= t\mu^*\boldsymbol{e} - A^{-1}(I - \Delta(e^{\boldsymbol{\lambda}t}))B\boldsymbol{e} \\ &= t\mu^*\boldsymbol{e} - \sum_{i=2}^{m}\boldsymbol{h}_i\boldsymbol{a}_iD'(1)\boldsymbol{e}(1-e^{\lambda_i t})/\lambda_i. \end{aligned}$$

The proof is completed. □

From Lemma 7, another expression of $\boldsymbol{x}_1(t)$ is

$$\boldsymbol{x}_1(t) = \mu^*t\boldsymbol{e} + [I - exp(Dt)]\,[\Pi - D]^{-1}\,D'(1)\boldsymbol{e}$$

which was given by Neuts [6]. If we define the equilibrium mean as $x_1(t) = \boldsymbol{\pi x}_1(t)$, it follows from $\boldsymbol{\pi h}_i = 0$ $(i \geq 2)$ that for all $t \geq 0$

$$x_1(t) = t\mu^*.$$

Since $e^{\lambda_i t}$ in the right hand side of (5.7) tends to zero as $t \to \infty$, the linear asymptote of $\boldsymbol{x}_1(t)$ is given by

$$\lim_{t\to\infty}(\boldsymbol{x}_1(t) - t\mu^*\boldsymbol{e}) = -\sum_{i=2}^{m}\boldsymbol{h}_i\boldsymbol{a}_i D'(1)\boldsymbol{e}/\lambda_i.$$

Next we will obtain the time-dependent form of the second factorial moment.

Theorem 2

$$\begin{aligned}\boldsymbol{x}_2(t) &= t^2\mu^{*2}\boldsymbol{e} + t\rho_1\boldsymbol{e} - 2t\mu^*\sum_{i=2}^{m}\boldsymbol{h}_i\boldsymbol{a}_i D'(1)\boldsymbol{e}/\lambda_i \\ &+ 2t\sum_{i=2}^{m}\boldsymbol{h}_i\boldsymbol{a}_i D'(1)\boldsymbol{e}\mu_i e^{\lambda_i t}/\lambda_i + 2A^{-1}B\sum_{i=2}^{m}\boldsymbol{h}_i\boldsymbol{a}_i D'(1)\boldsymbol{e}(1 - e^{\lambda_i t})/\lambda_i \\ &- \sum_{i=2}^{m}\boldsymbol{h}_i\gamma_i(1 - e^{\lambda_i t}).\end{aligned} \tag{5.8}$$

Proof. The coefficient vector of $\frac{1}{2}(z-1)^2$ is given by

$$\begin{aligned}\boldsymbol{x}_2(t) &= (2A^{-1}BA^{-1}BA^{-1} - A^{-1}CA^{-1})\Delta(e^{\boldsymbol{\lambda}t})A\boldsymbol{e} \\ &\quad - 2A^{-1}BA^{-1}\Delta(e^{\boldsymbol{\lambda}t})B\boldsymbol{e} - 2tA^{-1}BA^{-1}\Delta(e^{\boldsymbol{\lambda}t})\Delta(\boldsymbol{\mu})A\boldsymbol{e} \\ &\quad + A^{-1}\Delta(e^{\boldsymbol{\lambda}t})(C + 2t\Delta(\boldsymbol{\mu})B + \Delta(\boldsymbol{\rho}t + \boldsymbol{\mu}^{(2)}t^2)A)\boldsymbol{e}.\end{aligned}$$

In a similar manner to Theorem 1, we have

$$\begin{aligned}\boldsymbol{x}_2(t) &= 2A^{-1}BA^{-1}(I - \Delta(e^{\boldsymbol{\lambda}t}))B\boldsymbol{e} - A^{-1}((I - \Delta(e^{\boldsymbol{\lambda}t}))C\boldsymbol{e} \\ &\quad + 2tA^{-1}\Delta(e^{\boldsymbol{\lambda}t})\Delta(\boldsymbol{\mu})B\boldsymbol{e} - 2t\mu^*A^{-1}B\boldsymbol{e} + (\rho_1 t + \mu^{*2}t^2)\boldsymbol{e} \\ &= t^2\mu^{*2}\boldsymbol{e} + t\rho_1\boldsymbol{e} - 2t\mu^*A^{-1}B\boldsymbol{e} \\ &\quad + 2tA^{-1}\Delta(e^{\boldsymbol{\lambda}t})\Delta(\boldsymbol{\mu})B\boldsymbol{e} + 2A^{-1}B\sum_{i=2}^{m}\frac{1 - e^{\lambda_i t}}{\lambda_i}\boldsymbol{h}_i\boldsymbol{a}_i D'(1)\boldsymbol{e} \\ &\quad - \sum_{i=2}^{m}\boldsymbol{h}_i\gamma_i(1 - e^{\lambda_i t}).\end{aligned}$$

□

If we define the equilibrium second factorial moment as $x_2(t) = \boldsymbol{\pi x}_2(t)$, using the property $\boldsymbol{\pi h}_i = 0$ $(i \geq 2)$ again, we have for all $t \geq 0$

$$x_2(t) = t^2\mu^{*2} + t\rho_1 + 2\boldsymbol{\pi}A^{-1}B\sum_{i=2}^{m}\boldsymbol{h}_i\boldsymbol{a}_i D'(1)\boldsymbol{e}(1 - e^{\lambda_i t})/\lambda_i.$$

Using the fact that $\boldsymbol{\pi}A^{-1}B = \boldsymbol{\pi}(D'(1) - \mu^* I)[\Pi - D]^{-1}$ we have that from (4.16) and Lemma 7,

$$\begin{aligned}x_2(t) &= t^2\mu^{*2} + t(\boldsymbol{\pi}D''(1)\boldsymbol{e} - 2\boldsymbol{\pi}D'(1)Z_1D'(1)\boldsymbol{e} - 2(\mu^*)^2) \\ &- 2\boldsymbol{\pi}D'(1)[I - exp(Dt)][\Pi - D]^{-2}D'(1)\boldsymbol{e}.\end{aligned}$$

This equation was given in [7] page 285-286. Lastly we will discuss the limiting property of $\boldsymbol{x}_2(t)$. From (5.8) we have as $t \to \infty$,

$$\begin{aligned}\lim_{t\to\infty} & \left(\boldsymbol{x}_2(t) - t^2\mu^{*2}\boldsymbol{e} - t(\rho_1\boldsymbol{e} - 2\mu^* \sum_{i=2}^{m} \boldsymbol{h}_i\boldsymbol{a}_i D'(1)\boldsymbol{e}/\lambda_i)\right) \\ = & \ 2A^{-1}B\sum_{i=2}^{m} \boldsymbol{h}_i\boldsymbol{a}_i D'(1)\boldsymbol{e}/\lambda_i - \sum_{i=2}^{m} \boldsymbol{h}_i\gamma_i.\end{aligned}$$

6. Conclusion

In this paper we constantly supposed that the transition rate matrix has only simple eigenvalues. Although this assumption is not a severe restricton in applications, we could not remove it as our arguments essentially depend on it. However, compared to the results formerly obtained in [5] and [6], our formulae given in Theorem 1 and Theorem 2 have two merits. (i) Matrix exponential functions are replaced by ordinary exponentials. (ii) With respect to the second fuctorial moment, time-dependent exact formula is obtained.

Since our formulae contain many coefficients and parameters, we summerize here the computational algorithm.

Step 1 To calculate eigenvalues λ_i and eigenvectors $\boldsymbol{a}_i, \boldsymbol{h}_i$.

Step 2 To calculate $\mu_i, \boldsymbol{b}_i, \rho_i$ and $\boldsymbol{c}_i$ by using Lemma 6.

Step 3 To calculate γ_i by using (4.17).

Step 4 To calculate $\boldsymbol{x}_1(t)$ and $\boldsymbol{x}_2(t)$ by using Theorem 1 and Theorem 2.

7. Appendix

Proof of the Lemma 1.

We have only to solve the linear equation $(M(z) - \lambda(z)I)\boldsymbol{x} = \mathbf{o}$ in $\boldsymbol{x}$, applying to the coefficient matrix $M(z) - \lambda(z)I$, step by step, the classical sweeping out method in such a way that, when we put $z = 0$, it is precisely what we usually do to compute an eigenvector of $M(0)$ associated with $\lambda(0)$. Most of the main operations, i.e. swapping of rows or columns, addition of one row to another and multiplication of one row by an analytic function at $z = 0$, obviously maintain the analyticity of the entries of the matrix. The only operation which might require our particular attention is division of a row by one of the non-zero entries. At this point, we have to notice that such a division by the (i, j) element $m_{ij}(z)$ occurs only when $m_{ij}(0) \neq 0$ and that this means precisely the analyticity of $m_{ij}(z)^{-1}$ at $z = 0$. Thus the sweeping out procedure continues, keeping all the elements of the matrix analytic, until the matrix takes the form

$$\begin{pmatrix} 1 & 0 & 0 & \cdot & 0 & v_1(z) \\ 0 & 1 & 0 & \cdot & 0 & v_2(z) \\ \vdots & \vdots & \vdots & \cdot & \vdots & \vdots \\ \vdots & \vdots & \vdots & \cdot & 1 & v_{m-1}(z) \\ 0 & 0 & 0 & \cdot & 0 & v_m(z) \end{pmatrix} \tag{7.1}$$

as the simplicity of the eigenvalue $\lambda(0)$ of $M(0)$ implies that $M(0) - \lambda(0)I$ is of rank $m - 1$. But in a neighborhood of $z = 0$, the rank of $M(z) - \lambda(z)I$ must be less than or equal to $m - 1$, because $\lambda(z)$ is an eigenvalue of $M(z)$. Therefore the rank of $M(z) - \lambda(z)I$ (and also that of the matrix (7.1)) is constantly equal to $m - 1$ and $v_m(z) = 0$ in a neighborhood of $z = 0$. We suppose, without any loss of generality, that the matrix (7.1) is attained without swapping columns. Then a solution to the equation $(M(z) - \lambda(z)I)\boldsymbol{x} = \mathbf{o}$ is given by

$$(v_1(z), \ldots, v_{m-1}(z), -1)^t,$$

which is obviously analytic in a neighborhood of $z = 0$. As its value at $z = 0$ is an eigenvector of $M(0)$ associated with $\lambda(0)$ which is a simple eigenvalue, there must be a constant $c \neq 0$ such that

$$\boldsymbol{x}_0 = c(v_1(0), \ldots, v_{m-1}(0), -1)^t.$$

It follows that $\boldsymbol{x}(z) = c(v_1(z), \ldots, v_{m-1}(z), -1)^t$ satisfies every requirement of statement (i).

To prove (ii), we begin by supposing $\boldsymbol{y}(z)$ to be analytic at $z = 0$ satisfying $\boldsymbol{y}(0) = \boldsymbol{x}_0$ and $M(z)\boldsymbol{y}(z) = \lambda(z)\boldsymbol{y}(z)$ in a neighborhood of $z = 0$. As we have just shown above, the rank of $M(z) - \lambda(z)I$ is $m - 1$. It implies that $\boldsymbol{y}(z)$ must be proportional to $\boldsymbol{x}(z)$ at each z, so that we have a function $c(z)$ such that

$$\begin{aligned} \boldsymbol{y}(z) &= c(z)\boldsymbol{x}(z) \\ &= c(z)c(v_1(z), \ldots, v_{m-1}(z), -1)^t. \end{aligned}$$

As $y(z)$ is analytic at $z = 0$ and $c \neq 0$, $c(z)$ is easily seen to be analytic at $z = 0$. From $\boldsymbol{y}(0) = \boldsymbol{x}(0) = \boldsymbol{x}_0 \neq \mathbf{o}$, we obtain immediately that $c(0) = 1$. The converse part of the assertion being obvious, this completes the proof of this lemma. □

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References

[1] Cartan, H. *Théorie élémemtaire des fonctions analytiques d'une ou plusieurs variables complexes*, Hermann, Paris 1961.

[2] Hatori. H., and Mori, T., *Finite Markov chains,* Baifukan, Tokyo (in Japanese) 1982.

[3] Kemeny, J., and Snell, J.L., *Finite Markov chains,* McGraw-Hill, New York 1960.

[4] Lucantoni, D.M., New results on the single server queue with a batch Markovian arrival process, *Stochastic Models,* 7, (1991) 1-46

[5] Narayana, S and Neuts, M.F., The first two moments matrixes of the counts for the Markovian arrival process, *Stochastic Models,* 8, (1992) 459-477.

[6] Neuts, M.F., A versatile Markovian point process, *J. Appl. Prob.,* 16, (1979) 764-779.

[7] Neuts, M.F., *Structured Stochastic Matrices of M/G/1 Type and Their Applications,* Marcel Dekker, Inc., New York 1989.

[8] Nishimura, S., Eigenvalue Expression for Mean Queue Length of BMAP/G/1 Queue. (submitted)

[9] Ramaswami V., The N/G/1 queue and its detailed analysis, *Adv. Appl. Prob.,* 12, (1979) 222-261.

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