

PREEMPTIVE RESUME PRIORITIES IN AN N-CLASS STRUCTURED QUEUE WITH SERVER VACATIONS

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Abstract A queue with preemptive resume priorities between customers from n different classes, is studied. The customers arrive in batches of arbitrary size according to a Poisson process and the service times are arbitrarily distributed with a different distribution for each class. The server finally takes multiple vacations each time the system becomes empty. For this model, the Laplace transforms of the joint distributions of the system states and the elapsed service times of the customer in service and the customers in limbo are obtained, both in a transient state and in the steady state. Similar results are also derived for the model without vacations, and the mean performance measures for both models are extracted. These measures are finally compared with the corresponding measures for models with nonpreemptive priority and useful relationships are obtained which provide insight into the impact of the various strategies on the performance measures.

1. Introduction

In this work we study a single server queue with batch arrivals and multiple server vacations, accepting n different classes of customers P_i $i = 1, 2, \dots, n$ which can arrive in the same batch. For all $i < j$ the P_i customer has always preemptive resume priority for service in front of all P_j customers. Finally the customers are served one by one according to arbitrary distributions.

Queueing systems of such kind (structured priority queues) have been proved useful particularly to model communication systems, telephone switching systems etc. Practical examples of such situations can be found in Sidi and Segall [11,12] and in Takahashi and Takagi [18].

The first who studied a priority system with batch arrivals and two classes of customers arriving separately in different batches are Gaver [4] and Hawkes [5]. Jaiswal [6,7] and more recently Kella and Yechiali [9] analyzed priority systems with the customers to arrive one by one and not in batches. Sidi and Segall [11,12] studied a discrete-time structured priority queue and Takahashi and Takagi [18] analyzed a model similar to our model here, but with only two classes of customers and without vacations. Stuck and Arthurs [13] obtained a formula for the steady state waiting time in the structured priority model in continuous time. Takagi and Takahashi [15], using the delay-cycle approach, derived the individual waiting time and completion time distributions in the multiclass model with batch inputs and nonpreemptive (HL) or preemptive resume (PR) priorities. The results of [18], [13] and [15] may also be found in section 3.5 of Takagi [16]. Takahashi and Shimogawa [19] using a decomposition result and an approach based again on the delay cycle, analyzed the steady state waiting time process in the model with structured batch inputs and composite priorities. Finally Takahashi and Miyazawa [20], by deriving a distributional form of Little's law and using the results in [15], obtained the marginal queue length distribution for the multi-class batch Poisson arrival model with HL or PR priorities. Note here that although the combination of the results in [15] and [20] might enable one to obtain the steady state

marginal queue length distribution of the model (or any queue length moment), however this approach (the delay cycle approach) cannot be used to analyze the time dependent (or the steady state) joint queue length distribution. The aim of the work here is to derive these joint queue length distributions for our model, by using the supplementary variable technique. Langaris and Katsaros [10] are the first who studied, using the imbedded Markov chain technique, the joint system state probabilities and the busy period of a structured priority queue accepting an arbitrary number of customer classes with nonpreemptive priorities between them and without vacations. In a second work (Katsaros and Langaris [8]) the authors extended their results studying the state probabilities and the unfinished work for the model with multiple vacations but again under a nonpreemptive priority rule.

There has been an increasing interest recently in queueing systems with server vacations. For a survey of the earliest works on the subject see Doshi [2]. We have also to mention here the work of Baba [1] and more recently the works of Takagi [14,17].

In this paper and after the description of the model, we study in section 3, using the supplementary variable technique (See also Takahashi and Takagi [18]) the joint distribution of the system states and the elapsed service times of the customer in service and the customers in limbo, both in a transient state and in the steady state. Similar results for the model without vacations are derived in section 4 and the corresponding formulae are compared. Finally, in section 5, explicit relations for the mean number of customers, in both models, are obtained and used for comparisons and numerical calculations.

We have to point out here the difficulties that arise in the time-dependent analysis of the model with preemptive priorities. For such a model, at any time t , one has to deal not only with a customer, of type i say, in service but also at the same time, with some customers from low priority classes (type $j > i$ customers), which are in limbo with different elapsed service times for each one of them. Such kind of problems did not arise for the model of nonpreemptive priorities in which case the analysis is simpler.

2. The Model

Customers arrive in a single server queueing model according to the Poisson distribution with parameter λ , in batches of random size. Each batch contains customers of different types $P_i, i = 1, 2, \dots, n$, and a P_i customer has always preemptive resume priority for service in front of all P_j customers, for all $j > i$.

Denote by $X_i, i = 1, 2, \dots, n$, the number of P_i customers in an arbitrary batch and define

$$b(\underline{x}) = \Pr[X_1 = x_1, X_2 = x_2, \dots, X_n = x_n], \quad B(\underline{z}) = \sum_{\substack{\underline{x} \geq \underline{0} \\ \underline{z} \geq \underline{0}}} b(\underline{x}) z_1^{x_1} z_2^{x_2} \dots z_n^{x_n}$$

where in general $\underline{w} = (w_1, w_2, \dots, w_n)$.

The customers are served one by one according to an arbitrary distribution with probability density function (p.d.f.) $u_i(t)$, distribution function (D.F.) $U_i(t)$ and finite mean values \bar{u}_i and k^{th} moments $\bar{u}_i^{(k)}$ for the P_i class respectively.

We assume further that each time the system becomes empty the server takes a vacation of random length U_0 following an arbitrary distribution with p.d.f. $u_0(t)$, D.F. $U_0(t)$ and finite mean \bar{u}_0 and k^{th} moment $\bar{u}_0^{(k)}$. If the server finds no customers waiting after the end of a vacation, he begins another vacation and so on until he finds at least one customer waiting in the system. All processes so far considered are assumed to be independent.

Let $\mathbf{r}_i^k = (r_{i1}^k, r_{i2}^k, \dots, r_{ik}^k)$ be a size k ordered subset of the set $(i+1, i+2, \dots, n)$ for all $i = 1, 2, \dots, n-1, k = 1, 2, \dots, n-i$. To avoid heavy notation through the work we will use, for all $1 \leq i \leq n-1, 1 \leq k \leq n-i, 1 \leq m \leq k$, the following symbolism

$$\mathbf{r}_{im}^k = (r_{im}^k, r_{im+1}^k, \dots, r_{ik}^k), \quad \bar{\mathbf{r}}_i^k = (i, r_{i1}^k, r_{i2}^k, \dots, r_{ik}^k)$$

$$\begin{aligned} \tilde{x}_{\mathbf{r}_i^k} &= (x_{r_{i1}^k}, x_{r_{i2}^k}, \dots, x_{r_{ik}^k}), & \tilde{x}_{\bar{\mathbf{r}}_i^k} &= (x, x_{r_{i1}^k}, x_{r_{i2}^k}, \dots, x_{r_{ik}^k}) \\ \tilde{u}_i &= (0, 0, \dots, 0, \underset{\substack{\uparrow \\ \text{ith position}}}{1}, 0, \dots, 0), & \tilde{u}_{\mathbf{r}_i^k} &= (0, \dots, 0, \underset{\substack{\uparrow \\ r_{i1}^k}}{1}, 0, \dots, 0, \underset{\substack{\uparrow \\ r_{i2}^k}}{1}, 0, \dots, 0, \underset{\substack{\uparrow \\ r_{ik}^k}}{1}, 0, \dots, 0) \end{aligned}$$

and also for $i = 1, 2, \dots, n$

$$\begin{aligned} \tilde{k}_i &= (k_i, k_{i+1}, \dots, k_n), & \tilde{k}_i^* &= (\underbrace{0, 0, \dots, 0}_{i-1}, k_i, k_{i+1}, \dots, k_n) \\ \tilde{z}_i &= (z_i, z_{i+1}, \dots, z_n), & \tilde{z}_{im} &= (\underbrace{0, 0, \dots, 0}_{m-i}, z_m, z_{m+1}, \dots, z_n), & \tilde{z}_i^* &\equiv \tilde{z}_{1i} \\ \tilde{z}_i^{k_i} &= z_i^{k_i} z_{i+1}^{k_{i+1}} \dots z_n^{k_n}, & d \tilde{x}_{\mathbf{r}_i^k} &= dx_{r_{i1}^k} dx_{r_{i2}^k} \dots dx_{r_{ik}^k}. \end{aligned}$$

3. System state probabilities

Consider the vector $\tilde{\xi}(t) = (\xi_1(t), \xi_2(t), \dots, \xi_n(t))$, where $\xi_i(t)$ represents the number of P_i customers in the system at time t and define

$$\begin{aligned} q(t) &= \begin{cases} 0 & \text{if the server is on vacation at } t, \\ i & \text{if a } P_i \text{ customer is served at } t, \end{cases} \\ \mathbf{v}(t) &= \begin{cases} \mathbf{r}_i^k = (r_{i1}^k, r_{i2}^k, \dots, r_{ik}^k) & \text{if } k \text{ customers of priorities } r_{i1}^k, r_{i2}^k, \dots, r_{ik}^k \\ & \text{are in limbo (were preempted earlier) at } t, \\ 0 & \text{if no customer is in limbo at time } t. \end{cases} \end{aligned}$$

Let also

$$\begin{aligned} p_i(\tilde{k}_i^*, x, t) dx &= \Pr[q(t) = i, \mathbf{v}(t) = 0, \tilde{\xi}(t) = \tilde{k}_i^*, x < U_i^*(t) \leq x + dx] \\ (3.1) \quad p_{i\mathbf{r}_i^k}(\tilde{k}_i^*, x, \tilde{x}_{\mathbf{r}_i^k}, t) dx d \tilde{x}_{\mathbf{r}_i^k} &= \Pr[q(t) = i, \mathbf{v}(t) = \mathbf{r}_i^k, \tilde{\xi}(t) = \tilde{k}_i^*, x < U_i^*(t) \leq x + dx, \\ & \quad x_{r_{im}^k} < U_{r_{im}^k}^*(t) < x_{r_{im}^k} + dx_{r_{im}^k}, \quad m = 1, 2, \dots, k] \end{aligned}$$

where $U_0^*(t)$, $U_i^*(t)$ is the elapsed vacation time and the elapsed service time of the customer in service or in limbo at time t respectively. By defining,

$$\tilde{z}_0 \equiv \tilde{z}_1 = (z_1, z_2, \dots, z_n), \quad n_i(x) = \frac{u_i(x)}{1 - U_i(x)} \quad i = 0, 1, 2, \dots, n,$$

and the generating functions

$$\begin{aligned} P_i(\tilde{z}_i, x, t) &= \sum_{\substack{\tilde{k}_i^* \geq \tilde{u}_i \\ k_i^* \geq u_i}} p_i(\tilde{k}_i^*, x, t) \tilde{z}_i^{k_i} \quad i = 0, 1, 2, \dots, n \\ (3.2) \quad P_{i\mathbf{r}_i^k}(\tilde{z}_i, x, \tilde{x}_{\mathbf{r}_i^k}, t) &= \sum_{\substack{\tilde{k}_i^* \geq \tilde{u}_{\bar{\mathbf{r}}_i^k} \\ k_i^* \geq u_{\bar{\mathbf{r}}_i^k}}} p_{i\mathbf{r}_i^k}(\tilde{k}_i^*, x, \tilde{x}_{\mathbf{r}_i^k}, t) \tilde{z}_i^{k_i} \quad i = 1, 2, \dots, n - 1 \end{aligned}$$

$$q_{mi}(\tilde{z}_i, y, x, t) = \sum_{\substack{k_i^* \geq \tilde{z}_i \\ \tilde{z}_i}} p_{mi}(k_i^* + \tilde{u}_m, y, x, t) \tilde{z}_i^{k_i} \quad m < i$$

$$q_{m\mathbf{r}_i^k}(\tilde{z}_i, y, \tilde{x}_{\mathbf{r}_i^k}, t) = \sum_{\substack{k_i^* \geq \tilde{u}_{\mathbf{r}_i^k} \\ \tilde{z}_i}} p_{m\mathbf{r}_i^k}(k_i^* + \tilde{u}_m, y, \tilde{x}_{\mathbf{r}_i^k}, t) \tilde{z}_i^{k_i} \quad m < i,$$

we obtain, in a similar way as in Takahashi and Takagi [18],

$$\begin{aligned} \frac{\partial}{\partial t} P_0(\tilde{z}_0, x, t) + \frac{\partial}{\partial x} P_0(\tilde{z}_0, x, t) + [n_0(x) + \lambda - \lambda B(\tilde{z}_0)] P_0(\tilde{z}_0, x, t) &= 0 \\ \frac{\partial}{\partial t} P_i(\tilde{z}_i, x, t) + \frac{\partial}{\partial x} P_i(\tilde{z}_i, x, t) + [n_i(x) + \lambda - \lambda B(\tilde{z}_i^*)] P_i(\tilde{z}_i, x, t) \\ (3.3) \quad &= \sum_{m=1}^{i-1} \int_0^\infty q_{mi}(\tilde{z}_i, y, x, t) n_m(y) dy \quad i = 1, 2, \dots, n \\ \frac{\partial}{\partial t} P_{i\mathbf{r}_i^k}(\tilde{z}_i, x, \tilde{x}_{\mathbf{r}_i^k}, t) + \frac{\partial}{\partial x} P_{i\mathbf{r}_i^k}(\tilde{z}_i, x, \tilde{x}_{\mathbf{r}_i^k}, t) + [n_i(x) + \lambda - \lambda B(\tilde{z}_i^*)] P_{i\mathbf{r}_i^k}(\tilde{z}_i, x, \tilde{x}_{\mathbf{r}_i^k}, t) \\ &= \sum_{m=1}^{i-1} \int_0^\infty q_{m\mathbf{r}_i^k}(\tilde{z}_i, y, \tilde{x}_{\mathbf{r}_i^k}, t) n_m(y) dy \quad i = 1, 2, \dots, n-1, \end{aligned}$$

with boundary conditions

$$(3.4) \quad p_0(\tilde{z}_0, 0, t) = \sum_{i=0}^n \int_0^\infty p_i(\tilde{u}_i, x, t) n_i(x) dx, \quad p_0(k, 0, t) = 0 \quad k \neq \tilde{z}_0,$$

$$(3.5) \quad p_i(k_i^*, 0, t) = \sum_{m=0}^i \int_0^\infty p_m(k_i^* + \tilde{u}_m, x, t) n_m(x) dx \quad i = 1, 2, \dots, n,$$

$$(3.6) \quad \begin{aligned} p_{i\mathbf{r}_i^k}(k_i^*, 0, \tilde{x}_{\mathbf{r}_i^k}, t) &= \sum_{m=1}^i \int_0^\infty p_{m\mathbf{r}_i^k}(k_i^* + \tilde{u}_m, x, \tilde{x}_{\mathbf{r}_i^k}, t) n_m(x) dx \\ &+ \lambda \sum_{\substack{\tilde{u}_{\mathbf{r}_i^k} \leq m_{\mathbf{r}_i^k}^* \leq k_i^* \\ \tilde{z}_{\mathbf{r}_i^k} \leq \tilde{z}_{\mathbf{r}_i^k}^*}} p_{\mathbf{r}_i^k}(m_{\mathbf{r}_i^k}^*, \tilde{x}_{\mathbf{r}_i^k}, \tilde{x}_{\mathbf{r}_i^k}, t) b(k_i^* - m_{\mathbf{r}_i^k}^*) \quad i = 1, 2, \dots, n-1. \end{aligned}$$

Let us assume now that the system is empty at time $t = 0$ and the server has just started a vacation. Denote also by $P_i^*(\cdot, s)$, $P_{i\mathbf{r}_i^k}^*(\cdot, s)$, $q_{m\mathbf{r}_i^k}^*(\cdot, s)$ the Laplace transforms of $P_i(\cdot, t)$, $P_{i\mathbf{r}_i^k}(\cdot, t)$, $q_{m\mathbf{r}_i^k}(\cdot, t)$ respectively. Then from (3.3)

$$\begin{aligned} \frac{\partial}{\partial x} P_0^*(\tilde{z}_0, x, s) + [n_0(x) + s + \lambda - \lambda B(\tilde{z}_0)] P_0^*(\tilde{z}_0, x, s) &= P_0(\tilde{z}_0, x, 0) \\ \frac{\partial}{\partial x} P_i^*(\tilde{z}_i, x, s) + [n_i(x) + s + \lambda - \lambda B(\tilde{z}_i^*)] P_i^*(\tilde{z}_i, x, s) \\ (3.7) \quad &= \sum_{m=1}^{i-1} \int_0^\infty q_{mi}^*(\tilde{z}_i, y, x, s) n_m(y) dy \\ \frac{\partial}{\partial x} P_{i\mathbf{r}_i^k}^*(\tilde{z}_i, x, \tilde{x}_{\mathbf{r}_i^k}, s) + [n_i(x) + s + \lambda - \lambda B(\tilde{z}_i^*)] P_{i\mathbf{r}_i^k}^*(\tilde{z}_i, x, \tilde{x}_{\mathbf{r}_i^k}, s) \\ &= \sum_{m=1}^{i-1} \int_0^\infty q_{m\mathbf{r}_i^k}^*(\tilde{z}_i, y, \tilde{x}_{\mathbf{r}_i^k}, s) n_m(y) dy, \end{aligned}$$

while from (3.6)

$$\begin{aligned}
 P_{ir^k}^*(z_i, 0, \tilde{x}_{r^k}, s) &= \frac{1}{z_i} \int_0^\infty P_{ir^k}^*(z_i, x, \tilde{x}_{r^k}, s) n_i(x) dx \\
 &\quad - \int_0^\infty q_{ir^k}^*(z_{i+1}, x, \tilde{x}_{r^k}, s) n_i(x) dx \\
 (3.8) \quad &\quad + \lambda [B(z_i^*) - B(z_{i+1}^*)] P_{r_{i1}^k r_{i2}^k}^*(z_{r_{i1}^k}, x_{r_{i1}^k}, \tilde{x}_{r_{i2}^k}, s) \\
 &\quad + \sum_{m=1}^{i-1} \int_0^\infty [q_{mr^k}^*(z_i, y, \tilde{x}_{r^k}, s) - q_{mr^k}^*(z_{i+1}, y, \tilde{x}_{r^k}, s)] n_m(x) dx.
 \end{aligned}$$

Let now $u_i^*(s)$ be the Laplace transform of $u_i(t)$, $\rho_0 = 0$ and for $i = 1, 2, \dots, n$

$$\begin{aligned}
 \rho_i &= \lambda b_1 \bar{u}_1 + \lambda b_2 \bar{u}_2 + \dots + \lambda b_i \bar{u}_i, & e_i &= \underbrace{(1, 1, \dots, 1)}_{n-i+1} \\
 (3.9) \quad & & & \\
 b_i &= \frac{\partial B(z)}{\partial z_i} \Big|_{z=e_1}, & b_i^{(k)} &= \frac{\partial^k B(z)}{\partial z_i^k} \Big|_{z=e_1}, & b_{ij} &= \frac{\partial^2 B(z)}{\partial z_i \partial z_j} \Big|_{z=e_1}.
 \end{aligned}$$

We state here without proof a theorem from Katsaros and Langaris [8].

Theorem 3.1 For (a) $Re(s) > 0$, $|z_r| \leq 1$ $j+1 \leq r \leq n$ or (b) $Re(s) \geq 0$, $|z_r| \leq 1$ $j+1 \leq r \leq n$ $r \neq m$ $m = j+1, j+2, \dots, n$, $|z_m| < 1$ or (c) $Re(s) \geq 0$, $|z_r| \leq 1$ $j+1 \leq r \leq n$ and $\rho_{j-1} > 1$ or (d) $Re(s) \geq 0$, $|z_r| \leq 1$ $j+1 \leq r \leq n$ and $\rho_j > 1 \geq \rho_{j-1}$, the equation

$$(3.10) \quad z_j = u_j^*(s + \lambda - \lambda B(a_{j-1}(s, z_j)))$$

has for all $j = 1, 2, \dots, n$ one and only one root, $z_j = x_j(s, z_{j+1})$, $z_n = x_n(s)$ say, inside the region $|z_j| < 1$, where the vector $a_k(s, z_{k+1})$ is defined by

$$\begin{aligned}
 a_0(s, z_1) &= a_0(s, z_1, z_2, \dots, z_n) = (z_1, z_2, \dots, z_n) \\
 (3.11) \quad a_k(s, z_{k+1}) &= a_k(s, z_{k+1}, z_{k+2}, \dots, z_n) = a_{k-1}(s, x_k(s, z_{k+1}), z_{k+1}, \dots, z_n) \\
 a_n(s, z_{n+1}) &\equiv a_n(s) = a_{n-1}(s, x_n(s)) & k &= 1, 2, \dots, n-1.
 \end{aligned}$$

Specifically for $s = 0$, $z_r = 1$ $j+1 \leq r \leq n$ and $\rho_{j-1} \leq 1$, $x_j(0, e_{j+1})$ is the smallest positive real root of (3.10) with $x_j(0, e_{j+1}) < 1$ if $\rho_j > 1$ and $x_j(0, e_{j+1}) = 1$ for $\rho_j \leq 1$.

We have to point out here that the roots of the equation (3.10) have in fact a busy period interpretation. To explain the case, denote by B_j^m the duration of a busy period of P_j customers starting with m P_j customers, i.e. denote by B_j^m the time interval from the epoch at which there are m P_j customers in the system and one of them commences service, until the epoch at which, although the server is free to serve the next P_j customer, there are no more P_j customers in the system. One understands that there are no customers of classes $1, 2, \dots, j-1$ in the system, at the beginning and at the end of B_j^m .

If now we denote by $f_j^{(m)}(t)$ the p.d.f. of B_j^m and by $\bar{f}_j^{(m)}(s)$ its Laplace transform, then it can be shown (see Langaris and Katsaros [10]) that

$$\bar{f}_j^{(m)}(s) = x_j^m(s, e_{j+1}).$$

Now we will use Theorem 3.1 to show

Theorem 3.2 For $Re(s) \geq 0$ and $|z_i| \leq 1$ the following relations hold for all $i=1,2,\dots,n-1$

$$(3.12) \quad \int_0^\infty q_{i\mathbf{r}_i^k}^*(z_{i+1}, x, \mathbf{x}_{\mathbf{r}_i^k}, s) n_i(x) dx = \lambda [B(a_{\tilde{z}_i}(s, z_{i+1})) - B(a_{\tilde{z}_{i-1}}(s, z_{i+1}))] P_{r_{i1}^k r_{i2}^k}^*(z_{r_{i1}^k}, x_{r_{i1}^k}, \mathbf{x}_{\mathbf{r}_{i2}^k}, s)$$

$$(3.13) \quad P_{i\mathbf{r}_i^k}^*(z_i, x, \mathbf{x}_{\mathbf{r}_i^k}, s) = P_{i\mathbf{r}_i^k}^*(z_i, 0, \mathbf{x}_{\mathbf{r}_i^k}, s) \exp [-(s + \lambda - \lambda B(a_{\tilde{z}_{i-1}}(s, z_i)))x - \int_0^x n_i(y) dy]$$

$$(3.14) \quad P_{i\mathbf{r}_i^k}^*(z_i, 0, \mathbf{x}_{\mathbf{r}_i^k}, s) = \frac{\lambda [B(a_{\tilde{z}_{i-1}}(s, z_i)) - B(a_{\tilde{z}_i}(s, z_{i+1}))]}{1 - \frac{1}{z_i} u_i^*(s + \lambda - \lambda B(a_{\tilde{z}_{i-1}}(s, z_i)))} P_{r_{i1}^k r_{i2}^k}^*(z_{r_{i1}^k}, x_{r_{i1}^k}, \mathbf{x}_{\mathbf{r}_{i2}^k}, s).$$

Proof. From the third of (3.7) we have

$$\frac{\partial}{\partial x} P_{i\mathbf{r}_i^k}^*(z_1, x, \mathbf{x}_{\mathbf{r}_i^k}, s) + [n_1(x) + s + \lambda - \lambda B(z_1)] P_{i\mathbf{r}_i^k}^*(z_1, x, \mathbf{x}_{\mathbf{r}_i^k}, s) = 0,$$

and so (3.13) holds for $i = 1$.

Using now (3.13) (for $i = 1$) in (3.8)

$$P_{i\mathbf{r}_i^k}^*(z_1, 0, \mathbf{x}_{\mathbf{r}_i^k}, s) = \frac{1}{z_1} u_1^*(s + \lambda - \lambda B(z_1)) P_{i\mathbf{r}_i^k}^*(z_1, 0, \mathbf{x}_{\mathbf{r}_i^k}, s) - \int_0^\infty q_{i\mathbf{r}_i^k}^*(z_2, x, \mathbf{x}_{\mathbf{r}_i^k}, s) n_1(x) dx + \lambda [B(z_1) - B(z_2^*)] P_{r_{i1}^k r_{i2}^k}^*(z_{r_{i1}^k}, x_{r_{i1}^k}, \mathbf{x}_{\mathbf{r}_{i2}^k}, s),$$

and so

$$(3.15) \quad P_{i\mathbf{r}_i^k}^*(z_1, 0, \mathbf{x}_{\mathbf{r}_i^k}, s) = \frac{\lambda [B(z_1) - B(z_2^*)] P_{r_{i1}^k r_{i2}^k}^*(z_{r_{i1}^k}, x_{r_{i1}^k}, \mathbf{x}_{\mathbf{r}_{i2}^k}, s) - \int_0^\infty q_{i\mathbf{r}_i^k}^*(z_2, x, \mathbf{x}_{\mathbf{r}_i^k}, s) n_1(x) dx}{1 - \frac{1}{z_1} u_1^*(s + \lambda - \lambda B(z_1))}.$$

But according to Theorem 3.1, the denominator of (3.15) has one and only one zero in $|z_1| < 1$ which becomes equal to one when $s = 0$, $z_j = 1$ $j = 2, 3, \dots, n$. By putting this zero in the numerator of (3.15) we obtain relation (3.12) for $i=1$. Using finally (3.12) (with $i = 1$), we eliminate the integral from the numerator of (3.15) and obtain (3.14). Thus the theorem holds for $i=1$. Suppose now that it holds for $i=1,2,\dots,j-1$ too. Then from (3.12) and for all $m = 1, 2, \dots, j-1$,

$$(3.16) \quad \int_0^\infty q_{m\mathbf{r}_j^k}^*(z_{m+1j}, y, \mathbf{x}_{\mathbf{r}_j^k}, s) n_m(y) dy = \lambda [B(a_{\tilde{z}_m}(s, z_{m+1j})) - B(a_{\tilde{z}_{m-1}}(s, z_{mj}))] P_{j\mathbf{r}_j^k}^*(z_j, x, \mathbf{x}_{\mathbf{r}_j^k}, s),$$

and using this relation in (3.7) we obtain

$$\begin{aligned} \frac{\partial}{\partial x} P_{j\mathbf{r}_j^k}^*(z_j, x, \mathbf{x}_{\mathbf{r}_j^k}, s) + [n_j(x) + s + \lambda - \lambda B(z_j^*)] P_{j\mathbf{r}_j^k}^*(z_j, x, \mathbf{x}_{\mathbf{r}_j^k}, s) \\ = \lambda \sum_{m=1}^{j-1} [B(a_{\tilde{z}_m}(s, z_{m+1j})) - B(a_{\tilde{z}_{m-1}}(s, z_{mj}))] P_{j\mathbf{r}_j^k}^*(z_j, x, \mathbf{x}_{\mathbf{r}_j^k}, s) \\ = \lambda [B(a_{\tilde{z}_{j-1}}(s, z_j)) - B(z_j^*)] P_{j\mathbf{r}_j^k}^*(z_j, x, \mathbf{x}_{\mathbf{r}_j^k}, s). \end{aligned}$$

Thus

$$\frac{\partial}{\partial x} P_{j\mathbf{r}_j^k}^*(z_j, x, \tilde{x}_{\mathbf{r}_j^k}, s) + [n_j(x) + s + \lambda - \lambda B(a_{j-1}(s, z_j))] P_{j\mathbf{r}_j^k}^*(z_j, x, \tilde{x}_{\mathbf{r}_j^k}, s) = 0,$$

and so (3.13) holds for $i = j$ too. If now we substitute $P_{j\mathbf{r}_j^k}^*(\cdot)$ from (3.13) and $q_{m\mathbf{r}_j^k}^*(\cdot)$ $m = 1, 2, \dots, j - 1$ from (3.12), into (3.8) we arrive at

$$\begin{aligned} & P_{j\mathbf{r}_j^k}^*(z_j, 0, \tilde{x}_{\mathbf{r}_j^k}, s) \left(1 - \frac{1}{z_j} u_j^*(s + \lambda - \lambda B(a_{j-1}(s, z_j)))\right) \\ &= - \int_0^\infty q_{j\mathbf{r}_j^k}^*(z_{j+1}, x, \tilde{x}_{\mathbf{r}_j^k}, s) n_j(x) dx \\ &+ \lambda [B(a_{j-1}(s, z_j)) - B(a_{j-1}(s, z_{jj+1}))] P_{\mathbf{r}_{j1}^k \mathbf{r}_{j2}^k}^*(z_{\mathbf{r}_{j1}^k}, x_{\mathbf{r}_{j1}^k}, \tilde{x}_{\mathbf{r}_{j2}^k}, s). \end{aligned}$$

Thus using again Theorem 3.1 we obtain (3.12) and (3.14) for $i=j$ and the theorem has been proved.

Define now $a_{-1}(s, z_0) \equiv z_0$. To proceed further our analysis we need the following theorem.

Theorem 3.3 For $Re(s) \geq 0$, $|z_i| \leq 1$ and for all $i=0,1,2,\dots,n$,

$$(3.17) \quad \begin{aligned} & P_i^*(z_i, x, s) \\ &= \frac{R_i(s, z_i)}{1 - u_0^*(s + \lambda - \lambda B(a_n(s)))} \exp[-(s + \lambda - \lambda B(a_{i-1}(s, z_i)))x - \int_0^x n_i(y) dy], \quad x > 0 \end{aligned}$$

where $R_0(s, z_0) \equiv 1$ and

$$(3.18) \quad R_i(s, z_i) = \frac{u_0^*(s + \lambda - \lambda B(a_{i-1}(s, z_i))) - u_0^*(s + \lambda - \lambda B(a_i(s, z_{i+1})))}{1 - \frac{1}{z_i} u_i^*(s + \lambda - \lambda B(a_{i-1}(s, z_i)))}$$

$i = 1, 2, \dots, n.$

Proof. From (3.16) and for $k = 0$

$$\int_0^\infty q_{mi}^*(z_{m+1i}, y, x, s) n_m(y) dy = \lambda [B(a_m(s, z_{m+1i})) - B(a_{m-1}(s, z_{mi}))] P_i^*(z_i, x, s).$$

Using this relation, we obtain from (3.7) for all $i = 1, 2, \dots, n$,

$$(3.19) \quad \begin{aligned} P_0^*(z_0, x, s) &= (\epsilon(x) + p_0^*(0, 0, s)) \exp[-(s + \lambda - \lambda B(z_0))x - \int_0^x n_0(y) dy] \\ P_i^*(z_i, x, s) &= P_i^*(z_i, 0, s) \exp[-(s + \lambda - \lambda B(a_{i-1}(s, z_i)))x - \int_0^x n_i(y) dy], \end{aligned}$$

where $\epsilon(x)$ is the unit step function at $x = 0$. By forming now generating functions in (3.5)

we arrive at

$$(3.20) \quad \begin{aligned} P_i^*(z_i, 0, s) &= \frac{1}{z_i} \int_0^\infty P_i^*(z_i, x, s) n_i(x) dx - \int_0^\infty q_i^*(z_{i+1}, x, s) n_i(x) dx \\ &+ \sum_{m=1}^{i-1} \int_0^\infty [q_m^*(z_i, x, s) - q_m^*(z_{i+1}, x, s)] n_m(x) dx \\ &+ \int_0^\infty [P_0^*(z_{1i}, x, s) - P_0^*(z_{1i+1}, x, s)] n_0(x) dx \end{aligned}$$

with $i = 1, 2, \dots, n-1$, and

$$q_m^*(z_i, x, s) = \sum_{\substack{k_i^* \geq 0 \\ \tilde{z}_i^{k_i}}} p_m^*(k_i^* + u_m, x, s) \tilde{z}_i^{k_i}, \quad m < i.$$

Using (3.19) in (3.20) and starting with $i = 1$ one can prove inductively, in a similar way as in Theorem 3.2, that for all $i = 1, 2, \dots, n-1$,

$$(3.21) \quad \begin{aligned} &\int_0^\infty q_i^*(z_{i+1}, x, s) n_i(x) dx \\ &= [u_0^*(s + \lambda - \lambda B(a_i(s, z_{i+1}))) - u_0^*(s + \lambda - \lambda B(a_{i-1}(s, z_{i+1})))](1 + p_0^*(0, 0, s)) \end{aligned}$$

$$(3.22) \quad P_i^*(z_i, 0, s) = \frac{u_0^*(s + \lambda - \lambda B(a_{i-1}(s, z_i))) - u_0^*(s + \lambda - \lambda B(a_i(s, z_{i+1})))}{1 - \frac{1}{z_i} u_i^*(s + \lambda - \lambda B(a_{i-1}(s, z_i)))} (1 + p_0^*(0, 0, s)).$$

Finally from (3.5) and for $i = n$,

$$\begin{aligned} P_n^*(z_n, 0, s) &= \frac{1}{z_n} \int_0^\infty P_n^*(z_n, x, s) n_n(x) dx + \sum_{m=1}^{n-1} \int_0^\infty q_m^*(z_n, x, s) n_m(x) dx \\ &+ \int_0^\infty P_0^*(z_{1n}, x, s) n_0(x) dx - \sum_{m=0}^n \int_0^\infty p_m^*(u_m, x, s) n_m(x) dx, \end{aligned}$$

and so using (3.4), (3.19) and (3.21) we obtain

$$(3.23) \quad P_n^*(z_n, 0, s) = \frac{[1 + p_0^*(0, 0, s)] u_0^*(s + \lambda - \lambda B(a_{n-1}(s, z_n))) - p_0^*(0, 0, s)}{1 - \frac{1}{z_n} u_n^*(s + \lambda - \lambda B(a_{n-1}(s, z_n)))}.$$

By putting now the zero of the denominator in the numerator we arrive at

$$(3.24) \quad p_0^*(0, 0, s) = \frac{u_0^*(s + \lambda - \lambda B(a_n(s)))}{1 - u_0^*(s + \lambda - \lambda B(a_n(s)))}.$$

Replacing finally $p_0^*(0, 0, s)$ into (3.22) and (3.23) and using (3.19) we obtain relations (3.17), (3.18) and the theorem has been proved.

Note here that if we put $n = 1$ and $B(z) = z$ in (3.17) we obtain relations (2.14a) and (2.14b) of Takagi [14].

From Theorems 3.2 and 3.3 it is now clear that for $i = 1, 2, \dots, n-1$,

$$(3.25) \quad \begin{aligned} P_{i\mathbf{r}_i^k}^*(z_i, x, x_{\mathbf{r}_i^k}, s) &= \frac{1}{1 - u_0^*(s + \lambda - \lambda B(a_n(s)))} \\ &\cdot \prod_{m \in \mathbf{r}_i^k} Q_m(s, z_m) \exp[-(s + \lambda - \lambda B(a_{m-1}(s, z_m)))x_m - \int_0^{x_m} n_m(y) dy], \end{aligned}$$

with

$$(3.26) \quad Q_m(s, \tilde{z}_m) = \frac{\lambda[B(a_{m-1}(s, \tilde{z}_m)) - B(a_m(s, \tilde{z}_{m+1}))]}{1 - \frac{1}{z_m} u_m^*(s + \lambda - \lambda B(a_{m-1}(s, \tilde{z}_m)))} \quad \text{if } m \in \bar{\mathbf{r}}_i^k, \quad m \neq r_{ik}^k$$

$$Q_{r_{ik}^k}(s, \tilde{z}_{r_{ik}^k}) = R_{r_{ik}^k}(s, \tilde{z}_{r_{ik}^k}),$$

and so the generating functions of the system state probabilities in a transient state are completely known.

To obtain steady state results let us assume that the limits

$$\lim_{t \rightarrow \infty} p_i(k_i^*, x, t) = \bar{p}_i(k_i^*, x), \quad \lim_{t \rightarrow \infty} p_{ir_i^k}(k_i^*, x, \tilde{x}_{r_i^k}, t) = \bar{p}_{ir_i^k}(k_i^*, x, \tilde{x}_{r_i^k})$$

exist. Then by defining the corresponding generating functions $\bar{P}_i(\tilde{z}_i, x)$, $\bar{P}_{ir_i^k}(\tilde{z}_i, x, \tilde{x}_{r_i^k})$, and using the well known Tauberian theorem

$$(3.27) \quad \bar{P}_i(\tilde{z}_i, x) = \lim_{s \rightarrow 0} s P_i^*(\tilde{z}_i, x, s), \quad \bar{P}_{ir_i^k}(\tilde{z}_i, x, \tilde{x}_{r_i^k}) = \lim_{s \rightarrow 0} s P_{ir_i^k}^*(\tilde{z}_i, x, \tilde{x}_{r_i^k}, s).$$

Suppose now that $\rho = \lambda b_1 \bar{u}_1 + \lambda b_2 \bar{u}_2 + \dots + \lambda b_n \bar{u}_n > 1$. Then from Theorem 3.1 the zero $z_n = x_n(0)$ (for $s = 0$) lies inside the region $|z_n| < 1$ and so the denominator in (3.17) and (3.25) cannot be zero when s tends to zero. Thus from (3.27),

$$\bar{P}_i(\tilde{z}_i, x) = \bar{P}_{ir_i^k}(\tilde{z}_i, x, \tilde{x}_{r_i^k}) = 0.$$

If now $\rho \leq 1$ then $x_n(0) = 1$ and $B(a_n(0)) = B(e_1)$ i.e. $u_0^*(s + \lambda - \lambda B(a_n(s))) = 1$. Thus in this case, from (3.27),

$$(3.28) \quad \bar{P}_i(\tilde{z}_i, x) = \frac{1 - \rho}{\bar{u}_0} R_i(0, \tilde{z}_i) \exp[-(\lambda - \lambda B(a_{i-1}(0, \tilde{z}_i)))x - \int_0^x n_i(y) dy]$$

$$\bar{P}_{ir_i^k}(\tilde{z}_i, x, \tilde{x}_{r_i^k}) = \frac{1 - \rho}{\bar{u}_0} \prod_{m \in \bar{\mathbf{r}}_i^k} Q_m(0, \tilde{z}_m) \exp[-(\lambda - \lambda B(a_{m-1}(0, \tilde{z}_m)))x_m - \int_0^{x_m} n_m(y) dy]$$

Integrating now relations (3.28) for all $x, \tilde{x}_{r_i^k}$ we obtain

$$(3.29) \quad \bar{P}_i(\tilde{z}_i) = \frac{1 - \rho}{\bar{u}_0} R_i(0, \tilde{z}_i) \frac{1 - u_i^*(\lambda - \lambda B(a_{i-1}(0, \tilde{z}_i)))}{\lambda - \lambda B(a_{i-1}(0, \tilde{z}_i))} \quad i = 0, 1, \dots, n$$

$$\bar{P}_{ir_i^k}(\tilde{z}_i) = \frac{1 - \rho}{\bar{u}_0} \prod_{m \in \bar{\mathbf{r}}_i^k} Q_m(0, \tilde{z}_m) \frac{1 - u_m^*(\lambda - \lambda B(a_{m-1}(0, \tilde{z}_m)))}{\lambda - \lambda B(a_{m-1}(0, \tilde{z}_m))} \quad i = 1, 2, \dots, n - 1,$$

and so finally the generating function of the system state probabilities in a steady state is given, for $\rho < 1$, by

$$\bar{P}(\tilde{z}) = \sum_{i=0}^n \bar{P}_i(\tilde{z}_i) + \sum_{i=1}^{n-1} \sum_{\mathbf{r}^k} \bar{P}_{ir_i^k}(\tilde{z}_i),$$

and from (3.29) it is completely known. Using $n = 1$ and $B(z) = z$ in the above relation we obtain relation (2.30c) of Takagi [14].

4. The model without vacation-A decomposition result

Let us examine now the case at which the server does not leave the system for a vacation when he becomes idle, but he remains idle until the first arriving customer pushes him to start serving again. If now $p_0(t) = \Pr$ [server idle at t] and $p_0^*(s)$ is the corresponding Laplace transform, then it is easy to see that

$$(4.1) \quad \frac{dp_0(t)}{dt} + \lambda p_0(t) = \sum_{m=1}^n \int_0^{\infty} p_m(\underline{u}_m, x, t) n_m(x) dx$$

$$(4.2) \quad (s + \lambda)p_0^*(s) = 1 + \sum_{m=1}^n \int_0^{\infty} p_m^*(\underline{u}_m, x, s) n_m(x) dx,$$

while relations (3.5) and (3.20) become accordingly

$$(4.3) \quad p_i(k_i^*, 0, t) = \sum_{m=1}^i \int_0^{\infty} p_m(k_i^* + \underline{u}_m, x, t) n_m(x) dx + \lambda p_0(t) b(k_i^*) \quad i = 1, 2, \dots, n$$

$$(4.4) \quad \begin{aligned} P_i^*(z_i, 0, s) &= \frac{1}{z_i} \int_0^{\infty} P_i^*(z_i, x, s) n_i(x) dx - \int_0^{\infty} q_i^*(z_{i+1}, x, s) n_i(x) dx \\ &+ \sum_{m=1}^{i-1} \int_0^{\infty} [q_m^*(z_i, x, s) - q_m^*(z_{i+1}, x, s)] n_m(x) dx + \lambda p_0^*(s) [B(z_{1i}) - B(z_{1i+1})] \end{aligned} \quad i = 1, 2, \dots, n-1.$$

Thus using the second of (3.19) in (4.4) and starting with $i=1$ we can show that now

$$(4.5) \quad \int_0^{\infty} q_i^*(z_{i+1}, x, s) n_i(x) dx = \lambda [B(a_i(s, z_{i+1})) - B(a_{i-1}(s, z_{i+1}))] p_0^*(s)$$

$$(4.6) \quad P_i^*(z_i, 0, s) = \lambda \frac{B(a_{i-1}(s, z_i)) - B(a_i(s, z_{i+1}))}{1 - \frac{1}{z_i} u_i^*(s + \lambda - \lambda B(a_{i-1}(s, z_i)))} p_0^*(s), \quad i = 1, 2, \dots, n-1,$$

and in a similar way as in section 3

$$(4.7) \quad P_n^*(z_n, 0, s) = \frac{1 + \lambda p_0^*(s) B(a_{n-1}(s, z_n)) - (s + \lambda) p_0^*(s)}{1 - \frac{1}{z_n} u_n^*(s + \lambda - \lambda B(a_{n-1}(s, z_n)))}.$$

Using finally the root of the denominator in (4.7) we arrive at

$$(4.8) \quad p_0^*(s) = \frac{1}{s + \lambda - \lambda B(a_n(s))}.$$

From (4.6)–(4.8) it is now clear that the quantities $P_i^*(z_i, x, s)$ are again completely known. Thus Theorem 3.3 becomes for the model without vacation

Theorem 4.1 For $Re(s) \geq 0$ and $|z_i| \leq 1$

$$(4.9) \quad P_i^*(z_i, x, s) = \frac{Q_i(s, z_i)}{s + \lambda - \lambda B(a_n(s))} \exp[-(s + \lambda - \lambda B(a_{i-1}(s, z_i)))x - \int_0^x n_i(y) dy]$$

where $Q_i(s, z_i)$ is given, for each $i=1, 2, \dots, n$ by the first of (3.26).

Note here that Theorem 3.2 holds for the system without vacation in exactly the same form (as in section 3) and so using (4.9) we obtain

$$(4.10) \quad P_{\mathbf{r}_i^k}^*(z_i, x, x_{\mathbf{r}_i^k}, s) = \frac{1}{s + \lambda - \lambda B(\underline{a}_n(s))} \cdot \prod_{m \in \mathbf{r}_i^k} Q_m(s, z_m) \exp[-(s + \lambda - \lambda B(\underline{a}_{m-1}(s, z_m)))x_m - \int_0^{x_m} n_m(y)dy] \quad i = 1, 2, \dots, n - 1.$$

From (4.8)–(4.10) we can easily obtain as before the steady state results, $\bar{p}_0 = 1 - \rho$, and

$$(4.11) \quad \bar{P}_i(z_i) = (1 - \rho) Q_i(0, z_i) \frac{1 - u_i^*(\lambda - \lambda B(\underline{a}_{i-1}(0, z_i)))}{\lambda - \lambda B(\underline{a}_{i-1}(0, z_i))} \quad i = 1, \dots, n$$

$$\bar{P}_{\mathbf{r}_i^k}(z_i) = (1 - \rho) \prod_{m \in \mathbf{r}_i^k} Q_m(0, z_m) \frac{1 - u_m^*(\lambda - \lambda B(\underline{a}_{m-1}(0, z_m)))}{\lambda - \lambda B(\underline{a}_{m-1}(0, z_m))} \quad i = 1, \dots, n - 1.$$

Note here that relations (4.11) generalize (for arbitrary n now) the corresponding results of Takahashi and Takagi[18]. Thus if we put $n = 2$ in (4.8) and (4.11) here we obtain (2.25) and (2.26) of [18].

To compare now the obtained results for models with and without vacation, let us denote by $\bar{P}_i^{(v)}(z_i)$, $\bar{P}_{\mathbf{r}_i^k}^{(v)}(z_i)$ the functions $\bar{P}_i(z_i)$, $\bar{P}_{\mathbf{r}_i^k}(z_i)$ given by (3.29) and concerning the model with multiple vacations, and by $\bar{P}_i^{(w)}(z_i)$, $\bar{P}_{\mathbf{r}_i^k}^{(w)}(z_i)$ the corresponding quantities in the model without vacations (given by (4.11)). Then from (4.11) it is clear that for any i and \mathbf{r}_i^k

$$(4.12) \quad \bar{P}_{\mathbf{r}_i^k}^{(w)}(z_i) = \frac{1}{(1 - \rho)^k} \prod_{m \in \mathbf{r}_i^k} \bar{P}_m^{(w)}(z_m),$$

while from (3.29)

$$(4.13) \quad \bar{P}_{\mathbf{r}_i^k}^{(v)}(z_i) = \frac{1}{(1 - \rho)^k} \bar{P}_{\mathbf{r}_i^k}^{(v)}(z_{\mathbf{r}_i^k}) \prod_{\substack{m \in \mathbf{r}_i^k \\ m \neq \mathbf{r}_i^k}} \bar{P}_m^{(w)}(z_m).$$

By comparing also the first equations in (3.29) and (4.11) and using definitions (3.26) and (3.18) we obtain

$$(4.14) \quad \bar{P}_i^{(v)}(z_i) = \frac{u_0^*(\lambda - \lambda B(\underline{a}_{i-1}(0, z_i))) - u_0^*(\lambda - \lambda B(\underline{a}_i(0, z_{i+1})))}{\lambda [B(\underline{a}_{i-1}(0, z_i)) - B(\underline{a}_i(0, z_{i+1}))] \bar{u}_0} \bar{P}_i^{(w)}(z_i),$$

and, in general, from (4.12)–(4.14)

$$(4.15) \quad \bar{P}_{\mathbf{r}_i^k}^{(v)}(z_i) = \frac{u_0^*(\lambda - \lambda B(\underline{a}_{\mathbf{r}_i^k-1}(0, z_{\mathbf{r}_i^k}))) - u_0^*(\lambda - \lambda B(\underline{a}_{\mathbf{r}_i^k}(0, z_{\mathbf{r}_i^k+1})))}{\lambda [B(\underline{a}_{\mathbf{r}_i^k-1}(0, z_{\mathbf{r}_i^k})) - B(\underline{a}_{\mathbf{r}_i^k}(0, z_{\mathbf{r}_i^k+1}))] \bar{u}_0} \bar{P}_{\mathbf{r}_i^k}^{(w)}(z_i).$$

By observing finally that the fractional term in (4.15) depends only on the class of the last customer in limbo, we can write

$$(4.16) \quad \bar{P}_{(m)}^{(v)}(\tilde{z}) = \alpha(\tilde{z}_m) \bar{P}_{(m)}^{(w)}(\tilde{z}),$$

and for the generating function of the system state probabilities

$$(4.17) \quad \bar{P}^{(v)}(\tilde{z}) = \frac{1 - u_0^*(\lambda - \lambda B(\tilde{z}))}{\lambda(1 - B(\tilde{z}))\bar{u}_0} \bar{p}_0^{(w)} + \sum_{m=1}^n \alpha(\tilde{z}_m) \bar{P}_{(m)}^{(w)}(\tilde{z}),$$

where $\bar{p}_0^{(w)} = 1 - \rho$ and

$$(4.18) \quad \alpha(\tilde{z}_m) = \frac{u_0^*(\lambda - \lambda B(\tilde{a}_m(0, \tilde{z}_{m+1}))) - u_0^*(\lambda - \lambda B(\tilde{a}_{m-1}(0, \tilde{z}_m)))}{\lambda[B(\tilde{a}_m(0, \tilde{z}_{m+1})) - B(\tilde{a}_{m-1}(0, \tilde{z}_m))] \bar{u}_0}$$

$$\bar{P}_{(m)}^{(\cdot)}(\tilde{z}) = \bar{P}_m^{(\cdot)}(\tilde{z}) + \sum_{i=1}^{m-1} \sum_{\mathbf{r}_i^k} \bar{P}_{i\mathbf{r}_i^k}^{(\cdot)}(\tilde{z}) \Big|_{r_{ik}^k = m}.$$

Note here that in both models (with and without vacations) the server's busy period can be considered as the sum of the particular busy periods of each class of customers in descending order of priority, while a busy period of the m^{th} class customers is composed of a series of service completion times. Under this consideration the function $\bar{P}_{(m)}^{(\cdot)}(\tilde{z})$ is in fact the probability generating function (P.G.F.) of the number of customers in the system during an m^{th} class customer completion time. If finally we define $\tilde{z}_m = (z_m, 1, 1, \dots, 1)$ and substitute $\tilde{z}_{m+1} = \tilde{z}_{m+1}$ in (4.16) we arrive at

$$(4.19) \quad \bar{P}_{(m)}^{(v)}(z_1, z_2, \dots, z_m) = \bar{\alpha}(z_m) \bar{P}_{(m)}^{(w)}(z_1, z_2, \dots, z_m),$$

with

$$\bar{\alpha}(z_m) = \frac{1 - u_0^*(\lambda - \lambda B(\tilde{a}_{m-1}(0, \tilde{z}_m)))}{\lambda[1 - B(\tilde{a}_{m-1}(0, \tilde{z}_m))] \bar{u}_0}.$$

Denoting now by V_c the equilibrium forward recurrence time of the vacation and using the busy period results of Langaris and Katsaros [10], we can show that the function $\bar{\alpha}(z_m)$ is in fact the P.G.F. of the number of class m customers who arrive inside V_c and inside the busy period of P_1, P_2, \dots, P_{m-1} customers created from all P_1, P_2, \dots, P_{m-1} customers arriving in V_c .

From the analysis above, it is clear that relation (4.19) (and (4.16), (4.17) accordingly) can be considered as the generalization of the well known decomposition result holding for the M/G/1 queue with multiple vacations (see Fuhrmann [3] and the references therein). Thus if we assume $n = 1$ (only one class of customers) and single arrivals ($B(z) = z$) in our model then relations (4.17) and (4.19) become

$$\bar{P}^{(v)}(z) = \frac{1 - u_0^*(\lambda - \lambda z)}{\lambda(1 - z)\bar{u}_0} \bar{P}^{(w)}(z),$$

which is the decomposition property of the M/G/1 queue expressed by relation (1) in Fuhrmann [3].

5. Performance measures-Comparisons and conclusions

To obtain explicit formulae for the mean queue length, in both models, we will use relation (3.10). If in (3.10) we replace z_j with the corresponding zero $x_j(s, z_{j+1})$, then

$$x_j(s, z_{j+1}) = u_j^*(s + \lambda - \lambda B(a_j(s, z_{j+1}))) \quad j = 1, 2, \dots, n.$$

If now we put in the above relation, the root $x_{j+1}(s, z_{j+2})$ instead of z_{j+1} and after that $x_{j+2}(s, z_{j+3})$ instead of z_{j+2} and so on, and take in the obtained relations the first and second derivative with respect to z_m , we finally arrive (see Katsaros and Langaris [8] for details) at

$$(5.1) \quad \begin{aligned} \left. \frac{\partial B(a_{k-1}(0, z_k))}{\partial z_m} \right|_{z_k=e_k} &= \frac{b_m}{1 - \rho_{k-1}} \\ \left. \frac{\partial u_j^*(\lambda - \lambda B(a_{k-1}(0, z_k)))}{\partial z_m} \right|_{z_k=e_k} &= \frac{\lambda b_m \bar{u}_j}{1 - \rho_{k-1}} \\ \left. \frac{\partial^2 B(a_{k-1}(0, z_k))}{\partial z_m^2} \right|_{z_k=e_k} &= \frac{e_k^{(m)}}{1 - \rho_{k-1}} \\ \left. \frac{\partial^2 u_j^*(\lambda - \lambda B(a_{k-1}(0, z_k)))}{\partial z_m^2} \right|_{z_k=e_k} &= \frac{\lambda e_k^{(m)} \bar{u}_j}{1 - \rho_{k-1}} + \frac{\lambda^2 b_m^2 \bar{u}_j^{(2)}}{(1 - \rho_{k-1})^2}, \end{aligned}$$

where

$$(5.2) \quad e_k^{(m)} = b_m^{(2)} + \frac{2\lambda b_m}{1 - \rho_{k-1}} \sum_{j=1}^{k-1} b_{mj} \bar{u}_j + \frac{\lambda^2 b_m^2}{(1 - \rho_{k-1})^2} \sum_{i=1}^{k-1} \left(b_i \bar{u}_i^{(2)} + \bar{u}_i \sum_{j=1}^{k-1} b_{mj} \bar{u}_j \right).$$

Using now the above relations in (4.11) we obtain the mean number of the m^{th} class customers ($m = 1, 2, \dots, n$) in the model without vacations

$$(5.3) \quad E(L_m^{(w)}) = \frac{\lambda b_m}{1 - \rho_{m-1}} \left(\bar{u}_m + \frac{\lambda b_m \bar{u}_m^{(2)}}{2(1 - \rho_m)} \right) + \frac{1 - \rho_{m-1}}{2b_m(1 - \rho_m)} e_m^{(m)} - \frac{b_m^{(2)}}{2b_m}.$$

For the model with multiple vacations, using again relations (5.1) in (3.29) (or in (4.14), (4.15)) and comparing the result with (5.3) we obtain

$$(5.4) \quad E(L_m^{(v)}) = E(L_m^{(w)}) + \frac{\lambda b_m(1 - \rho)}{2(1 - \rho_{m-1})(1 - \rho_m)} \frac{\bar{u}_0^{(2)}}{\bar{u}_0} \quad m = 1, 2, \dots, n.$$

Relation (5.4) shows the way in which the vacation affects the mean number of class m customers in the system. Note here that, according to (5.4), when ρ becomes equal to one we have $E(L_m^{(v)}) = E(L_m^{(w)})$ for all $m = 1, 2, \dots, n - 1$. This can be explained by the fact that with $\rho = 1$, the lowest priority queue ($m = n$) becomes saturated, the opportunity for a server vacation does not exist, and so the vacation time does not affect the state of the system anymore.

To compare now the mean performance measures obtained above, with the corresponding measures for models with nonpreemptive priority, we have to use relations (6.9) and (5.8) of Langaris and Katsaros [10] and Katsaros and Langaris [8] respectively.

If we denote by $E(\mathcal{L}_m^{(v)})$ and $E(\mathcal{L}_m^{(w)})$ the mean number of class m customers in the model with nonpreemptive priority and multiple vacation and in the model with nonpreemptive priority but without vacations respectively, then after manipulations in (6.9) and (5.8) of [10] and [8] we obtain

$$(5.5) \quad E(\mathcal{L}_m^{(v)}) = E(\mathcal{L}_m^{(w)}) + \frac{\lambda b_m (1 - \rho)}{2(1 - \rho_{m-1})(1 - \rho_m)} \frac{\bar{u}_0^{(2)}}{\bar{u}_0} \quad m = 1, 2, \dots, n,$$

which is again relation (5.4). Thus the first interesting result is that the vacation affects the mean number of class m customers in the models of preemptive and nonpreemptive priority in exactly the same way.

By using also formula (5.3) and the corresponding formula for $E(\mathcal{L}_m^{(w)})$ (relation (6.9) in [10]) and after heavy algebra, we arrive at

$$(5.6) \quad E(\mathcal{L}_m^{(w)}) = E(L_m^{(w)}) + \frac{\lambda^2 b_m}{2(1 - \rho_{m-1})(1 - \rho_m)} \sum_{i=m+1}^n b_i \bar{u}_i^{(2)} - \frac{\lambda b_m \bar{u}_m \rho_{m-1}}{1 - \rho_{m-1}} \quad m = 1, \dots, n.$$

Relation (5.6) shows explicitly the way in which the mean number of class m customers changes when we pass from the model with preemptive priority to the model with nonpreemptive priority. As the third term in the right hand side of (5.6) vanishes when $m = 1$ ($\rho_0 = 0$), and the second term vanishes when $m = n$, we conclude that for the class with the highest priority the preemptive priority model always ensures shorter queue length, while for the class with the lowest priority, the queue length is shorter in the case of nonpreemptive priority.

For the intermediate classes, $m = 2, 3, \dots, n - 1$, the sign of the difference $E(\mathcal{L}_m^{(w)}) - E(L_m^{(w)})$ depends on the sign of the term

$$(5.7) \quad \varphi_m = \sum_{i=m+1}^n b_i \bar{u}_i^{(2)} - 2\bar{u}_m (1 - \rho_m) \sum_{i=1}^{m-1} b_i \bar{u}_i,$$

obtained from (5.6) after manipulations. Thus, if $\varphi_m < 0$, the nonpreemptive model is better for the class m customers, while the preemptive model is better in case of $\varphi_m > 0$. Note here that possible ways to make φ_m negative are, either by increasing the mean service time \bar{u}_m of the P_m customers, or by increasing the mean number b_i of P_i customers in a batch (or the mean service time \bar{u}_i) for $i \leq m - 1$ and to decrease the corresponding quantities b_i for $i \geq m + 1$.

Using finally (5.4) and (5.5) in (5.6) we realize that relation (5.6) holds, in exactly the same form, between $E(\mathcal{L}_m^{(v)})$ and $E(L_m^{(v)})$ too. Thus all observations made above hold for models with server vacations as well.

We have used relations (5.3)-(5.6) to construct Table I which gives values of $E(L_m^{(w)})$, $E(L_m^{(v)})$, $E(\mathcal{L}_m^{(w)})$, $E(\mathcal{L}_m^{(v)})$ for a model of five classes. We have assumed that the service times and the vacation length follow exponential distributions,

$$u_i(t) = \frac{1}{u_i} e^{-\frac{1}{u_i} t} \quad 0 \leq i \leq 5.$$

We have also assumed that, if Y denotes the batch size and X_{mi} $i = 1, 2, 3, 4, 5$ the number of P_i customers in a batch of size m , then

$$P(Y = m) = \frac{1}{2^m} \quad m = 1, 2, \dots$$

$$P(X_{mi} = k_i \quad i = 1, 2, 3, 4, 5) = \frac{m!}{k_1! k_2! k_3! k_4! k_5!} p_1^{k_1} p_2^{k_2} p_3^{k_3} p_4^{k_4} p_5^{k_5},$$

$\bar{u}_0 \setminus \rho$	0.4		0.6		0.8		0.9		0.99	
	nonpreemptive	preemptive	nonpreemptive	preemptive	nonpreemptive	preemptive	nonpreemptive	preemptive	nonpreemptive	preemptive
Without Vacation	0.1322	0.1043	0.2291	0.1636	0.3505	0.2286	0.4215	0.2634	0.4918	0.2963
	0.1802	0.1623	0.3267	0.2785	0.5342	0.4303	0.6698	0.5241	0.8155	0.6216
	0.2434	0.2386	0.4792	0.4579	0.8876	0.8181	1.2096	1.0908	1.6129	1.4230
	0.3289	0.3418	0.7373	0.7615	1.7121	1.7231	2.7906	2.7503	4.6875	4.5130
	0.4486	0.4863	1.2277	1.3385	4.5156	4.8000	12.9086	13.3714	190.3922	191.1462
0.01	0.1332	0.1054	0.2302	0.1647	0.3512	0.2293	0.4219	0.2639	0.4918	0.2963
	0.1814	0.1636	0.3281	0.2799	0.5354	0.4314	0.6705	0.5248	0.8156	0.6217
	0.2449	0.2401	0.4812	0.4599	0.8894	0.8199	1.2108	1.0920	1.6131	1.4232
	0.3308	0.3437	0.7402	0.7644	1.7156	1.7265	2.7934	2.7531	4.6880	4.5134
	0.4510	0.4886	1.2323	1.3431	4.5244	4.8089	12.9214	13.3843	190.4113	191.1652
1	0.2365	0.2087	0.3382	0.2727	0.4267	0.3048	0.4654	0.3073	0.4967	0.3012
	0.3044	0.2865	0.4702	0.4220	0.6463	0.5423	0.7384	0.5927	0.8237	0.6298
	0.3939	0.3890	0.6766	0.6553	1.0686	0.9991	1.3320	1.2130	1.6291	1.4392
	0.5147	0.5276	1.0258	1.0500	2.0540	2.0650	3.0701	3.0298	4.7344	4.5599
	0.6839	0.7216	1.6892	1.8000	5.4044	5.6889	14.1943	14.6571	192.2961	193.0500
5	0.6539	0.6261	0.7745	0.7091	0.7314	0.6095	0.6410	0.4829	0.5165	0.3209
	0.8013	0.7834	1.0444	0.9962	1.0945	0.9905	1.0128	0.8671	0.8564	0.6625
	0.9953	0.9905	1.4661	1.4447	1.7926	1.7231	1.8210	1.7022	1.6937	1.5037
	1.2577	1.2706	2.1796	2.2038	3.4215	3.4325	4.1881	4.1478	4.9220	4.7474
	1.6251	1.6627	3.5354	3.6462	8.9600	9.2444	19.3371	19.8000	199.9115	200.6654
10	1.1757	1.1478	1.3200	1.2545	1.1124	0.9905	0.8605	0.7024	0.5412	0.3456
	1.4224	1.4046	1.7621	1.7139	1.6547	1.5507	1.3557	1.2101	0.8973	0.7034
	1.7472	1.7424	2.4529	2.4316	2.6976	2.6281	2.4324	2.3136	1.7744	1.5845
	2.1865	2.1994	3.6220	3.6462	5.1309	5.1419	5.8556	5.5453	5.1564	4.9819
	2.8016	2.8392	5.8431	5.9538	13.4044	13.6889	25.7657	26.2286	209.4307	210.1846

Table I : Values of $E(L_m^{(i)})$, $E(L_m^{(i)})$ for $p_i=0.2$, $\bar{u}_i=0.5$, $i=1, 2, \dots, 5$

with $\sum k_i = m$ and $\sum p_i = 1$. Thus

$$B(z_1, z_2, z_3, z_4, z_5) = \sum_{m=1}^{\infty} \frac{1}{2^m} (p_1 z_1 + p_2 z_2 + p_3 z_3 + p_4 z_4 + p_5 z_5)^m.$$

In Table I one can observe how the mean number of class m customers in the system changes, when we pass from a model with nonpreemptive priority to a model with preemptive priority, or when we pass from a model without vacation to a model with increasing vacation length. Finally one can observe the behavior of the four models when the traffic intensity ρ increases.

Concerning the comparison between the preemptive and nonpreemptive priority models, Table I supports the observations made before on the sign of φ_m . For example, with $\rho = 0.4$ we obtain from (5.7) $\varphi_2 = 0.432$, $\varphi_3 = 0.096$, $\varphi_4 = -0.208$ while for $\rho = 0.9$, $\varphi_2 = 0.472$, $\varphi_3 = 0.216$, $\varphi_4 = 0.032$. This means that for $\rho = 0.4$ the preemptive priority model is expected to be better for the first three classes (P_1, P_2, P_3) while for $\rho = 0.9$ this model is expected to be better for the first four classes, something which now is verified from the numbers in Table I as well.

Another thing which seems 'strange' in Table I is that for both models (preemptive and nonpreemptive) and for large mean vacation time \bar{u}_0 , a reduction in the mean number of the higher priority customers is observed when (and although) the traffic intensity increases. Thus, for example, for $\bar{u}_0 = 5$, $E(L_1^{(v)})$ is reduced from 0.709 to 0.6095 when we pass from $\rho = 0.6$ to $\rho = 0.8$ and a similar reduction is observed in the case of nonpreemptive priority too. This behavior can be explained by the fact that, under light traffic, the duration of the server busy period is small and the server becomes very often idle and ready to depart for a vacation. Thus a large vacation time affects drastically the performance measures in such a model. When now the traffic intensity increases, the duration of the busy period increases and the server does not have so often now the opportunity to start a vacation. Thus in this second case the effect of a large vacation on the performance measures is expected to be smaller than before.

We can also explain this 'strange' behaviour, in a more mathematical way, by using formula (5.5). For the model without vacation, when ρ increases, an increase in $E(\mathcal{L}_m^{(w)})$ is always expected, but, if at the same time we face a large reduction on the second term in the right hand side of (5.5), then we can arrive at a smaller $E(\mathcal{L}_m^{(v)})$. Thus in the case of $\bar{u}_0 = 5$, $E(\mathcal{L}_1^{(w)})$ increases from 0.1636 to 0.2286 (+0.065) when ρ passes from 0.6 to 0.8 but at the same time the second term in the right hand side of (5.5) is reduced from 0.5455 to 0.3809 (-0.1646) and so finally $E(\mathcal{L}_m^{(v)})$ decreases.

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