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ON THE AIRLINE HUB PROBLEM: THE CONTINUOUS MODEL

Atsuo Suzuki
Nanzan University

Zvi Drezner
California State University

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Abstract The location of hubs in an area where airports are evenly spread is considered. Two models are presented and analyzed. The first assumes that customers fly to the closest hub, then to the hub closest to their destination, and then to their destination. The second model assumes that customers select one hub, fly to that hub and then fly to their destination. The hub is selected such that the total distance to the destination via that hub is minimized.

1. Introduction

The location of hub facilities was recently discussed in many papers [2, 6, 7, 10, 11, 12, 13, 14]. In these papers it is assumed that a finite set of airports exist in the area and a subset of these airports is to be selected as hubs. The discrete models in these papers are formulated as combinatorial problems and many kinds of heuristics or branch and bound method are proposed. In such a formulation, it is sometimes difficult to obtain an insight of the solution.

In this paper the continuous problem is considered. It is assumed that customers and airports cover a given area. A set of n airports have to be selected as hubs. To consider the continuous model, an insight of the solution is easily obtained. Also, it gives an good approximate solution when the size of the problem is too big for the discrete method to obtain the exact solution.

The other types of location problems are usually treated as discrete problems. i.e. demand is aggregated into demand points. However, many studies have been done on the continuous version of such problems in order to gain insight into the solution patterns and to get approximate solutions for very large problems. Location problems using area demand are discussed in [19]. Examples include: the p-median problem in a square [5]; the p-center problem in an area [18]; the competitive location problem in a square [9]; The p-dispersion problem in a square [3]; competitive location problems in the plane [4].

In the continuous hub problem, two hub selection models are considered. In the first model, each customer who travels between two airports, will fly to the closest hub, then fly to the closest hub to his destination and then travel to his destination. The distance between two hubs may have a different weight than the distance between an airport and a hub because the operating cost for travel between hubs should be lower by the economy of scale. Many former studies [2, 6, 7, 10, 11, 12, 13, 14] adopt this model. This is an appropriate model for the analysis of a situation in a large area in order to estimate the best number of hubs to be selected. The actual selection should be done at a second phase using an exact data set and a more elaborate optimal or heuristic approach. The present study also gives an insight about the expected distribution of hubs. This may help in determining good heuristic solutions for the exact discrete problem. This formulation can also be used to establish a system of airports (each of which is a hub) to serve an area of customers. The travel to and from an airport should have a significantly different weight for the distance because of the difference in speed of travel between the airports as compared with travel to the airport.

In the second model it is assumed that a customer flies to a hub and from that hub he flies directly to his destination. Customers use only one hub on their way to their destination. In this model intra hubs distances are irrelevant to the model. Each customer selects the hub that minimizes the total distance to that hub and from there to his destination. It is a new hub location model proposed in [15]. In this model, the airline companies also save many routes because they have to maintain only routes to hubs and not between any two points. Customers should like this arrangement because they have to change airplanes only once to get to their destination.

In reality both models of operations exist, and also direct flights that do not involve hubs also operate. In order to solve the hub problem, all existing flights that do not involve hubs should be taken out from the demand to fly between two airports and the remaining demand should be satisfied by selecting the appropriate subset of hubs.

The discrete version of the second model is presented in [15]. In this paper we analyze both models when demand is continuous in an area.

2. The First Model

2.1. Problem formulation

The following notations are needed for the formulation of the models. These notations are used in both models.

S : the study area,(For simplicity, the study area has an area of 1.)
n : the number of hubs
X_i : the location of hub number i for $i = 1, \ldots, n$. $X_i = (x_i, y_i)$
$d(X, Y)$: the Euclidean distance between points X and Y
K : the weight for the intra-hub distance
V_i : the Voronoi region of X_i
$|V_i|$: the area of the Voronoi region V_i
W_{ij} : the boundary between V_i and V_j (a segment or an empty set)
L_{ij} : the length of W_{ij} (may be zero).

The Voronoi region of X_i is the set of all points in the plane for which X_i is the closest hub. In this case, because the customers spread only in S, we consider the intersection of the Voronoi region and S. So every Voronoi region is a polygon [8]. Since all the customers access to the nearest hub, the territory of each hub is its Voronoi region. A point on the boundary between two Voronoi regions may arbitrarily access any of the equally distant hubs. It is usually assumed that these customers split their services between the equally distant hubs. However, since our problem is continuous and the measure of all the points on the boundary is zero, their choice does not influence the calculation of the travel patterns of customers and the evaluation of the total cost.

Using these properties, our objective function, the average travelling distance of customers, is calculated as follows: Since the total area of S is assumed one, the probability that a trip is originated at V_i is $|V_i|$. The probability that a trip is originated at V_i and terminates at V_j is $|V_i| \cdot |V_j|$. The average distance for all customers in S is therefore:

$$\sum_{i=1}^{n}\sum_{j=1}^{n}\left[\frac{\int_{V_i} d(X,X_i)dxdy}{|V_i|} + Kd(X_i,X_j) + \frac{\int_{V_j} d(X,X_j)dxdy}{|V_j|}\right]|V_i|\cdot|V_j| \qquad (2.1)$$

The first and third terms in equation (2.1) represent the average distance from a point in the Voronoi region to its own hub. Expanding the terms in (2.1) and recalling that the sum of the areas is one, leads to the following average distance to be minimized:

$$f(X_1,\ldots,X_n) = 2\sum_{i=1}^{n}\int_{V_i} d(X,X_i)dxdy + K\sum_{i\neq j} d(X_i,X_j)|V_i|\cdot|V_j| \qquad (2.2)$$

2.2. Calculating the average distance

In the following we calculate the derivative of $f(X_1,\ldots,X_n)$ by x_k for some k. Similar equations exist for the derivative by y_k. When x_k is changed, the boundary of the Voronoi diagram changes. In order to calculate the derivative of $f(X_1,\ldots,X_n)$ by x_k for some k, a few particular formulas are required. The first formula is straight forward:

$$\frac{\partial}{\partial x_i}d(X_i,X_j) = \frac{x_i - x_j}{d(X_i,X_j)} \qquad (2.3)$$

We now turn to calculating the change in the area of V_i by moving hub X_k for $k \neq i$. Suppose that X_k is moved a distance ϵ towards X_i. The segment W_{ik} (if it is not empty) moves a distance of $\frac{\epsilon}{2}$ because it is the perpendicular bisector between X_i and X_k. The area of V_i is reduced by $\frac{\epsilon L_{ik}}{2}$ where L_{ik} is the length of W_{ik}. This entails the relationship

$$\frac{\partial|V_i|}{\partial x_k}\frac{x_i - x_k}{d(X_i,X_j)} + \frac{\partial|V_i|}{\partial y_k}\frac{y_i - y_k}{d(X_i,X_j)} = -\frac{L_{ik}}{2} \qquad (2.4)$$

Now suppose that X_k is moved perpendicular to the direction toward X_i. The change in the area is now zero. This leads to the relationship

$$\frac{\partial|V_i|}{\partial x_k}\frac{y_i - y_k}{d(X_i,X_k)} - \frac{\partial|V_i|}{\partial y_k}\frac{x_i - x_k}{d(X_i,X_k)} = 0 \qquad (2.5)$$

Solving equations (2.4) and (2.5) leads to:

$$\frac{\partial|V_i|}{\partial x_k} = -\frac{L_{ik}(x_i - x_k)}{2d(X_i,X_k)} \quad \text{for } k \neq i \qquad (2.6)$$

In order to calculate the derivative of the area of V_i by x_i recall that the sum of all areas is constant and therefore

$$\sum_{k=1}^{n}\frac{\partial|V_i|}{\partial x_k} = 0$$

By substituting Equation (2.6):

$$\frac{\partial|V_i|}{\partial x_i} = \sum_{k\neq i}\frac{L_{ik}(x_i - x_k)}{2d(X_i,X_k)} \qquad (2.7)$$

For the following calculations the function $\phi(\alpha)$ is defined:

$$\phi(\alpha) = \int_o^\alpha \sqrt{1+x^2}\,dx = \frac{\ln[\sqrt{1+\alpha^2}+\alpha] + \alpha\sqrt{1+\alpha^2}}{2} \tag{2.8}$$

which leads to

$$\int_0^\alpha \sqrt{\beta^2+x^2}\,dx = \beta^2\phi(\frac{\alpha}{\beta}) \tag{2.9}$$

We now turn to calculate $\frac{\partial}{\partial x_k}\int_{V_i} d(X,X_i)dxdy$ for $k \neq i$. Equations similar to equations (2.4) and (2.5) are constructed. The distance $d(X,X_i)$ is independent of x_k. When X_k is moved a distance ϵ towards X_i, the segment W_{ik} (if it is not empty) moves a distance of $\frac{\epsilon}{2}$ because it is the perpendicular bisector between X_i and X_k. The integration region is reduced by a strip of length L_{ik} and width $\frac{\epsilon}{2}$. The integral is therefore reduced by the integral of $d(X,X_i)$ over that strip. The distance between X_i and the center of the strip is $\frac{d(X_i,X_k)}{2}$ because the strip is the perpendicular bisector between X_i and X_k. The reduction in the integral is therefore by equation (2.9):

$$\frac{\epsilon}{2}\int_{-\frac{L_{ik}}{2}}^{\frac{L_{ik}}{2}} \sqrt{\left[\frac{d(X_i,X_k)}{2}\right]^2 + z^2}\,dz = \frac{\epsilon}{4}\left[d(X_i,X_k)\right]^2\phi\left(\frac{L_{ik}}{d(X_i,X_k)}\right)$$

where $\phi(\alpha)$ is defined by (2.8). Similarly to the construction of equation (2.6) from equations (2.4) and (2.5) we get for this case:

$$\frac{\partial}{\partial x_k}\int_{V_i} d(X,X_i)dxdy = -\frac{d(X_i,X_k)}{4}(x_i-x_k)\phi\left(\frac{L_{ik}}{d(X_i,X_k)}\right) \quad \text{for } k \neq i \tag{2.10}$$

To calculate $\frac{\partial}{\partial x_i}\int_{V_i} d(X,X_i)dxdy$ observe that on the segment W_{ik}: $d(X,X_i) = d(X,X_k)$ because it is the perpendicular bisector. Therefore,

$$\frac{\partial}{\partial x_i}\int_{V_i} d(X,X_i)dxdy = \sum_{k\neq i}\frac{d(X_i,X_k)}{4}(x_i-x_k)\phi\left(\frac{L_{ik}}{d(X_i,X_k)}\right) + \int_{V_i}\frac{x_i-x}{d(X,X_i)}dxdy \tag{2.11}$$

The above equations provide all the necessary derivatives needed to calculate the derivative of $f(X_1,\ldots,X_n)$ in equation (2.2):

$$\begin{aligned}\frac{\partial}{\partial x_k}f(X_1,\ldots,X_n) &= 2\sum_{i=1}^{n}\frac{\partial}{\partial x_k}\int_{V_i} d(X,X_i)dxdy + 2K\sum_{i\neq k}\frac{x_k-x_i}{d(X_i,X_k)}|V_i|\cdot|V_k| \\ &+ 2K\sum_{i=1}^{n}\sum_{j=1}^{n} d(X_i,X_j)|V_i|\frac{\partial}{\partial x_k}|V_j|\end{aligned} \tag{2.12}$$

It is recommended that the Voronoi diagrams should be calculated using the error-free Voronoi diagram program [16, 17].

2.3. Examples

First we find a simpler formula for the case of symmetric patterns where hubs are moved simultaneously to retain symmetry and therefore the Voronoi regions remain unchanged. When the Voronoi regions V_i remain unchanged, equation (2.12) simplifies to:

$$\frac{\partial}{\partial x_k} f(X_1, \ldots, X_n) = 2 \int_{V_k} \frac{x_k - x}{d(X, X_k)} dx dy + 2K \sum_{i \neq k} \frac{x_k - x_i}{d(X_i, X_k)} |V_i| \cdot |V_k| \tag{2.13}$$

Example 1: Two hubs in a square (the axis case)

The square is centered at $(0,0)$, and is defined as $-0.5 \leq x, y \leq 0.5$. Let the two hubs be symmetrically located on the x-axis at $(-a, 0)$ and $(a, 0)$. Each Voronoi region has an area of 0.5. By equation (2.13):

$$\frac{\partial f}{\partial x_2} = 2 \int_{-0.5}^{0.5} \int_{0}^{0.5} \frac{(a-x) dx dy}{\sqrt{(x-a)^2 + y^2}} + \frac{K}{2} = -2 \int_{-0.5}^{0.5} [\sqrt{(0.5-a)^2 + y^2} - \sqrt{a^2 + y^2}] dy + \frac{K}{2} \tag{2.14}$$

The optimal value of a is determined by the equation (using equation (2.9)):

$$K = 8 \int_{0}^{0.5} \left[\sqrt{(0.5-a)^2 + y^2} - \sqrt{a^2 + y^2} \right] dy = 8 \left[(0.5-a)^2 \phi\left(\frac{1}{1-2a}\right) - a^2 \phi\left(\frac{1}{2a}\right) \right] \tag{2.15}$$

where $\phi(\alpha)$ is defined by (2.8). Also note that $\lim_{a \to 0} a^2 \phi(\frac{\gamma}{a}) = \frac{\gamma^2}{2}$. For $a = 0$, $K = 2\phi(1) - 1 = 1.2956$. Therefore, if $K > 1.2956$ the two hubs are located at the center and only for $K < 1.2956$ two separate hub locations are optimal. For $K = 1$, $a = 0.06808$.

Example 2: Two hubs in a square (the diagonal case)

The square, with area of 1, is centered at $(0,0)$. The square is rotated by 45° so it is easier to construct the equations. The square is defined by the four equations: $\pm x \pm y \leq \frac{\sqrt{2}}{2}$. Let the two hubs be symmetrically located on the x-axis at $(-a, 0)$ and $(a, 0)$. Each Voronoi region has an area of 0.5. By equation (2.13):

$$\begin{aligned} \frac{\partial f}{\partial x_2} &= 4 \int_{0}^{\frac{\sqrt{2}}{2}} \int_{0}^{\frac{\sqrt{2}}{2}-y} \frac{(a-x) dx dy}{\sqrt{(x-a)^2 + y^2}} + \frac{K}{2} \\ &= -4 \int_{0}^{\frac{\sqrt{2}}{2}} \left[\sqrt{(\frac{\sqrt{2}}{2} - y - a)^2 + y^2} - \sqrt{a^2 + y^2} \right] dy + \frac{K}{2} \end{aligned} \tag{2.16}$$

The first integral can also be expressed in terms of $\phi(\alpha)$ by some algebraic manipulations:

$$\int_{0}^{\alpha} \sqrt{(\beta - x)^2 + x^2}\, dx = \frac{\beta^2 \sqrt{2}}{4} \left[\phi\left(\frac{2\alpha - \beta}{\beta}\right) + \phi(1) \right]$$

The optimal value of a is determined by the equation:

$$\begin{aligned} K &= 8 \int_{0}^{\frac{\sqrt{2}}{2}} \left[\sqrt{(\frac{\sqrt{2}}{2} - y - a)^2 + y^2} - \sqrt{a^2 + y^2} \right] dy \\ &= 2\sqrt{2} \left(\frac{\sqrt{2}}{2} - a\right)^2 \left[\phi\left(\frac{1 + a\sqrt{2}}{1 - a\sqrt{2}}\right) + \phi(1) \right] - 8a^2 \phi\left(\frac{\sqrt{2}}{2a}\right) \end{aligned} \tag{2.17}$$

Table 1: Optimal solutions for two hubs in a square

	Axis Location		Diagonal Location	
K	a	$D(a)$	a	$D(a)$
0.0	0.250	0.593	0.249	0.602
0.1	0.232	0.617	0.230	0.626
0.2	0.215	0.640	0.211	0.648
0.3	0.198	0.660	0.193	0.669
0.4	0.180	0.679	0.175	0.687
0.5	0.163	0.696	0.156	0.704
0.6	0.145	0.712	0.137	0.718
0.7	0.126	0.725	0.118	0.731
0.8	0.108	0.737	0.099	0.742
0.9	0.088	0.747	0.079	0.751
1.0	0.068	0.755	0.058	0.758
1.1	0.047	0.760	0.036	0.762
1.2	0.024	0.764	0.012	0.765

where $\phi(a)$ is defined by (2.8). For $a = 0$ $K = 2\sqrt{2}\phi(1) - 2 = 1.2465$, and for $K = 1$ $a = 0.0582$.

The two cases were solved for various values of K. In all cases the solution on the axis is slightly better than the solution when the hubs are located on the diagonal. The results are presented in Table 1.

Example 3: Three hubs in a square

We also experimented with the case of three hubs located in a square. This example requires the use of the complete formula (2.12) because the Voronoi diagrams change when the locations of the hubs change. The locations of the hubs for various values of K are summarized in Table 2 and depicted in Figure 1.

Note that for $K \geq 1.3$ the solution is to locate all three hubs at the center of the square yielding a distance of 0.7652.

Example 4: Two hubs in a rectangle

Assume a rectangle with length b and height $1/b$ for $b \geq 1$. Let the two hubs be located at $(-a, 0)$ and $(a, 0)$. Then the equivalent equation (2.14) is:

$$\frac{\partial f}{\partial x_2} = 2\int_{-\frac{1}{2b}}^{\frac{1}{2b}}\int_{0}^{\frac{b}{2}} \frac{(a-x)dxdy}{\sqrt{(x-a)^2+y^2}} + \frac{K}{2} = -2\int_{-\frac{1}{2b}}^{\frac{1}{2b}} \left[\sqrt{(\frac{b}{2}-a)^2+y^2} - \sqrt{a^2+y^2}\right] dy + \frac{K}{2} \tag{2.18}$$

The optimal value of a is determined by the equation:

$$K = 8\int_{0}^{\frac{1}{2b}} \left[\sqrt{(\frac{b}{2}-a)^2+y^2} - \sqrt{a^2+y^2}\right] dy = 8\left[\left(\frac{b}{2}-a\right)^2 \phi\left(\frac{1}{b^2-2ab}\right) - a^2\phi\left(\frac{1}{2ab}\right)\right] \tag{2.19}$$

Since $\phi(x) \approx x$ for a small x, this equation reduces to $a = \frac{(2-K)b}{8}$ for a large b. We also find the largest K, $K_{\max}$, for which the solution consists of two different hubs. This

Table 2: Location of three hubs in a square

K	Hub #1		Hub #2		Hub #3		Dist-
	x	y	x	y	x	y	ance
0.0	.5000	.7909	.2341	.3095	.7659	.3095	.4712
0.1	.5000	.7653	.2583	.3080	.7417	.3080	.5155
0.2	.5000	.7415	.2821	.3100	.7179	.3100	.5562
0.3	.5000	.7192	.3064	.3133	.6936	.3133	.5932
0.4	.5000	.6981	.3309	.3197	.6691	.3197	.6266
0.5	.5000	.6766	.3553	.3265	.6447	.3265	.6565
0.6	.5000	.6567	.3794	.3374	.6206	.3374	.6828
0.7	.5000	.6386	.4042	.3543	.5958	.3543	.7054
0.7	.5000	.6386	.4042	.3543	.5958	.3543	.7054
0.8	.5000	.6186	.4276	.3718	.5724	.3718	.7245
0.9	.5000	.5971	.4496	.3923	.5504	.3923	.7398
1.0	.5000	.5759	.4696	.4183	.5306	.4183	.7514
1.1	.5000	.5503	.4859	.4453	.5141	.4453	.7594
1.2	.5000	.5256	.4963	.4732	.5037	.4732	.7639

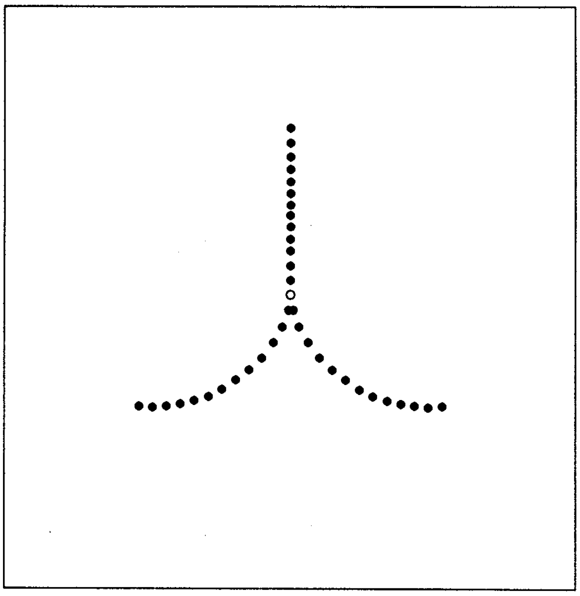

Figure 1: Location of three hubs in a square for different values of K
$\bullet$: hub location, $\circ$: center of square

maximal K is obtained for $a = 0$:

$$K_{\max} = b^2 \ln\left\{\frac{\sqrt{1+b^4}+1}{b^2}\right\} + \frac{\sqrt{1+b^2}-1}{b^2} \tag{2.20}$$

The limit of $K_{\max}$ as b goes to infinity is "2".

3. The Second Model

3.1. Problem formulation

Assume that each customer will use only one hub to get to its destination. That means, that one of the n hubs is selected by the customer, and the customer travels to that hub and then travels from that hub to his destination. The objective in selecting a hub is to minimize travel, i.e. the sum of distances from the customer's location to the hub and from the hub to its destination is minimized. Consider first m customers located at $A_i = (a_j, b_j)$ for $j = 1, \ldots m$. In this case, the objective function to be minimized is:

$$F(X_1, \cdots, X_n) = \sum_{s=1}^{m}\sum_{t=1}^{m} \min_{1\le i\le n} \{d(X_i, A_s) + d(X_i, A_t)\} \tag{3.1}$$

Note that by this formulation a customer cannot travel directly to its destination but must use a hub airport. When a continuous area is involved, the sums are replaced by integrals and the analysis is similar to that of the first model. In the following we consider the two-hub case. Two cases in a square are analyzed: hubs located on an axis, and hubs located on a diagonal. The square is centered at (0,0). The last case is that hubs are located in a rectangle.

3.2. The case of hubs located on the axis

When the hubs are on an axis, the sides of the square are parallel to the axes and the two hubs are located at $(-a, 0)$ and $(a, 0)$. Consider a customer located at (u, v) for $u, v \geq 0$. The other three quadrants offer similar results. For some destinations the customer will use the "closer" hub at $(a, 0)$ and for some others the customer will use the "farther" hub $(-a, 0)$. The boundary between the destinations that use the closer hub and the destinations that use the farther hub is determined by the following equation. Assume a destination (x, y). The boundary is the solution to:

$$\sqrt{(u-a)^2+v^2} + \sqrt{(x-a)^2+y^2} = \sqrt{(u+a)^2+v^2} + \sqrt{(x+a)^2+y^2} \tag{3.2}$$

Let $2\Delta = \sqrt{(u+a)^2+v^2} - \sqrt{(u-a)^2+v^2}$. Note that by the triangle inequality $\Delta \leq a$. This leads to:

$$\frac{x^2}{\Delta^2} - \frac{y^2}{a^2-\Delta^2} = 1 \tag{3.3}$$

The calculation of the average travel distance for a given a, u, and v (and consequently a given Δ) is now presented. In the area bounded by the hyperbola (3.3) (that includes the point $(-a, 0)$) distances to $(-a, 0)$ should be used (because the route via hub $(-a, 0)$ is shorter than the route via hub $(a, 0)$), and outside this hyperbolic area distances to $(a, 0)$ should be used. It is easier to integrate using the distances to $(a, 0)$ over the complete square, and using the *difference* between the distances to $(a, 0)$ and $(-a, 0)$ when integrating over the hyperbolic region. This leads to the following formula for the average distance $D(a, u, v)$ for a given a, positive quadrant (u, v) and consequently a given Δ:

$$\begin{aligned} D(a,u,v) &= 2\int_{-0.5}^{0.5}\int_{0}^{0.5}\left[\sqrt{(x-a)^2+y^2} + \sqrt{(u-a)^2+v^2}\right]dydx \\ &+ 2\int_{-0.5}^{-\Delta}\int_{0}^{\min\{0.5,\frac{\sqrt{(x^2-\Delta^2)(a^2-\Delta^2)}}{\Delta}\}}\left[\sqrt{(x+a)^2+y^2}+\sqrt{(u+a)^2+v^2}\right. \\ &- \left.\sqrt{(x-a)^2+y^2}-\sqrt{(u-a)^2+v^2}\right]dydx \end{aligned}$$

which leads to:

$$\begin{aligned} D(a,u,v) &= \sqrt{(u-a)^2+v^2} + 2\int_{-0.5}^{0.5}\int_{0}^{0.5}\sqrt{(x-a)^2+y^2}\,dydx \\ &+ 2\int_{-0.5}^{-\Delta}\int_{0}^{\min\{0.5,\frac{\sqrt{(x^2-\Delta^2)(a^2-\Delta^2)}}{\Delta}\}}\left[\sqrt{(x+a)^2+y^2} - \sqrt{(x-a)^2+y^2} + 2\Delta\right]dydx \end{aligned} \tag{3.4}$$

The average distance, $D(a)$, is obtained by integrating over all positive (u, v):

$$D(a) = 4\int_{0}^{0.5}\int_{0}^{0.5} D(a,u,v)dudv \tag{3.5}$$

For $a = 0$ the formula is simplified because the last integral in (3.4) vanishes and:

$$\begin{aligned} D(0) &= 8\int_{0}^{0.5}\int_{0}^{0.5}\sqrt{x^2+y^2}dxdy = 2\int_{0}^{1}\int_{0}^{1}\sqrt{x^2+y^2}dxdy \\ &= 4\int_{0}^{\frac{\pi}{4}}\int_{0}^{\frac{1}{\cos\theta}} r^2drd\theta = \frac{4}{3}\int_{0}^{\frac{\pi}{4}}\frac{d\theta}{\cos^3\theta} = \frac{\ln(\sqrt{2}+1)+\sqrt{2}}{3} = 0.7652. \end{aligned} \tag{3.6}$$

The average distance as a function of a is depicted in Figure 2. We used Gaussian quadrature formulas based on Legendre polynomials with 20 points for the outer three dimensions, and used 48 integration points for the inner dimension (y) [1]. The total number of integration points is therefore $20^3 \cdot 48 = 384,000$. Obtaining one value of $D(a)$ took about 10 seconds on a 486 IBM compatible computer.

3.3. The case of hubs located on the diagonal

As before, the square, with area of 1, is centered at $(0, 0)$. The square is rotated by 45° so it is easier to construct the equations. The square is defined by the four equations: $\pm x \pm y \leq \frac{\sqrt{2}}{2}$. Let the two hubs be symmetrically located on the x-axis at $(-a, 0)$ and $(a, 0)$.

Very similar derivations leading to equations (3.4-3.5) lead to:

$$D(a,u,v) = \sqrt{(u-a)^2+v^2} + 2\int_{-\sqrt{0.5}}^{\sqrt{0.5}}\int_{0}^{\sqrt{0.5}-|x|}\sqrt{(x-a)^2+y^2}\,dydx \tag{3.7}$$

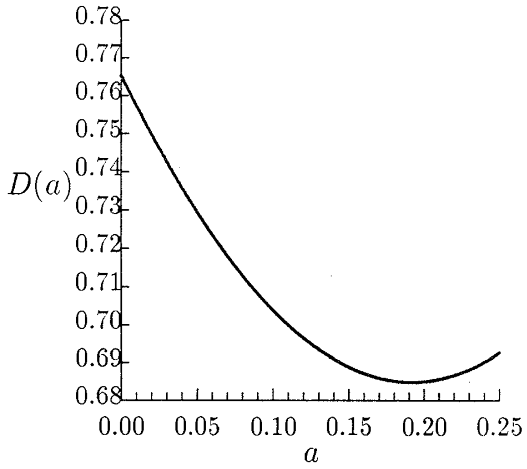

Figure 2: The average distance $D(a)$ as a function of a in the axis case

$$+2\int_{-\sqrt{0.5}}^{-\Delta}\int_{0}^{\min\{\sqrt{0.5}-|x|,\frac{\sqrt{(x^2-\Delta^2)(a^2-\Delta^2)}}{\Delta}\}}\left[\sqrt{(x+a)^2+y^2}-\sqrt{(x-a)^2+y^2}+2\Delta\right]dydx$$

The average distance, $D(a)$, is obtained by integrating over all positive (u, v):

$$D(a) = 4\int_{0}^{\sqrt{0.5}}\int_{0}^{\sqrt{0.5}-v} D(a,u,v)dudv \tag{3.8}$$

We found the optimal solution for in this case numerically. The optimal location was at a distance of 0.1949 from the center with a minimal average distance of 0.6820. This average distance is better than the average distance of 0.6846 that was obtained for the axis location of hubs. This confirms the result in [15] where it is reported that a solution to 100 and 200 airports randomly selected in a square was on the diagonal of the square at a distance very close to the optimal location obtained here.

3.4. The rectangle case

Assume that the area is a rectangle of length b and height $\frac{1}{b}$. The equations can be easily modified to finding the best location for the hub. In Table 3 we report the best location for the hub (the minimal point in Figure 2, but for various values of b). Since the best location and the minimal distance are quite proportional to b, we report in Table 3 those values divided by b. The minimal average distance is about 40% of the rectangle's length for long and narrow rectangles. The best locations for the hubs are about 20% of the rectangle's length from the center.

We approximated the United States by a rectangle and found the locations for two hubs (20% of the length of the rectangle off its center). See Figure 3. The rectangle was superimposed on the map by free-hand drawing and no calculations were used for its determination. The locations are very close to Indianapolis, Indiana, and quite close to Denver, Colorado (a little to the west of it). United Airline has four major hub airports in the United States. These are San Francisco, Denver, Chicago and Washington D. C. Considering San Francisco for transpacific routes and Washington D. C. for transatlantic route, their hub airports are quite near to our results obtained above. (Chicago is approximately 300 kilometers NNW of Indianapolis.)

Table 3: Best location and minimal distance for a rectangle

b	$\frac{a}{b}$	$\frac{D(a)}{b}$
1.0	0.1917	0.6846
1.1	0.1929	0.6159
1.2	0.1941	0.5663
1.3	0.1951	0.5297
1.4	0.1962	0.5023
1.5	0.1974	0.4814
2.0	0.2009	0.4278
2.5	0.2021	0.4087
3.0	0.2023	0.4005
4.0	0.2008	0.3943
5.0	0.1987	0.3923
6.0	0.1969	0.3914
8.0	0.1948	0.3908
10.0	0.1939	0.3906
20.0	0.1932	0.3904

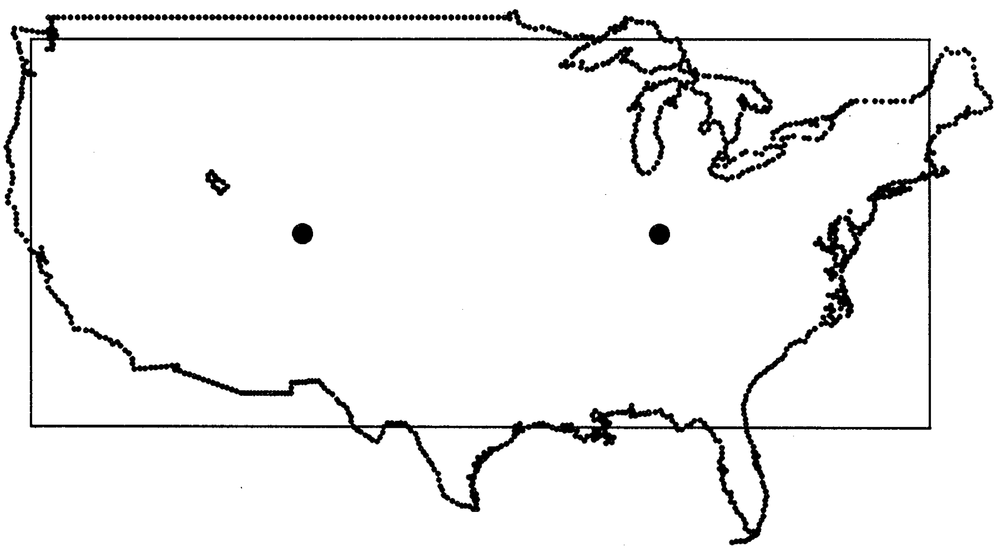

Figure 3: Two Hubs Approximately Located in the United States
• : a hub airport

4. Conclusions

The hub selection problem when demand is evenly spread in a given area is formulated and solved. Two formulations are considered. One model assumes that a customer will use two hubs, the one closest to him and the one closest to his destination. A customer will travel to the closest hub, then to the other hub and then to his destination. The second model assumes that a customer selects only one hub, travels to that hub and then to his destination. The customer selects the hub that provides the shortest total distance to his destination.

The problem of two hubs to be selected in a square are analyzed. For the first model, the best location for the two hubs is on the axis parallel to the square's side. For the second model the best location is on the diagonal of the square. Both models were also analyzed for a rectangle, and the best axis location is found as a function of the shape of that rectangle. For the first model, three-hub case is also analyzed. The best hub location varied according to the factor of economy of scale.

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Atsuo Suzuki
Department of Information Systems & Quantitative Sciences
Nanzan University
Nagoya, Aichi, 466 Japan
email: atsuo@iq.nanzan-u.ac.jp

Zvi Drezner
Department of Management Science/Information Systems
California State University-Fullerton
Fullerton, CA 92634
email: drezner@fullerton.edu