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A METHOD TO ESTIMATE RESULTS OF UNKNOWN COMPARISONS IN BINARY AHP

Kazutomo Nishizawa
Nihon University

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Abstract In this paper we propose an algorithm to estimate results of unknown comparisons in binary AHP. For incomplete comparison case, which includes unknown pairwise comparisons, there are various methods to calculate weights, for example, Harker method, Two-Stage method and so on. However these methods are to estimate weights of activities but not to estimate results of unknown comparisons directly. We intend to estimate results of unknown comparisons directly. For this purpose we introduce a kind of syllogism which is to connect two activities with unknown comparisons by an arrow so that there are no cycles in the graph corresponding to the given comparison matrix. We apply our algorithm, first, to estimate results of unknown comparisons and compare our weights with Harker's and Two-Stage's, for four examples, respectively. Next, we also apply our algorithm to construct comparison matrix using a personal computer and the performance of our algorithm is discussed based on simulation using random number. From results of these examples and simulation, we can illustrate an effectiveness of the proposed algorithm.

1. Introduction

In this paper we propose an algorithm to estimate results of unknown comparisons in an incomplete binary AHP[1] and illustrate applications.

In binary AHP, result of pairwise comparison is represented by θ or $1/\theta$, where θ is a parameter whose value is grater than 1. If activity i is important than j, we have $A(i,j) = \theta$ and $A(j,i) = 1/\theta$ where $A(i,j)$ is an element of a comparison matrix A ($i = 1 \sim n$, $j = 1 \sim n$ where n is the number of activities). If activities i and j are equally important then we have $A(i,j) = 1$. Of course $A(i,i) = 1$. (Here we represent (i,j) element of a matrix X by $X(i,j)$.)

For incomplete comparison case, which includes unknown pairwise comparisons, there are various methods to calculate weights of activity, for example, Harker method [2], Two-Stage method [3] and so on. However Harker method and Two-Stage method are not to estimate results of unknown comparisons since an element of estimated comparison matrix does not exactly coincide with 1, θ or $1/\theta$. Our method is to estimate results of unknown comparisons by any values of 1, θ or $1/\theta$.

For this purpose we introduce a directed graph which corresponding to comparison matrix A. If $A(i,j) = \theta$ ($1/\theta$) then i and j are connected by an arrow "$i \rightarrow j$" ("$i \leftarrow j$"). However if $A(i,j) = 1$ then i and j are connected by a straight line "$i - j$" and unknown comparisons are not connected by lines. In previous study we proposed a criterion of consistency by number of cycles [4] and how to find cycles of various lengths. A basic idea of our method is based on two principles, a kind of syllogism and rule of avoiding inconsistency; that is, if "$i \rightarrow j$" and "$j \rightarrow k$" and the result of comparison of (i,k) is unknown, then we estimate i must be important than k ("$i \rightarrow k$") unless "$i \rightarrow k$" induces new inconsistencies.

In order to implement the estimation of unknown comparisons based on the directed graph, we introduce $n \times n$ matrix P and $n \times n$ matrix V. If $A(i,j) = \theta$ $(1/\theta)$ then let $P(i,j) = 1$ (-1), and if $A(i,j) = 1$ then let $P(i,j) = 0$. If comparison between i and j is unknown then we assume $P(i,j) = 9$. If $A(i,j) = \theta$ then let $V(i,j) = 1$, otherwise $V(i,j) = 0$. In our estimating algorithm matrix P and matrix V play important roles.

In this paper, we describe how to estimate results of unknown comparisons in Section 2. In Section 3, we apply the proposed algorithm to estimate results of unknown comparisons and illustrate four examples. In Section 4, we also apply our algorithm to construct comparison matrix using a personal computer and the performance of our algorithm is discussed. Finally, we conclude this investigation in Section 5.

2. A Method to Estimate Results of Unknown Comparisons

In this paper we often use terms "path" and "cycle" in a graph. These are most important then we define them through matrix P as follows;

path: A series of points $i, i_1, \cdots, i_{m-1}, j$ is called path starting at i and ending at j if and only if $P(i,i_1) = P(i_1,i_2) = \cdots = P(i_{m-1},j) = 1$ (some of points $i, i_1, \cdots, i_{m-1}, j$ are permitted to coincide).

cycle: A path whose starting point coincides with ending point (that is $i = j$) is called cycle.

diagonal line of cycle: Let $S = (i_0, i_1, \cdots, i_{m-1}, i_0)$ be a cycle, then "$i_j \to i_k$" $(k \neq j+1 \pmod{m})$ is called diagonal line of S.

First we consider a simple case with three activities shown in Fig. 2.1. For this case we can simply estimate "$i \to j$" by syllogism (of course such estimation "$j \to i$" cannot be considered). But in Fig. 2.2 we cannot apply syllogism, so the relation of (k,j) is left unknown.

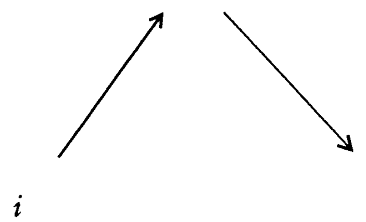

Fig. 2.1 Can apply syllogism

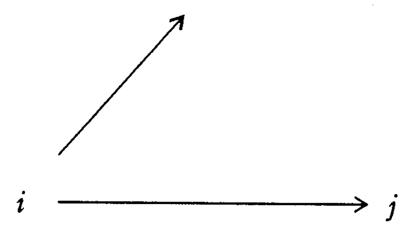

Fig. 2.2 Cannot apply syllogism

Note that in Fig. 2.1 a path "$i \to k \to j$" (starting at i and ending at j), represents $V^2(i,j) > 0$ [4], and the fact that the comparison of (i,j) is unknown is represented by $P(i,j) = 9$. So the both conditions $V^2(i,j) > 0$ and $P(i,j) = 9$ give the estimation "$i \to j$".

Next we consider a case of four activities. For example in Fig. 2.3 we can estimate not only "$i \to l$" ($V^2(i,l) > 0$ and $P(i,l) = 9$) and "$k \to j$" ($V^2(k,j) > 0$ and $P(k,j) = 9$) but also can estimate "$i \to j$" which is based on the conditions $V^3(i,j) > 0$ and $P(i,j) = 9$.

Generally if there is a path of length m (starting at i and ending at j), then we have $V^m(i,j) > 0$ so $V^m(i,j) > 0$ [4] and $P(i,j) = 9$ is the condition of estimation "$i \to j$".

However in Fig. 2.4 the situation is rather complicated. We can consider the possibility "$i \to l$" with $V^2(i,l) > 0$ and $P(i,l) = 9$, but "$i \to l$" induces another inconsistency $i\ l\ j\ i$ (that is "$i \to l$" induces a cycle $(i\ l\ j)$ in the graph), which is represented by $V^2(l,i) > 0$ [4]. As a result, we can estimate "$i \to l$" if the following condition is hold; $V^2(i,l) > 0$, $P(i,l) = 9$ and $V^2(l,i) = 0$.

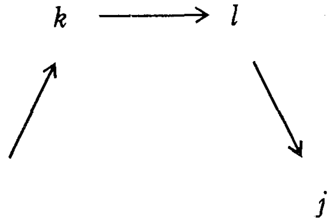

Fig. 2.3 Can apply syllogism multiply

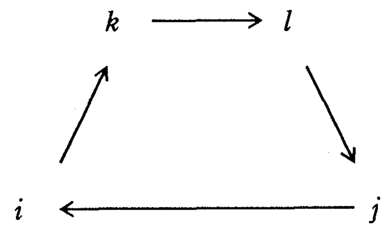

Fig. 2.4 An incomplete directed graph already exist cycle

Considering as above, we can set up our estimating principle as follows; If and only if there exists a path (of length $m \geq 2$) starting at i and ending at j and there are not any paths starting at j and ending at i, we estimate i is more important than j (that is "$i \to j$").

This principle is very simple, but implementing it to construct an efficient algorithm we use matrix P and V based on a given comparison matrix A and further introduce an $n \times n$ matrix C called check matrix. Based on C we have estimated matrix P_{est} and have estimated comparison matrix A_{est}. The following is our fundamental algorithm;

(2.1) Prepare matrix $P, P_{\text{est}}\ (= P)$ and V based on given matrix A $(i = 1 \sim n, j = 1 \sim n)$.
If $A(i,j) = \theta, 1/\theta$ and 1 then let $P(i,j) = 1, -1$ and 0, respectively.
If $A(i,j)$ is unknown then let $P(i,j) = 9$.
If $A(i,j) = \theta$ then let $V(i,j) = 1$, otherwise $V(i,j) = 0$.

(2.2) Let each element of check matrix C be zero.

(2.3) For each $\alpha = 2 \sim n - 1$.
Calculate V^α and find a pair of i and j $(i = 1 \sim n,\ j = 1 \sim n)$, with $V^\alpha(i,j) > 0$ and $P(i,j) = 9$, then let $C(i,j) = 1$.

(2.4) If $C(i,j) = 1$ and $C(j,i) = 0$ $(i = 1 \sim n,\ j = 1 \sim n)$ then let $P_{\text{est}}(i,j) = 1$ and $P_{\text{est}}(j,i) = -1$.

(2.5) If $P_{\text{est}}(i,j) = 1\ (-1)$ then let $A_{\text{est}}(i,j) = \theta\ (1/\theta)$, otherwise $A_{\text{est}}(i,j) = 1$ $(i = 1 \sim n,\ j = 1 \sim n)$.

It is clear that the above algorithm actually implements our principle; from (2.3) if and only if there exists a path (of length $m \geq 2$) starting at i and ending at j, we have $P(i,j) = 1$, and so $C(i,j) = 1$ and $C(j,i) = 0$ represent the condition of estimating "$i \to j$" in our principle.

Theorem 1. *By our algorithm* (2.1)∼(2.5) *we have unique estimated matrix* A_{est} *from given a comparison matrix* A*, and even if we again apply the algorithm* (2.1)∼(2.5) *to the estimated matrix* A_{est}*, the* A_{est} *remains unchanged* (*that is* $(A_{\text{est}})_{\text{est}} = A_{\text{est}}$).

Proof. It is clear the procedure (2.3) uniquely determines the check matrix C so that by (2.4) and (2.5) A_{est} is determined uniquely from C. This proves the uniqueness of A_{est}.

The latter part is also clear. Let denote the check matrix corresponding to A_{est} by C'. From (2.4) we have $P_{\text{est}}(i,j) = 1$ for every pair (i,j) satisfying the condition $C(i,j) = 1$ and $C(j,i) = 0$. So $C'(i,j) = 0$ for every pair (i,j) from (2.3) (that is C' is zero matrix), which means that A_{est} is unchanged by algorithm (2.1)∼(2.5). ■

By the above algorithm and Theorem 1 we can further describe the following important theorem.

Theorem 2. *If the original data is inconsistent, that is the graph corresponding to comparison matrix* A *include a cycle* S*, diagonal lines of* S *do not newly arise in the graph* *correspond to A_{est}, that is we cannot estimate importance among member of S.*

Proof. Assume comparison i and j is unknown and i and j are member of cycle. For pair (i,j), $C(i,j) = 1$ and $C(j,i) = 1$ then not satisfying estimate condition $C(i,j) = 1$ and $C(j,i) = 0$. ■

For example it is clear that in Fig. 2.4, shown above, we cannot estimate importance of pairs (k,j) and (i,l). Further we show next example in Fig. 2.5.

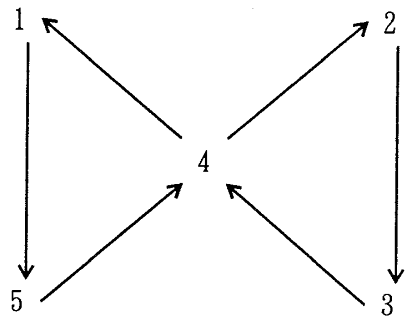

Fig. 2.5 An example of directed graph for Theorem 2

In Fig. 2.5 we have cycle of length 6, (1 5 4 2 3 4). Based on Theorem 2 we cannot estimate importance of $(1,2)$, $(1,3)$, $(2,5)$ and $(3,5)$. We describe details in § 3.4 Example 4.

3. Examples to Estimate Unknown Comparisons in Binary AHP

Now we show four examples in binary AHP to illustrate usefulness of the proposed estimating algorithm. All these examples have six activities, $n = 6$, so we consider 6×6 incomplete comparison matrix A. Example 1 is consistent, or acyclic graph, and all unknown comparisons are completely estimated. Example 2 is consistent but some unknown comparisons are not estimated. Example 3 is inconsistent, or cyclic graph, but all unknown comparisons are completely estimated without forming new cycles. Example 4 is inconsistent and some comparisons are not estimated because of forming new cycles.

The procedures to estimate results of unknown comparisons for each example as follows:

(1) Given incomplete comparison matrix A and corresponding directed graph.
(2) Prepare matrix P and matrix V based on A.
(3) Examine consistency of A by finding cycles corresponding directed graph.
(4) Obtain estimated matrix A_{est} form A by proposed algorithm.
(5) Obtain A_{HM} by Harker method and A_{TSM} by Two-stage method form A.
(6) Calculate the weights from A_{est}, A_{HM} and A_{TSM}, respectively where $\theta = 2$.
(7) Compare with the weights, obtained (6), normalized sum of them is equal to 1.

In this study we calculate the weights of comparison matrix where $\theta = 2$. A consistent case the weights are not depend on θ (see [5]), however it is not clear an inconsistent case. We usually calculate where $\theta = 2$ and we consider it is appropriate value.

3.1 Example 1

In Example 1, all unknown comparisons are completely estimated by using the proposed algorithm. An incomplete comparison matrix A is shown as (3.1) where the symbol "□" represents unknown comparison. The number of unknown comparisons, in this case, is nine. Corresponding directed graph is shown in Fig. 3.1.

$$(3.1)\qquad A=\begin{bmatrix} 1 & \square & \square & \square & \theta & \theta \\ \square & 1 & \theta & 1/\theta & \square & \square \\ \square & 1/\theta & 1 & \square & \square & \theta \\ \square & \theta & \square & 1 & 1/\theta & \square \\ 1/\theta & \square & \square & \theta & 1 & \square \\ 1/\theta & \square & 1/\theta & \square & \square & 1 \end{bmatrix}$$

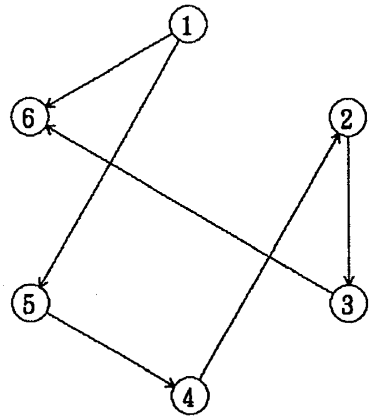

Fig. 3.1 Directed Graph for Example 1

From (3.1) we have matrix P and matrix V, shown as below.

$$(3.2)\qquad P=\begin{bmatrix} 0 & 9 & 9 & 9 & 1 & 1 \\ 9 & 0 & 1 & -1 & 9 & 9 \\ 9 & -1 & 0 & 9 & 9 & 1 \\ 9 & 1 & 9 & 0 & -1 & 9 \\ -1 & 9 & 9 & 1 & 0 & 9 \\ -1 & 9 & -1 & 9 & 9 & 0 \end{bmatrix}$$

$$(3.3)\qquad V=\begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

First, we examine consistency of comparison matrix A based on cycles. To find cycles of length 3~6 in Fig. 3.1, we apply the algorithm which proposed in previous paper. As a result, there are no cycles in Fig. 3.1 then we judge A is consistent.

Secondly, we estimate unknown comparisons in (3.1) by our algorithm (2.1)~(2.4) for $\alpha = 2 \sim 5$.

For $\alpha = 2$, we need P and V^2, shown as (3.4).

$$(3.4)\qquad V^2=\begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

By the proposed algorithm we have four sets (1,4), (2,6), (4,3) and (5,2), satisfying (2.3) for $\alpha = 2$. Then we have $C(1,4) = 1$, $C(2,6) = 1$, $C(4,3) = 1$ and $C(5,2) = 1$.

For $\alpha = 3$, we need P and V^3, shown as (3.5).

$$V^3 = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \tag{3.5}$$

Then we have three sets (1,2), (4,6) and (5,3), satisfying (2.3) for $\alpha = 3$ and let be $C(1,2) = 1$, $C(4,6) = 1$ and $C(5,3) = 1$.

For $\alpha = 4$, we need P and V^4, shown as (3.6).

$$V^4 = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \tag{3.6}$$

Then we have two sets (1,3) and (5,6), satisfying (2.3) for $\alpha = 4$ and let be $C(1,3) = 1$ and $C(5,6) = 1$.

For $\alpha = 5$, we need P and V^5, shown as (3.7).

$$V^5 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \tag{3.7}$$

There are no another sets satisfying (2.3) for $\alpha = 5$.

Thirdly, from above process, we have matrix C, shown as below, then let us decide results of unknown comparisons.

$$C = \begin{bmatrix} 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \tag{3.8}$$

As there are no sets, i and j, satisfying $C(i,j) \neq 0$ and $C(j,i) \neq 0$, we can estimate unknown comparisons without forming new cycles. For example, since $C(1,2) = 1$ and $C(2,1) = 0$ then let be $P_{\text{est}}(1,2) = 1$ and $P_{\text{est}}(2,1) = -1$. Thus we have estimated matrix P_{est}, shown as below.

(3.9)
$$P_{\text{est}} = \begin{bmatrix} 0 & 1 & 1 & 1 & 1 & 1 \\ -1 & 0 & 1 & -1 & -1 & 1 \\ -1 & -1 & 0 & -1 & -1 & 1 \\ -1 & 1 & 1 & 0 & -1 & 1 \\ -1 & 1 & 1 & 1 & 0 & 1 \\ -1 & -1 & -1 & -1 & -1 & 0 \end{bmatrix}$$

Then all unknown comparisons in (3.2) are completely estimated. Based on (3.9), we have A_{est}, completed estimating unknown comparisons of A, shown as (3.10).

(3.10)
$$A_{\text{est}} = \begin{bmatrix} 1 & \theta & \theta & \theta & \theta & \theta \\ 1/\theta & 1 & \theta & 1/\theta & 1/\theta & \theta \\ 1/\theta & 1/\theta & 1 & 1/\theta & 1/\theta & \theta \\ 1/\theta & \theta & \theta & 1 & 1/\theta & \theta \\ 1/\theta & \theta & \theta & \theta & 1 & \theta \\ 1/\theta & 1/\theta & 1/\theta & 1/\theta & 1/\theta & 1 \end{bmatrix}$$

The value of $A_{\text{est}}(i,j)$ is the power $P_{\text{est}}(i,j)$ of θ , $(i = 1 \sim n,\ j = 1 \sim n)$, that is, if $P_{\text{est}}(i,j) = 1\ (-1)$ then $A_{\text{est}}(i,j) = \theta\ (1/\theta)$, and if $P_{\text{est}}(i,j) = 0$ then $A_{\text{est}}(i,j) = 1$. Of course directed graph of A_{est} is acyclic and so consistent, and we have values of weights of all activities analytically (not numerically) by the formula given in [5].

On the other hand, from (3.1), we have A_{HM} by Harker Method and have A_{TSM} by Two-Stage Method shown as below.

(3.11)
$$A_{\text{HM}} = \begin{bmatrix} 4 & 0 & 0 & 0 & \theta & \theta \\ 0 & 4 & \theta & 1/\theta & 0 & 0 \\ 0 & 1/\theta & 4 & 0 & 0 & \theta \\ 0 & \theta & 0 & 4 & 1/\theta & 0 \\ 1/\theta & 0 & 0 & \theta & 4 & 0 \\ 1/\theta & 0 & 1/\theta & 0 & 0 & 4 \end{bmatrix}$$

(3.12)
$$A_{\text{TSM}} = \begin{bmatrix} 1 & \theta^{2/3} & \theta^{2/3} & \theta^{2/3} & \theta & \theta \\ \theta^{-2/3} & 1 & \theta & 1/\theta & 1 & \theta^{2/3} \\ \theta^{-2/3} & 1/\theta & 1 & 1 & 1 & \theta \\ \theta^{-2/3} & \theta & 1 & 1 & 1/\theta & \theta^{2/3} \\ 1/\theta & 1 & 1 & \theta & 1 & \theta^{2/3} \\ 1/\theta & \theta^{-2/3} & 1/\theta & \theta^{-2/3} & \theta^{-2/3} & 1 \end{bmatrix}$$

Finally, we calculate the weights from A_{est}, A_{HM} and A_{TSM}, respectively where $\theta = 2$. The results, compered with each weights, are listed in Table 3.1.

As a result, for consistent and completely estimated case, an order of six activities are coincide with three kinds of method's. It is interesting that the calculated weights from A_{est} by our algorithm are agree well with the weights from A_{HM} by Harker method.

3.2 Example 2

Next example is consistent but incompletely estimated. An incomplete comparison matrix A is shown as (3.13) and corresponding directed graph is shown in Fig. 3.2. The number of

Table 3.1 Results of weights for Example 1

	From A_{est}	From A_{HM}	From A_{TSM}
①	0.2751	0.2752	0.2481
②	0.1375	0.1375	0.1644
③	0.1092	0.1092	0.1475
④	0.1733	0.1731	0.1664
⑤	0.2183	0.2182	0.1755
⑥	0.0866	0.0867	0.0981
λ max	6.2710	6.2163	6.2661

unknown comparisons is nine.

(3.13)
$$A = \begin{bmatrix} 1 & \square & \theta & \square & \theta & \square \\ \square & 1 & \square & \theta & 1/\theta & \square \\ 1/\theta & \square & 1 & \square & \square & \theta \\ \square & 1/\theta & \square & 1 & \square & \theta \\ 1/\theta & \theta & \square & \square & 1 & \square \\ \square & \square & 1/\theta & 1/\theta & \square & 1 \end{bmatrix}$$

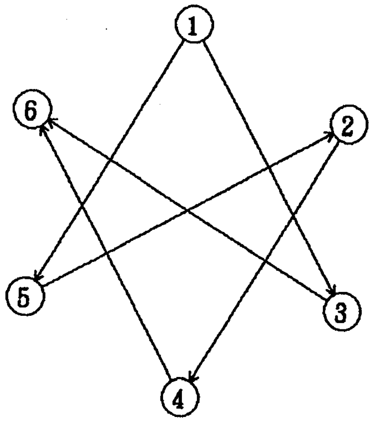

Fig. 3.2 Directed graph for Example 2

From (3.13) we have matrix P and matrix V, shown as below.

(3.14)
$$P = \begin{bmatrix} 0 & 9 & 1 & 9 & 1 & 9 \\ 9 & 0 & 9 & 1 & -1 & 9 \\ -1 & 9 & 0 & 9 & 9 & 1 \\ 9 & -1 & 9 & 0 & 9 & 1 \\ -1 & 1 & 9 & 9 & 0 & 9 \\ 9 & 9 & -1 & -1 & 9 & 0 \end{bmatrix}$$

(3.15)
$$V = \begin{bmatrix} 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

By the same way in Example 1, we examine consistency of comparison matrix A. As a result, there are no cycles in Fig. 3.2 then we judge A is consistent.

By the proposed algorithm, we have matrix C shown as below.

$$C = \begin{bmatrix} 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \tag{3.16}$$

Then we have estimated matrix P_{est} shown as below.

$$P_{\text{est}} = \begin{bmatrix} 0 & 1 & 1 & 1 & 1 & 1 \\ -1 & 0 & 9 & 1 & -1 & 1 \\ -1 & 9 & 0 & 9 & 9 & 1 \\ -1 & -1 & 9 & 0 & -1 & 1 \\ -1 & 1 & 9 & 1 & 0 & 1 \\ -1 & -1 & -1 & -1 & -1 & 0 \end{bmatrix} \tag{3.17}$$

We cannot estimate three elements $P(2,3)$, $P(3,4)$ and $P(3,5)$. We consider that these results of comparisons are equally important, let $P_{\text{est}}(2,3) = P_{\text{est}}(3,2) = 0$, $P_{\text{est}}(3,4) = P_{\text{est}}(4,3) = 0$ and $P_{\text{est}}(3,5) = P_{\text{est}}(5,3) = 0$, shown as below.

$$P_{\text{est}} = \begin{bmatrix} 0 & 1 & 1 & 1 & 1 & 1 \\ -1 & 0 & 0 & 1 & -1 & 1 \\ -1 & 0 & 0 & 0 & 0 & 1 \\ -1 & -1 & 0 & 0 & -1 & 1 \\ -1 & 1 & 0 & 1 & 0 & 1 \\ -1 & -1 & -1 & -1 & -1 & 0 \end{bmatrix} \tag{3.18}$$

Thus we have matrix A_{est}, from (3.18), shown as below.

$$A_{\text{est}} = \begin{bmatrix} 1 & \theta & \theta & \theta & \theta & \theta \\ 1/\theta & 1 & 1 & \theta & 1/\theta & \theta \\ 1/\theta & 1 & 1 & 1 & 1 & \theta \\ 1/\theta & 1/\theta & 1 & 1 & 1/\theta & \theta \\ 1/\theta & \theta & 1 & \theta & 1 & \theta \\ 1/\theta & 1/\theta & 1/\theta & 1/\theta & 1/\theta & 1 \end{bmatrix} \tag{3.19}$$

On the other hand, from (3.13), we have A_{HM} by Harker Method and have A_{TSM} by Two-Stage Method shown as below.

$$A_{\text{HM}} = \begin{bmatrix} 4 & 0 & \theta & 0 & \theta & 0 \\ 0 & 4 & 0 & \theta & 1/\theta & 0 \\ 1/\theta & 0 & 4 & 0 & 0 & \theta \\ 0 & 1/\theta & 0 & 4 & 0 & \theta \\ 1/\theta & \theta & 0 & 0 & 4 & 0 \\ 0 & 0 & 1/\theta & 1/\theta & 0 & 4 \end{bmatrix} \tag{3.20}$$

$$A_{\mathrm{TSM}} = \begin{bmatrix} 1 & \theta^{2/3} & \theta & \theta^{2/3} & \theta & \theta^{4/3} \\ \theta^{-2/3} & 1 & 1 & \theta & 1/\theta & \theta^{2/3} \\ 1/\theta & 1 & 1 & 1 & 1 & \theta \\ \theta^{-2/3} & 1/\theta & 1 & 1 & 1 & \theta \\ 1/\theta & \theta & 1 & 1 & 1 & \theta^{2/3} \\ \theta^{-4/3} & \theta^{-2/3} & 1/\theta & 1/\theta & \theta^{-2/3} & 1 \end{bmatrix} \tag{3.21}$$

Then we calculate the weights from A_{est}, A_{HM} and A_{TSM}, respectively where $\theta = 2$. The results are listed in Table 3.2.

Table 3.2 Results of weights for Example 2

	From A_{est}	From A_{HM}	From A_{TSM}
①	0.2782	0.3527	0.2673
②	0.1573	0.1409	0.1641
③	0.1533	0.1392	0.1549
④	0.1245	0.0883	0.1473
⑤	0.1988	0.2235	0.1745
⑥	0.0879	0.0553	0.0910
λ max	6.1898	6.0574	6.1789

As a result, using our algorithm, we cannot estimate results of three comparisons, $A(2,3), A(3,4)$ and $A(3,5)$. If possible to obtain these comparisons, we can completely estimated all unknown comparisons. However an order of six activities are coincide with Harker Method's and Two-Stage Method's. It is interesting that the value of unestimated elements of A_{est} are equal to 1, and same elements of A_{TSM}, except $A_{\mathrm{TSM}}(4,5)$, also equal to 1.

3.3 Example 3

Next example is inconsistent, or cyclic graph, but all unknown comparisons are completely estimated without forming new cycles. An incomplete comparison matrix A is shown (3.22) and the corresponding directed graph is shown in Fig. 3.3. The number of unknown comparisons, in this case, is five.

$$A = \begin{bmatrix} 1 & \square & \theta & \square & \theta & \theta \\ \square & 1 & \square & \theta & 1/\theta & \theta \\ 1/\theta & \square & 1 & \square & 1/\theta & \theta \\ \square & 1/\theta & \square & 1 & \theta & \theta \\ 1/\theta & \theta & \theta & 1/\theta & 1 & \square \\ 1/\theta & 1/\theta & 1/\theta & 1/\theta & \square & 1 \end{bmatrix} \tag{3.22}$$

From (3.22) we have matrix P and matrix V, shown as below.

$$P = \begin{bmatrix} 0 & 9 & 1 & 9 & 1 & 1 \\ 9 & 0 & 9 & 1 & -1 & 1 \\ -1 & 9 & 0 & 9 & -1 & 1 \\ 9 & -1 & 9 & 0 & 1 & 1 \\ -1 & 1 & 1 & -1 & 0 & 9 \\ -1 & -1 & -1 & -1 & 9 & 0 \end{bmatrix} \tag{3.23}$$

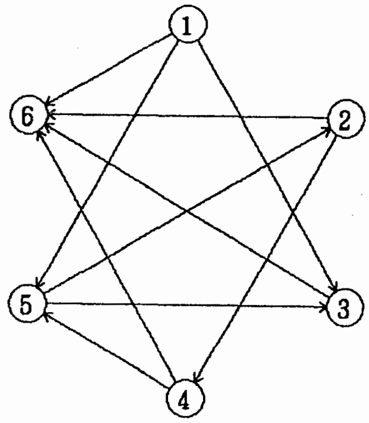

Fig. 3.3 Directed graph for Example 3

(3.24)
$$V = \begin{bmatrix} 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

By the same way in the previous examples, we examine consistency of comparison matrix A. As a result, we have one cycle of length 3, (2 4 5), in Fig. 3.3 then we judge A is inconsistent.

By our algorithm, we have matrix C, shown as below, then let us decide results of unknown comparisons.

(3.25)
$$C = \begin{bmatrix} 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Thus we have matrix P_{est}, shown as below, whose unknown comparisons are completely estimated.

(3.26)
$$P_{\text{est}} = \begin{bmatrix} 0 & 1 & 1 & 1 & 1 & 1 \\ -1 & 0 & 1 & 1 & -1 & 1 \\ -1 & -1 & 0 & -1 & -1 & 1 \\ -1 & -1 & 1 & 0 & 1 & 1 \\ -1 & 1 & 1 & -1 & 0 & 1 \\ -1 & -1 & -1 & -1 & -1 & 0 \end{bmatrix}$$

Then all unknown comparisons in (3.23) are completely estimated. From (3.26), we have A_{est}, shown as below.

(3.27)
$$A_{\text{est}} = \begin{bmatrix} 1 & \theta & \theta & \theta & \theta & \theta \\ 1/\theta & 1 & \theta & \theta & 1/\theta & \theta \\ 1/\theta & 1/\theta & 1 & 1/\theta & 1/\theta & \theta \\ 1/\theta & 1/\theta & \theta & 1 & \theta & \theta \\ 1/\theta & \theta & \theta & 1/\theta & 1 & \theta \\ 1/\theta & 1/\theta & 1/\theta & 1/\theta & 1/\theta & 1 \end{bmatrix}$$

Directed graph of A_{est} is inconsistent since forming cycle (2 4 5), but there are no another new cycles.

From (3.22), we have A_{HM} by Harker Method and have A_{TSM} by Two-Stage Method shown as below.

$$(3.28) \qquad A_{\text{HM}} = \begin{bmatrix} 3 & 0 & \theta & 0 & \theta & \theta \\ 0 & 3 & 0 & \theta & 1/\theta & \theta \\ 1/\theta & 0 & 3 & 0 & 1/\theta & \theta \\ 0 & 1/\theta & 0 & 3 & \theta & \theta \\ 1/\theta & \theta & \theta & 1/\theta & 2 & 0 \\ 1/\theta & 1/\theta & 1/\theta & 1/\theta & 0 & 2 \end{bmatrix}$$

$$(3.29) \qquad A_{\text{TSM}} = \begin{bmatrix} 1 & \theta^{1/2} & \theta & \theta^{1/2} & \theta & \theta \\ \theta^{-1/2} & 1 & \theta^{1/2} & \theta & 1/\theta & \theta \\ 1/\theta & \theta^{-1/2} & 1 & \theta^{-1/2} & 1/\theta & \theta \\ \theta^{-1/2} & 1/\theta & \theta^{1/2} & 1 & \theta & \theta \\ 1/\theta & \theta & \theta & 1/\theta & 1 & \theta^{4/5} \\ 1/\theta & 1/\theta & 1/\theta & 1/\theta & \theta^{-4/5} & 1 \end{bmatrix}$$

Then we calculate the weights from A_{est}, A_{HM} and A_{TSM}, respectively where $\theta = 2$. The results are listed in Table 3.3.

Table 3.3 Results of weights for Example 3

	From A_{est}	From A_{HM}	From A_{TSM}
①	0.2706	0.2302	0.2433
②	0.1793	0.1928	0.1827
③	0.1065	0.1126	0.1211
④	0.1793	0.1917	0.1826
⑤	0.1793	0.1890	0.1823
⑥	0.0849	0.0837	0.0879
λ max	6.3896	6.3472	6.3405

As a result, in Table 3.3, from A_{est}, ②, ④ and ⑤ have the same weight but from A_{HM} and from A_{TSM}, ②, ④ and ⑤ have different weights but near. It may be consider that ②, ④ and ⑤ are equally important since cycle of length 3, (2 4 5), already forms and in corresponding directed graph of A, all unknown comparisons are completely estimated with forming no another new cycles.

3.4 Example 4

The last example is inconsistent and its some unknown comparisons are not estimated because of forming new cycles. The incomplete comparison matrix A is shown as (3.30) and corresponding directed graph is shown in Fig. 3.4. The number of unknown comparisons, in this case, is five.

$$(3.30) \qquad A = \begin{bmatrix} 1 & \square & \square & 1/\theta & \theta & \theta \\ \square & 1 & \theta & 1/\theta & \square & \theta \\ \boxminus & 1/\theta & 1 & \theta & \square & \theta \\ \theta & \theta & 1/\theta & 1 & 1/\theta & \square \\ 1/\theta & \boxminus & \square & \theta & 1 & \theta \\ 1/\theta & 1/\theta & 1/\theta & \boxminus & 1/\theta & 1 \end{bmatrix}$$

From (3.30) we have matrix P and matrix V, shown as below.

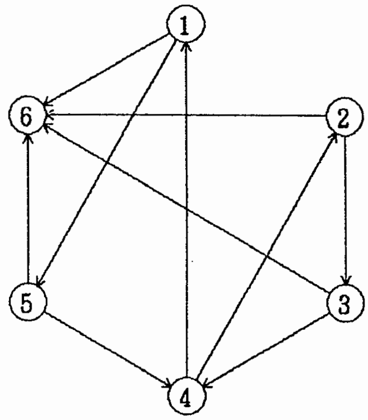

Fig. 3.4 Directed graph for Example 4

$$P = \begin{bmatrix} 0 & 9 & 9 & -1 & 1 & 1 \\ 9 & 0 & 1 & -1 & 9 & 1 \\ 9 & -1 & 0 & 1 & 9 & 1 \\ 1 & 1 & -1 & 0 & -1 & 9 \\ -1 & 9 & 9 & 1 & 0 & 1 \\ -1 & -1 & -1 & 9 & -1 & 0 \end{bmatrix} \tag{3.31}$$

$$V = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \tag{3.32}$$

By the same way as the previous examples, we examine consistency of comparison matrix A. As a result, we have two cycles of length 3, (1 5 4) and (2 3 4), in Fig. 3.4 then we judge A is inconsistent.

These two cycles are connected and forming cycle of length 6, (1 5 4 2 3 4). Based on Theorem 2, mentioned in Section 2, we can predict that we cannot estimate unknown comparison pairs (1,2), (1,3), (2,5) and (3,5), since there are member of cycle.

By our algorithm, we have matrix C, shown as below, then let us decide results of unknown comparisons.

$$C = \begin{bmatrix} 0 & 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \tag{3.33}$$

We can estimate $P_{\text{est}}(4,6) = 1$ and $P_{\text{est}}(6,4) = -1$, satisfying (2.4). According to our prediction we cannot estimate four elements $P(1,2)$, $P(1,3)$, $P(2,5)$ and $P(3,5)$, because forming new cycles since, for example, $C(1,2) = 1$ and $C(2,1) = 1$, not satisfying (2.4). We consider that these results of comparisons are equally important, then $P_{\text{est}}(1,2) = P_{\text{est}}(2,1) = 0$, $P_{\text{est}}(1,3) = P_{\text{est}}(3,1) = 0$, $P_{\text{est}}(2,5) = P_{\text{est}}(5,2) = 0$ and $P_{\text{est}}(3,5) = P_{\text{est}}(5,3) = 0$. Thus we have estimated matrix P_{est} shown as below.

$$(3.34)\qquad P_{\text{est}} = \begin{bmatrix} 0 & 0 & 0 & -1 & 1 & 1 \\ 0 & 0 & 1 & -1 & 0 & 1 \\ 0 & -1 & 0 & 1 & 0 & 1 \\ 1 & 1 & -1 & 0 & -1 & 1 \\ -1 & 0 & 0 & 1 & 0 & 1 \\ -1 & -1 & -1 & -1 & -1 & 0 \end{bmatrix}$$

Thus we have matrix A_{est} from (3.34) shown as below.

$$(3.35)\qquad A_{\text{est}} = \begin{bmatrix} 1 & 1 & 1 & 1/\theta & \theta & \theta \\ 1 & 1 & \theta & 1/\theta & 1 & \theta \\ 1 & 1/\theta & 1 & \theta & 1 & \theta \\ \theta & \theta & 1/\theta & 1 & 1/\theta & \theta \\ 1/\theta & 1 & 1 & \theta & 1 & \theta \\ 1/\theta & 1/\theta & 1/\theta & 1/\theta & 1/\theta & 1 \end{bmatrix}$$

Directed graph of A_{est} is inconsistent since forming cycle (1 5 4) and (2 3 4), but there are no another new cycles.

From (3.30) we have A_{HM} by Harker Method and have A_{TSM} by Two-Stage Method shown as below.

$$(3.36)\qquad A_{\text{HM}} = \begin{bmatrix} 3 & 0 & 0 & 1/\theta & \theta & \theta \\ 0 & 3 & \theta & 1/\theta & 0 & \theta \\ 0 & 1/\theta & 3 & \theta & 0 & \theta \\ \theta & \theta & 1/\theta & 2 & 1/\theta & 0 \\ 1/\theta & 0 & 0 & \theta & 3 & \theta \\ 1/\theta & 1/\theta & 1/\theta & 0 & 1/\theta & 2 \end{bmatrix}$$

$$(3.37)\qquad A_{\text{TSM}} = \begin{bmatrix} 1 & 1 & 1 & 1/\theta & \theta & \theta \\ 1 & 1 & \theta & 1/\theta & 1 & \theta \\ 1 & 1/\theta & 1 & \theta & 1 & \theta \\ \theta & \theta & 1/\theta & 1 & 1/\theta & \theta^{4/5} \\ 1/\theta & 1 & 1 & \theta & 1 & \theta \\ 1/\theta & 1/\theta & 1/\theta & \theta^{-4/5} & 1/\theta & 1 \end{bmatrix}$$

Then we calculate the weights from A_{est}, A_{HM} and A_{TSM}, respectively where $\theta = 2$. The results are listed in Table 3.4.

Table 3.4 Results of weights for Example 4

	From A_{est}	From A_{HM}	From A_{TSM}
①	0.1795	0.1778	0.1799
②	0.1795	0.1778	0.1799
③	0.1822	0.1833	0.1821
④	0.1934	0.1980	0.1905
⑤	0.1822	0.1833	0.1821
⑥	0.0832	0.0799	0.0855
λ max	6.5125	6.5177	6.5147

As a result, in Table 3.4, it is impossible to order ① and ②, and ③ and ⑤, since having same weight. It may be consider that such result in Example 4 is caused by already include cycles before estimating. Note that the value of weights from A_{est} are similar to from A_{TSM} since the elements of A_{est} are coincide with A_{TSM} except $A_{\text{TSM}}(4,6)$.

4. Constructing Comparison Matrix Using The Proposed Algorithm

In previous section we apply our algorithm to estimate results of unknown comparisons for incomplete comparison matrix. In this section we apply our algorithm to construct comparison matrix. In general a number of all comparisons is $n(n-1)/2$. If n becomes large, it is the donkey work to construct comparison matrix. But if we introduce the principle of successive experiment with our algorithm then we can reduce a number of comparisons remarkably.

Now we show procedures to construct a comparison matrix and show the performance of our algorithm.

4.1 Procedures to Construct Comparison Matrix

The principle of successive experiment is the following; First we select randomly one pair of activities (i_1, j_1) and compare i_1 and j_1 to have the value of $A(i_1, j_1)$. Secondly select another pair (i_2, j_2) to get $A(i_2, j_2), \cdots$. But often the *k-th* step if we can estimate by our algorithm the value of comparisons for unselected pairs based on the information obtained during these k steps, we have the value of A for these pairs $(k = 1, 2, \cdots)$. So after the *k-th* step we know the values of A for not only really compared pairs but also estimated pair comparisons. At $(k+1)$*-th* step we select randomly one of the rest unknown pairs. Repeat these procedures till all values of comparison matrix A are known.

The procedures to construct $n \times n$ comparison matrix are shown as follows:

(4.1) Prepare matrix P and matrix V.

(4.2) Randomly select an unknown pair of activities i and j.

(4.3) Compare i and j, and if i (j) is important than j (i) then input $P(i,j) = 1$ (-1) and $P(j,i) = -1$ (1).

(4.4) Estimate unknown comparisons by using the proposed algorithm.

(4.5) Repeat (4.2) to (4.4) till all pairs of activities are known.

(4.6) Construct comparison matrix A, if $P(i,j) = 1$ (-1) then let be $A(i,j) = \theta$ $(1/\theta)$ $(i = 1 \sim n,\ j = 1 \sim n)$.

4.2 The Performance of Constructing Comparison Matrix with Our Estimation

Next we examine the number of pairs really compared by using our algorithm based on principle of successive experiment. Based on procedures (4.1) to (4.6), mentioned above, several simulations were carried out for $n = 5$ to 50. In our simulation, the results of comparison are to determine randomly.

Thus we construct 1000 kinds of comparison matrix for each n and have a number r of comparisons necessary to completely construct comparison matrix. The results show in Fig. 4.1.

In Fig. 4.1 the vertical axis represents a number of comparisons which necessary to completely construct comparison matrix and the horizontal axis represents an order of comparison matrix (n). The symbol "○" represents a number of all pairwise comparisons ($n(n-1)/2$). Performing 1000 cases, the symbol "△" represents maximum number of comparisons, the symbol "●" represents average number of comparisons and the symbol "▽" represents minimum number of comparisons. The chief values in Fig. 4.1 are shown in Table 4.1.

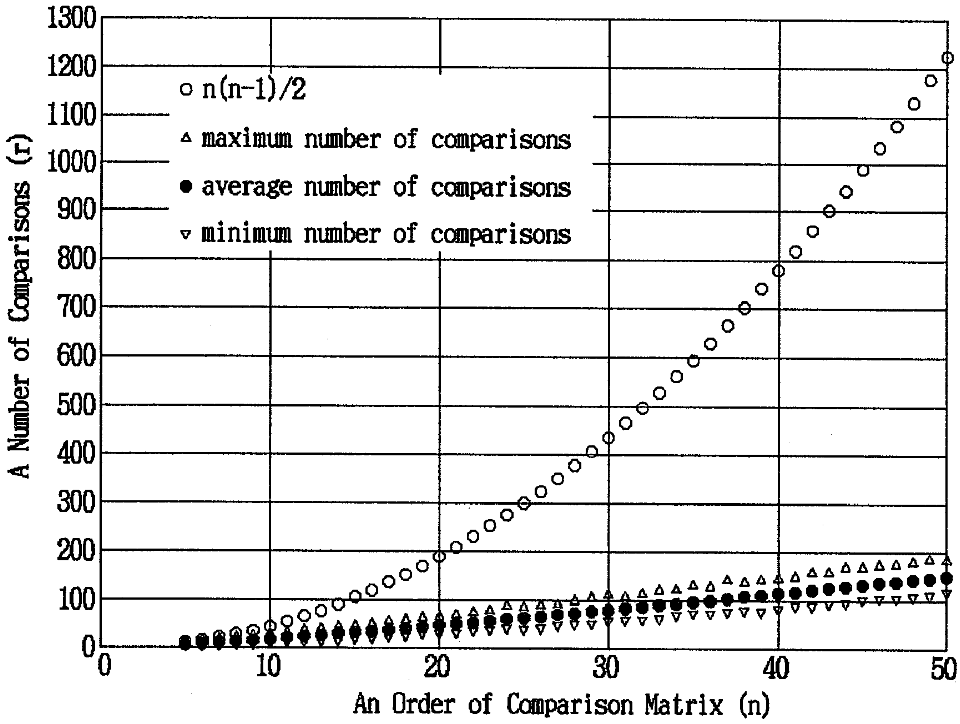

Fig. 4.1 Results of simulation by using random number

As a result, from Fig. 4.1 and Table 4.1, we can confirm to reduce a number of pairwise comparisons by using the proposed algorithm.

For example $n = 5$, there is no effect of our algorithm since maximum number of comparisons is equal to all comparisons. However for $n = 50$, maximum number of comparisons is 185 then it is about 15% of all comparisons. It is clear that, in Fig. 4.1, an order of all comparisons is n^2, on the other hand, using our algorithm an order of number of comparisons is n.

It may be considered that maximum number of comparisons corresponding to n, if randomly compared, is an aim of number to be necessary completely estimated unknown

Table 4.1 A number of pairwise comparisons using the proposed algorithm

n	$n(n-1)/2$	Minimum	Average	Maximum
5	10	4	6.36	10
10	45	10	18.61	29
15	105	20	32.51	46
20	190	31	47.49	66
25	300	40	63.61	85
30	435	58	80.27	111
35	595	75	97.06	131
40	780	83	113.96	145
45	990	103	131.91	169
50	1225	119	149.52	185

comparisons.

5. Conclusion

In this paper we propose an algorithm to estimate results of unknown comparisons in binary AHP and apply two kind of applications.

First applying our algorithm for incomplete comparison examples, we can estimate results of unknown comparisons without deteriorating consistency. Obtained weights, a consistent incomplete comparison and completely estimated case, are coincide with Harker method's. An inconsistent incomplete case, the form of estimated comparison matrix is similar Two-Stage method's.

Next applying our algorithm to construct comparison matrix and carrying out a simulation by using random number, we can confirm to reduce a number of pairwise comparisons. Thus for large n, it is possible to easy to construct comparison matrix with our estimation.

From these resutls, we can illustrate an effectiveness of the proposed algorithm.

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Kazutomo NISHIZAWA
Department of Mathematical Engineering
College of Industrial Technology, Nihon University
1-2-1, Izumicho, Narashino, Chiba 275, Japan
E-mail: nisizawa@su.cit.nihon-u.ac.jp