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A SINGLE FACILITY MINISUM LOCATION PROBLEM UNDER THE *A*-DISTANCE

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Abstract Given demand points on a plane, we consider where a new facility should be located. In this article, we consider a single facility minisum location problem under the A-distance, and study properties of optimal solutions for the problem, and propose "The Edge Tracing Algorithm" to find all optimal solutions.

1. Introduction

Given demand points on a plane, we consider where a new facility should be located. This problem is called a single facility location problem, and usually formulated as a minimization problem with an objective function involving distances between the facility and demand points.

When a distance between two points is defined on the plane, minisum and minimax criteria have been used in location problems. Using minisum criterion, the optimal location is given by a point which minimizes the weighted sum of distances between the facility to be located and all demand points. Minisum criterion is used in location problems for the public facility[3]. Using minimax criterion, the optimal location is given by a point which minimizes the maximum distance among weighted distances between the facility to be located and all demand points. Minimax criterion is used in location problems for the emergency facility[6].

On the other hand, various distances are used in location analyses [2,3,6,8-11]. For example, the ℓ_1 distance (the rectilinear distance) is used in [11], ℓ_p distances are used in [2] where the ℓ_2 distance is the Euclidean distance ($p = 2$), and the one-infinity norm is used in [10].

P.Widmayer et al. generalized the rectilinear distance and proposed the A-distance [12]. The A-distance is used in many distance problems, e.g. Voronoi diagrams, minimum spanning trees, minimum distances between convex polygons and other sets of points[12], and minimax location problems[6] in location analyses.

In this article, we consider a single facility location problem under the A-distance.

In section 2, we give some definitions and results for the A-distance. In section 3, which is our main part, we formulate a single facility minisum location problem under the A-distance, and give some properties of the optimal solution, and propose "The Edge Tracing Algorithm" to find all optimal solutions to that problem. In section 4, we give a numerical example. Finally, in section 5, we give some conclusions.

2. The A-Metric

In $\boldsymbol{R}^2$, we assume that $m(\geq 2)$ orientations $0 \leq \alpha_1 < \alpha_2 < \cdots < \alpha_m < \pi$ are given where α_i's are angles with the positive direction of the x-axis. We set $A = \{\alpha_1, \alpha_2, \cdots, \alpha_m\}$. If the orientation of a line (a half line, a line segment) belongs to A, we call the line (the half line, the line segment) an A-oriented line (half line, line segment). We set $B = \{L : L$ is an A-oriented line segment. $\}$ and $[\boldsymbol{x}_1, \boldsymbol{x}_2] = \{\boldsymbol{x} = \lambda\boldsymbol{x}_1 + (1-\lambda)\boldsymbol{x}_2 : 0 \leq \lambda \leq 1\}$ for $\boldsymbol{x}_1, \boldsymbol{x}_2 \in \boldsymbol{R}^2$. The A-distance is defined as follows:

Definition 1 (The A-Distance) *For any $\boldsymbol{x}_1, \boldsymbol{x}_2 \in \boldsymbol{R}^2$, we define the A-distance between $\boldsymbol{x}_1$ and $\boldsymbol{x}_2$, $d_A(\boldsymbol{x}_1, \boldsymbol{x}_2)$, as*

$$d_A(\boldsymbol{x}_1, \boldsymbol{x}_2) = \begin{cases} d_2(\boldsymbol{x}_1, \boldsymbol{x}_2), & [\boldsymbol{x}_1, \boldsymbol{x}_2] \in B \\ \min\limits_{\boldsymbol{x}_3 \in \boldsymbol{R}^2} \left\{ d_2(\boldsymbol{x}_1, \boldsymbol{x}_3) + d_2(\boldsymbol{x}_3, \boldsymbol{x}_2) : [\boldsymbol{x}_1, \boldsymbol{x}_3], [\boldsymbol{x}_3, \boldsymbol{x}_2] \in B \right\}, & \textit{otherwise} \end{cases}$$

where d_2 is the Euclidean metric. We call d_A the A-metric. In fact, d_A is a metric in $\boldsymbol{R}^2$ [12].

Theorem 1 ([12]) *For any A and $\boldsymbol{x}_1, \boldsymbol{x}_2 \in \boldsymbol{R}^2$, $d_A(\boldsymbol{x}_1, \boldsymbol{x}_2)$ is always realized by a polygonal line segment which consists of at most two A-oriented line segments.*

In the following, we assume that A is given.

Definition 2 (The A-Circle) *For $\boldsymbol{y} \in \boldsymbol{R}^2$ and a constant $c > 0$,*

$$\{\boldsymbol{x} \in \boldsymbol{R}^2 : d_A(\boldsymbol{y}, \boldsymbol{x}) = c\}$$

is called the A-circle with radius c at center $\boldsymbol{y}$.

We set $\alpha_{m+k} = \pi + \alpha_k$, $k = 1, 2, \cdots, m$, $\alpha_0 = \alpha_{2m} - 2\pi$, and $\alpha_{2m+1} = \alpha_1 + 2\pi$. In this case, it follows that $0 \leq \alpha_1 < \alpha_2 < \cdots < \alpha_m < \pi \leq \alpha_{m+1} < \cdots < \alpha_{2m} < 2\pi$. Moreover, we set $\boldsymbol{a}_j = (\cos\alpha_j, \sin\alpha_j)$ for each α_j. By Definition 1, we can show two following lemmas easily.

Lemma 1 ([8]) *For $\boldsymbol{x} = (x^1, x^2)$ and $\boldsymbol{y} = (y^1, y^2)$ such that $\boldsymbol{x} \in \boldsymbol{y} + \mathcal{C}\{\boldsymbol{a}_j, \boldsymbol{a}_{j+1}\}$, where $\mathcal{C}\{\boldsymbol{a}_j, \boldsymbol{a}_{j+1}\} = \{\lambda\boldsymbol{a}_j + \mu\boldsymbol{a}_{j+1} : \lambda, \mu \geq 0\}$,*

$$(2.1) \qquad d_A(\boldsymbol{x}, \boldsymbol{y}) = \frac{(x^1 - y^1)(\sin\alpha_{j+1} - \sin\alpha_j) + (x^2 - y^2)(\cos\alpha_j - \cos\alpha_{j+1})}{\sin(\alpha_{j+1} - \alpha_j)}.$$

Lemma 2 ([8]) *For each $\boldsymbol{y} \in \boldsymbol{R}^2$, $f(\boldsymbol{x}) = d_A(\boldsymbol{x}, \boldsymbol{y})$ is a convex function.*

3. The Minisum Location Problem

In this section, we consider a single facility minisum location problem under the A-distance. In $\boldsymbol{R}^2$, we assume that n demand points $\boldsymbol{y}_i = (y_i^1, y_i^2)$, $i = 1, 2, \cdots, n$ are given. Let w_i be a positive weight for each $\boldsymbol{y}_i$ and $\boldsymbol{x}$ be the location of the facility to be located. The problem is formulated as

$$(3.1) \qquad \min_{\boldsymbol{x} \in \boldsymbol{R}^2} F(\boldsymbol{x})$$

where $F(\boldsymbol{x}) = \sum_{i=1}^n w_i d_A(\boldsymbol{x}, \boldsymbol{y}_i)$. Since F is a convex function by Lemma 2, (3.1) is a convex programming problem. Furthermore, there exists an optimal solution for (3.1). Let S^* be a set of optimal solutions for (3.1). We set $L_{ij} = \{\boldsymbol{y}_i + \gamma\boldsymbol{a}_j : \gamma \in \boldsymbol{R}\}$ for each $\boldsymbol{y}_i$ and α_j, i.e. L_{ij} is an $\boldsymbol{a}_j$-oriented line which passes $\boldsymbol{y}_i$. For $j \neq j'$, we call a point $L_{ij} \bigcap L_{i'j'}$ *an intersection point.* Let I be a set of intersection points, i.e.

$$I = \bigcup_{i,i',j\neq j'} (L_{ij} \bigcap L_{i'j'}).$$

We call a convex polygon $S \subset \boldsymbol{R}^2$ *a region* if all boundary lines of S are some of L_{ij}'s and $intS \neq \emptyset$, where $intS$ is the interior of S, and $(intS) \bigcap L_{ij} = \emptyset$ for all L_{ij}(Figure 1).

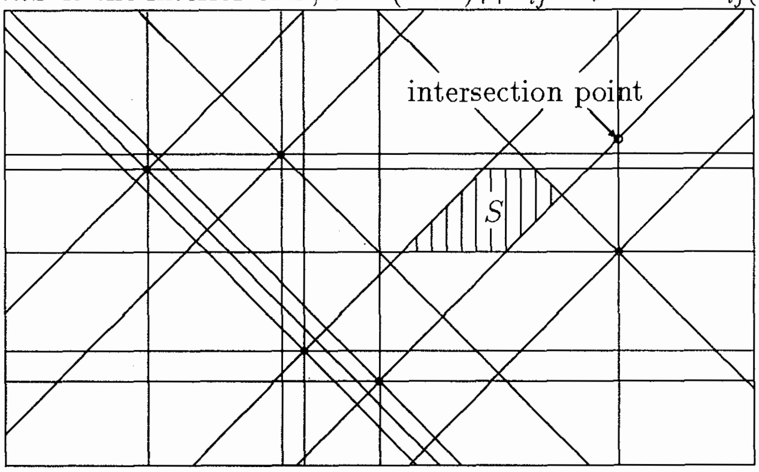

Figure 1. $A = \{0, \frac{\pi}{4}, \frac{\pi}{2}, \frac{3\pi}{4}\}$, • : demand points

Since any region S is represented as

$$S = \bigcap_{i=1}^{n} (\boldsymbol{y}_i + \mathcal{C}\{\boldsymbol{a}_{j_i}, \boldsymbol{a}_{j_i+1}\})$$

for some j_i $(1 \leq j_i \leq 2m)$, F is linear in $\boldsymbol{x}$ on each region S by (2.1). For any $\boldsymbol{x} \notin \bigcup_{i,j} L_{ij}$, a region S whose interior contains $\boldsymbol{x}$ is determined uniquely by $\boldsymbol{y}_i + \mathcal{C}\{\boldsymbol{a}_{j_i}, \boldsymbol{a}_{j_i+1}\}$, $i = 1, 2, \cdots, n$ which contain $\boldsymbol{x}$. In this case, we call a region S *a region characterized by* $\boldsymbol{x}$, and denote it by $S(\boldsymbol{x})$.

Theorem 2

$$S^* \bigcap I \neq \emptyset.$$

Proof For $\boldsymbol{x}^* \in S^* \backslash I$, two following cases may happen.

Case 1. For some adjacent intersection points $\boldsymbol{x}_1$ and $\boldsymbol{x}_2$, we assume that $\boldsymbol{x}^* = \lambda \boldsymbol{x}_1 + (1-\lambda)\boldsymbol{x}_2$ for some $\lambda (0 < \lambda < 1)$. Since F is constant on $[\boldsymbol{x}_1, \boldsymbol{x}_2]$, intersection points $\boldsymbol{x}_1$ and $\boldsymbol{x}_2$ are optimal.

Case 2. For some region S, we assume that $\boldsymbol{x}^* \in intS$. Since F is constant on S, intersection points which are vertices of a convex polygon S are also optimal. □

Theorem 3 *Let* S_1, S_2 *be adjacent bounded regions.*

$$S_1 \subset S^* \Longrightarrow S^* \bigcap (intS_2) = \emptyset.$$

Proof Assume that $\bar{\boldsymbol{x}} \in S^* \bigcap (intS_2)$. Since F is differentiable at $\boldsymbol{x}^* \in intS_1$, $\nabla F(\boldsymbol{x}^*) = \mathbf{o}$. It follows

$$\sum_{i=1}^{n} \frac{\sin \alpha_{j_i+1} - \sin \alpha_{j_i}}{\sin(\alpha_{j_i+1} - \alpha_{j_i})} = 0, \tag{3.2}$$

$$\sum_{i=1}^{n} \frac{\cos \alpha_{j_i} - \cos \alpha_{j_i+1}}{\sin(\alpha_{j_i+1} - \alpha_{j_i})} = 0, \tag{3.3}$$

for some j_i $(1 \le j_i \le 2m)$. Let $\boldsymbol{x}_1, \boldsymbol{x}_2$ be end points of a line segment $S_1 \bigcap S_2$ and α_{j_0} $(1 \le j_0 \le m)$ be an orientation of a line which passes $\boldsymbol{x}_1$ and $\boldsymbol{x}_2$. We set

$$\begin{aligned} I_{\alpha_{j_0}} &= \left\{ i : \boldsymbol{y}_i = \frac{\boldsymbol{x}_1 + \boldsymbol{x}_2}{2} + \gamma \boldsymbol{a}_{j_0} \text{ for some } \gamma > 0, 1 \le i \le n \right\}, \\ I_{\alpha_{m+j_0}} &= \left\{ i : \boldsymbol{y}_i = \frac{\boldsymbol{x}_1 + \boldsymbol{x}_2}{2} + \gamma \boldsymbol{a}_{m+j_0} \text{ for some } \gamma > 0, 1 \le i \le n \right\}. \end{aligned}$$

Note that $I_{\alpha_{j_0}} \bigcup I_{\alpha_{m+j_0}} \ne \emptyset$. For $i \in I_{\alpha_{j_0}}$, (a) $\boldsymbol{x}^* \in \boldsymbol{y}_i + \mathcal{C}\{\boldsymbol{a}_{m+j_0}, \boldsymbol{a}_{m+j_0+1}\}$ or (b) $\boldsymbol{x}^* \in \boldsymbol{y}_i + \mathcal{C}\{\boldsymbol{a}_{m+j_0-1}, \boldsymbol{a}_{m+j_0}\}$ holds. It is sufficient to show the case (a). We assume that the case (a) holds.

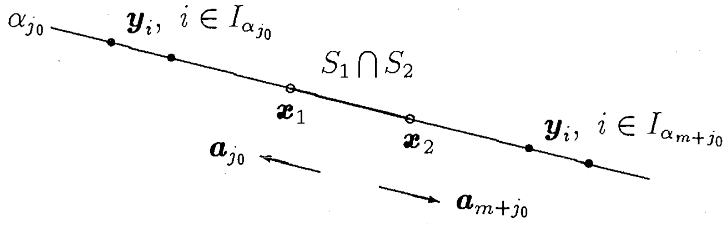

Figure 2.

From (3.2) and (3.3),

$$\sum_{i \notin I_{\alpha_{j_0}} \bigcup I_{\alpha_{m+j_0}}} w_i \frac{\sin \alpha_{j_i+1} - \sin \alpha_{j_i}}{\sin(\alpha_{j_i+1} - \alpha_{j_i})} + \sum_{i \in I_{\alpha_{j_0}}} w_i \frac{\sin \alpha_{m+j_0+1} - \sin \alpha_{m+j_0}}{\sin(\alpha_{m+j_0+1} - \alpha_{m+j_0})} + \sum_{i \in I_{\alpha_{m+j_0}}} w_i \frac{\sin \alpha_{j_0} - \sin \alpha_{j_0-1}}{\sin(\alpha_{j_0} - \alpha_{j_0-1})} = 0,$$

$$\sum_{i \notin I_{\alpha_{j_0}} \bigcup I_{\alpha_{m+j_0}}} w_i \frac{\cos \alpha_{j_i} - \cos \alpha_{j_i+1}}{\sin(\alpha_{j_i+1} - \alpha_{j_i})} + \sum_{i \in I_{\alpha_{j_0}}} w_i \frac{\cos \alpha_{m+j_0} - \cos \alpha_{m+j_0+1}}{\sin(\alpha_{m+j_0+1} - \alpha_{m+j_0})} + \sum_{i \in I_{\alpha_{m+j_0}}} w_i \frac{\cos \alpha_{j_0-1} - \cos \alpha_{j_0}}{\sin(\alpha_{j_0} - \alpha_{j_0-1})} = 0.$$

Therefore $\nabla F(\overline{\boldsymbol{x}})$ can be represented as

$$\nabla F(\overline{\boldsymbol{x}}) = \sum_{i \in I_{\alpha_{j_0}} \bigcup I_{\alpha_{m+j_0}}} w_i \nabla d_A(\overline{\boldsymbol{x}}, \boldsymbol{y}_i) - \sum_{i \in I_{\alpha_{j_0}} \bigcup I_{\alpha_{m+j_0}}} w_i \nabla d_A(\boldsymbol{x}^*, \boldsymbol{y}_i).$$

For $i \in I_{\alpha_{j_0}}$,

$$\nabla d_A(\boldsymbol{x}^*, \boldsymbol{y}_i) \in int\mathcal{C}\{\boldsymbol{a}_{m+j_0}, \boldsymbol{a}_{m+j_0+1}\},\ \nabla d_A(\overline{\boldsymbol{x}}, \boldsymbol{y}_i) \in int\mathcal{C}\{\boldsymbol{a}_{m+j_0-1}, \boldsymbol{a}_{m+j_0}\}.$$

For $i \in I_{\alpha_{m+j_0}}$,

$$\nabla d_A(\boldsymbol{x}^*, \boldsymbol{y}_i) \in int\mathcal{C}\{\boldsymbol{a}_{j_0-1}, \boldsymbol{a}_{j_0}\},\ \nabla d_A(\overline{\boldsymbol{x}}, \boldsymbol{y}_i) \in int\mathcal{C}\{\boldsymbol{a}_{j_0}, \boldsymbol{a}_{j_0+1}\}.$$

Since

$$\begin{aligned} \sum_{i \in I_{\alpha_{j_0}} \bigcup I_{\alpha_{m+j_0}}} w_i \nabla d_A(\boldsymbol{x}^*, \boldsymbol{y}_i) &\in int\mathcal{C}\left\{ \boldsymbol{a}_{j_0}, \boldsymbol{a}_{m+j_0}, \boldsymbol{x}^* - \frac{\boldsymbol{x}_1 + \boldsymbol{x}_2}{2} \right\}, \\ \sum_{i \in I_{\alpha_{j_0}} \bigcup I_{\alpha_{m+j_0}}} w_i \nabla d_A(\overline{\boldsymbol{x}}, \boldsymbol{y}_i) &\in int\mathcal{C}\left\{ \boldsymbol{a}_{j_0}, \boldsymbol{a}_{m+j_0}, \overline{\boldsymbol{x}} - \frac{\boldsymbol{x}_1 + \boldsymbol{x}_2}{2} \right\} \end{aligned}$$

and

$$intC\left\{\boldsymbol{a}_{j_0}, \boldsymbol{a}_{m+j_0}, \boldsymbol{x}^* - \frac{\boldsymbol{x}_1+\boldsymbol{x}_2}{2}\right\} \bigcap intC\left\{\boldsymbol{a}_{j_0}, \boldsymbol{a}_{m+j_0}, \overline{\boldsymbol{x}} - \frac{\boldsymbol{x}_1+\boldsymbol{x}_2}{2}\right\} = \emptyset,$$

we have

$$\nabla F(\overline{\boldsymbol{x}}) = \sum_{i\in I_{\alpha_{j_0}} \bigcup I_{\alpha_{m+j_0}}} w_i \nabla d_A(\overline{\boldsymbol{x}}, \boldsymbol{y}_i) - \sum_{i\in I_{\alpha_{j_0}} \bigcup I_{\alpha_{m+j_0}}} w_i \nabla d_A(\boldsymbol{x}^*, \boldsymbol{y}_i) \neq \mathbf{o}.$$

Thus $\overline{\boldsymbol{x}}$ is not optimal. This contradicts the assumption. □

Let $\{P_\lambda, \lambda \in \Lambda\}$ be a set of all convex polygons such that they include all demand points and their all boundary lines are A-oriented lines, where Λ is an index set. We set

$$P = \bigcap_{\lambda\in\Lambda} P_\lambda.$$

P is the smallest convex polygon such that it includes all demand points and its all boundary lines are A-oriented lines(Figure 3). Note that boundary lines of P are A-oriented supporting lines to $\{\boldsymbol{y}_1, \boldsymbol{y}_2, \cdots, \boldsymbol{y}_n\}$.

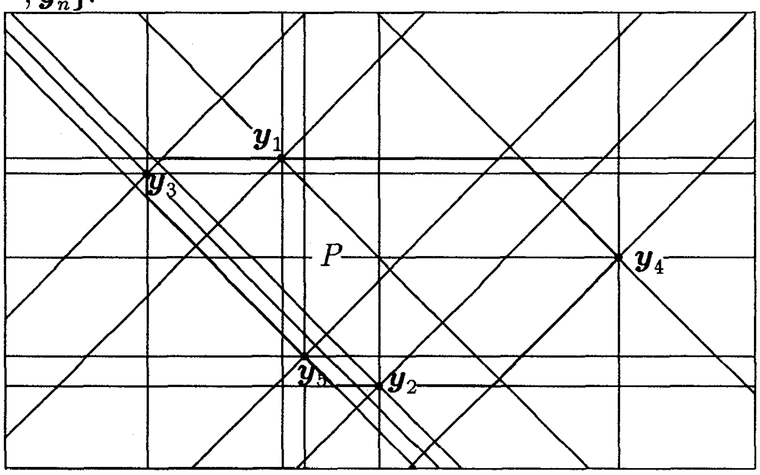

Figure 3. P for $\{\boldsymbol{y}_1, \boldsymbol{y}_2, \cdots, \boldsymbol{y}_5\}$, $A = \{0, \frac{\pi}{4}, \frac{\pi}{2}, \frac{3\pi}{4}\}$

Theorem 4

$$S^* \subset P.$$

Proof We assume that $\overline{\boldsymbol{x}} \notin P$ is optimal. There exists a line L such that L is an A-oriented supporting line to P and separates $\overline{\boldsymbol{x}}$ from P and $\overline{\boldsymbol{x}}$ is not on L. By the rotation of the plane and the translation, without loss of generality, we assume that $\overline{\boldsymbol{x}}$ is the origin and L is $y = c$ for some $c > 0$. According to such rotation and translation, we transform A-orientations and reset $\alpha_1, \alpha_2, \cdots, \alpha_m$. Note that $\{\boldsymbol{y}_1, \boldsymbol{y}_2, \cdots, \boldsymbol{y}_n\} \subset \{(x, y) : y > 0\}$ and $\alpha_1 = 0$. For each $\boldsymbol{y}_i$, $\boldsymbol{y}_i \in \mathcal{C}\{\boldsymbol{a}_{j_i}, \boldsymbol{a}_{j_i+1}\}$ for some j_i $(1 \le j_i \le m-1)$ and $\boldsymbol{y}_i$ is not on a line $y = 0$. Now we consider the A-circle with radius $d_A(\overline{\boldsymbol{x}}, \boldsymbol{y}_i)$ at center $\boldsymbol{y}_i$. For the simplicity of the notation, we set

$$\beta_j = \frac{\alpha_{m+j} + \alpha_{m+j+1}}{2}.$$

(i) When $\boldsymbol{y}_i \in intC\{\boldsymbol{a}_{j_i}, \boldsymbol{a}_{j_i+1}\}$ for some j_i $(1 \le j_i \le m-1)$, for sufficiently small $\varepsilon > 0$ and any

$$\boldsymbol{a} \in \left\{\boldsymbol{x} \in \boldsymbol{R}^2 : (\cos\beta_{j_i}, \sin\beta_{j_i})\, \boldsymbol{x}^T < 0\right\},$$

we have

$$d_A(\overline{\boldsymbol{x}} + \varepsilon \boldsymbol{a}, \boldsymbol{y}_i) < d_A(\overline{\boldsymbol{x}}, \boldsymbol{y}_i).$$

(ii) When $\boldsymbol{y}_i = \gamma \boldsymbol{a}_{j_i}$ for some $\gamma > 0, j_i$ $(2 \le j_i \le m)$, for sufficiently small $\varepsilon > 0$ and any

$$\boldsymbol{a} \in \left\{ \boldsymbol{x} \in \boldsymbol{R}^2 : (\cos \beta_{j_i - 1}, \sin \beta_{j_i - 1})\, \boldsymbol{x}^T < 0 \right\} \bigcap \left\{ \boldsymbol{x} \in \boldsymbol{R}^2 : (\cos \beta_{j_i}, \sin \beta_{j_i})\, \boldsymbol{x}^T < 0 \right\},$$

we have

$$d_A(\overline{\boldsymbol{x}} + \varepsilon \boldsymbol{a}, \boldsymbol{y}_i) < d_A(\overline{\boldsymbol{x}}, \boldsymbol{y}_i).$$

Since

$$(0,1) \in \bigcap_{j=1}^{m} \left\{ \boldsymbol{x} \in \boldsymbol{R}^2 : (\cos \beta_j, \sin \beta_j)\, \boldsymbol{x}^T < 0 \right\},$$

we have

$$d_A(\overline{\boldsymbol{x}} + \varepsilon(0,1), \boldsymbol{y}_i) < d_A(\overline{\boldsymbol{x}}, \boldsymbol{y}_i), \quad i = 1, 2, \cdots, n$$

for sufficiently small $\varepsilon > 0$. Therefore

$$F(\overline{\boldsymbol{x}} + \varepsilon(0,1)) < F(\overline{\boldsymbol{x}}).$$

This contradicts the assumption. □

By Theorem 2 and 4, there exists an optimal solution which is an intersection point in P. So we consider the determination of such an optimal solution by the iterative method that traces only intersection points in P, where an initial point is any demand point. Now we assume that we have a point $\boldsymbol{x}^{(r)}$ after rth iteration. Note that $\boldsymbol{x}^{(r)} \in I$. We say $\boldsymbol{a}_j$ satisfies condition (Q) for $\boldsymbol{x}^{(r)}$ if

$$\exists \boldsymbol{y}_i \text{ s.t. } \boldsymbol{y}_i = \boldsymbol{x}^{(r)} + \gamma \boldsymbol{a}_j \text{ for some } \gamma \in \boldsymbol{R}$$

and

$$\exists \varepsilon > 0 \text{ s.t. } \boldsymbol{x}^{(r)} + \varepsilon \boldsymbol{a}_j \in P.$$

We set

$$J = \{ j : \boldsymbol{a}_j \text{ satisfies condition (Q) for } \boldsymbol{x}^{(r)}.\}.$$

For the objective function F, we represent the right differential coefficient of F at $\boldsymbol{x}_0 \in \boldsymbol{R}^2$ with respect to $\mathbf{o} \neq \boldsymbol{a} \in \boldsymbol{R}^2$ as $\partial_+ F(\boldsymbol{x}_0; \boldsymbol{a})$, and set

$$u^{(r)} = \min_{j \in J} \left\{ \partial_+ F(\boldsymbol{x}^{(r)}; \boldsymbol{a}_j) \right\}. \tag{3.4}$$

If $u^{(r)} \ge 0$, then $\boldsymbol{x}^{(r)}$ is optimal by the convexity of F.

By Theorem 3, 4 and the proof of Theorem 3, S^* is an intersection point or an A-oriented line segment whose end points are adjacent intersection points or a region.

Before we state an algorithm, we consider the determination of P. We sort L_{ij}'s according to x-intercept or y-intercept. For each α_j, if $\alpha_j \neq \frac{\pi}{2}$, then L_{ij} is $-x \tan \alpha_j + y = y_i^2 - y_i^1 \tan \alpha_j$ and we set $b_{ij} = y_i^2 - y_i^1 \tan \alpha_j$. Otherwise L_{ij} is $x = y_i^1$ and we set $b_{ij} = y_i^1$. For each j, we sort all different lines among L_{ij}'s according to b_{ij}'s in ascending order. Let $\ell_1^j, \ell_2^j, \cdots, \ell_{n_j}^j$ be those lines, where ℓ_i^j is the ith line among α_j-oriented sorted lines. Note that $n_j \le n$. Now we assume that $0 \le \alpha_1 < \cdots < \alpha_{q-1} < \frac{\pi}{2}$, $\alpha_q = \frac{\pi}{2}$, $\frac{\pi}{2} < \alpha_{q+1} < \cdots < \alpha_m < \pi$. We arrange ℓ_i^j, $i = 1, n_j; j = 1, 2, \cdots, m$ as

$$\ell_1^1, \ell_1^2, \cdots, \ell_1^{q-1}; \ell_{n_q}^q, \ell_{n_{q+1}}^{q+1}, \cdots, \ell_{n_m}^m; \ell_{n_1}^1, \ell_{n_2}^2, \cdots, \ell_{n_{q-1}}^{q-1}; \ell_1^q, \ell_1^{q+1}, \cdots, \ell_1^m. \tag{3.5}$$

The kth line in (3.5) is $\boldsymbol{a}_k$-oriented($k = 1, 2, \cdots, 2m$). It is the bottom line if $1 \le k \le q-1$ or $m+q \le k \le 2m$, and it is the top line if $q \le k \le m+q-1$. The top line means the northmost or eastmost line among drawn lines with a same orientation, and the bottom line means the southmost or westmost line among drawn lines with a same orientation. Note that an $\boldsymbol{a}_{m+j}$-oriented line is also an $\boldsymbol{a}_j$-oriented line ($j = 1, 2, \cdots, m$). In (3.5), ℓ_i^j's are arranged as if they wrap P counterclockwise. The complexity for sorting n real numbers is $O(n \log n)$ [1]. So the complexity of the above sorting is $O(n \log n)$.

For example, we consider the case $n = 3, m = 3$(Figure 4). We have

$$\ell_1^1; \ell_3^2, \ell_3^3; \ell_3^1; \ell_1^2, \ell_1^3$$

as arranged lines. Note that $n_1 = n_2 = n_3 = 3$ and $q = 2$. A line ℓ_1^1 is the bottom $\boldsymbol{a}_1$-oriented line. Lines ℓ_3^2, ℓ_3^3 are the top $\boldsymbol{a}_2$-oriented line and the top $\boldsymbol{a}_3$-oriented line respectively. A line ℓ_3^1 is the top $\boldsymbol{a}_4$-oriented line. Lines ℓ_1^2, ℓ_1^3 are the bottom $\boldsymbol{a}_5$-oriented line and the bottom $\boldsymbol{a}_6$-oriented line respectively.

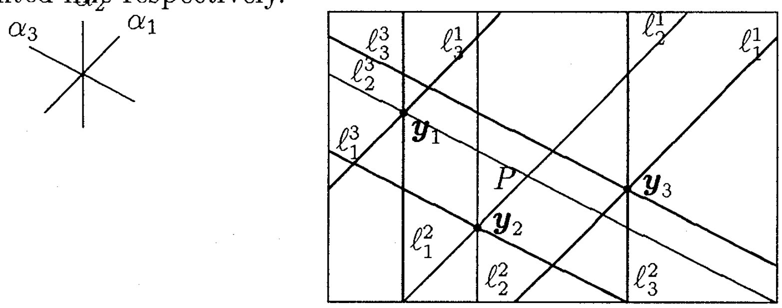

Figure 4. $\bullet$: demand points

Now coefficients of L_{ij}'s are stored. When we consider P, "=" in ℓ_1^j $(1 \le j \le m)$ is replaced by "$\ge$", and "=" in $\ell_{n_j}^j (1 \le j \le m)$ is replaced by "$\le$". P is a region determined by its system of inequalities. Note that this system of inequalities may contain redundant inequalities.

For the simplicity of the notation, let $\ell(1), \ell(2), \cdots, \ell(2m)$ be lines in (3.5). Especially, we set $\ell(2m+1) = \ell(1)$ and $\ell(2m+2) = \ell(2)$.

The procedure for the determination of P

Step 1. Determine an intersection point of $\ell(1)$ and $\ell(2)$, and let $\boldsymbol{z}_1$ be its intersection point. Set $j = 2$.

Step 2. Determine an intersection point of $\ell(j)$ and $\ell(j+1)$, and let $\boldsymbol{z}_j$ be its intersection point.

Step 3. If $\boldsymbol{z}_j = \boldsymbol{z}_{j-1}$ then remove $\ell(j)$.

Step 4. If $j = 2m+1$ then stop otherwise set $j = j+1$ and go to Step 2.

Let $\ell(j_1)$, $\ell(j_2)$, $\cdots$, $\ell(j_p)$ be lines left after the above procedure. P is represented by a system of inequalities corresponding to those lines. Now its system of inequalities does not contain redundant inequalities.

The Edge Tracing Algorithm

Step 1. Choose any demand point as an initial point $\boldsymbol{x}^{(0)}$. (We choose a demand point with the largest weight.) Set $r = 0$.

Step 2. Calculate $u^{(r)}$.

Step 3. If $u^{(r)} > 0$ then stop. $\boldsymbol{x}^{(r)}$ is an optimal solution.

Step 4. If $u^{(r)} = 0$ then stop. If the number of $\boldsymbol{a}_k$'s which satisfy $u^{(r)} = 0$, i.e. $\partial_+ F(\boldsymbol{x}^{(r)}; \boldsymbol{a}_k) = 0$, is

1. one, then any point $\boldsymbol{x} \in [\boldsymbol{x}^{(r)}, \boldsymbol{x}_k^{(r)}]$, where $\boldsymbol{x}_k^{(r)}$ is an $\boldsymbol{a}_k$-oriented adjacent intersection point to $\boldsymbol{x}^{(r)}$, is optimal.
2. two, then for sufficiently small $\varepsilon > 0$ and $\boldsymbol{a}_{k_1}, \boldsymbol{a}_{k_2}$ which satisfy $u^{(r)} = 0$, any point $\boldsymbol{x} \in S(\boldsymbol{x}^{(r)} + \varepsilon(\boldsymbol{a}_{k_1} + \boldsymbol{a}_{k_2}))$ is optimal.

Step 5. Otherwise, i.e. $u^{(r)} < 0$, choose any $\boldsymbol{a}_k$ which satisfies $u^{(r)} = \partial_+ F(\boldsymbol{x}^{(r)}; \boldsymbol{a}_k)$, and let $\boldsymbol{x}^{(r+1)}$ be an $\boldsymbol{a}_k$-oriented adjacent intersection point to $\boldsymbol{x}^{(r)}$. Set $r = r + 1$, and go to Step 2.

The Edge Tracing Algorithm is convergent in finite iterations because $\boldsymbol{x}^{(r)}$ in The Edge Tracing Algorithm is different from $\boldsymbol{x}^{(0)}, \boldsymbol{x}^{(1)}, \cdots, \boldsymbol{x}^{(r-1)}$ and the number of intersection points is finite and $F(\boldsymbol{x}^{(0)}) > F(\boldsymbol{x}^{(1)}) > \cdots > F(\boldsymbol{x}^{(r)})$ from Step 5. The number of intersection points is $O(n^2)$. For a given $\boldsymbol{x} \in \boldsymbol{R}^2$, the complexity for calculating $F(\boldsymbol{x})$ is $O(n)$, so the complexity for determining $u^{(r)}$ in Step 2 is $O(n)$. The complexity for determining $S(\boldsymbol{x}^{(r)} + \varepsilon(\boldsymbol{a}_{k_1} + \boldsymbol{a}_{k_2}))$ in 2 of Step 4 is $O(n)$, so the complexity of Step 4 is $O(n)$. The complexity for determining $\boldsymbol{x}^{(r+1)}$ is $O(1)$ when we have sorted lines, so the complexity of Step 5 is $O(1)$. Therefore the complexity of The Edge Tracing Algorithm is $O(n^3)$.

If $\boldsymbol{a}_k$ which satisfies $u^{(r)} = \partial_+ F(\boldsymbol{x}^{(r)}; \boldsymbol{a}_k)$ is determined in Step 5 of The Edge Tracing Algorithm, we need to determine $\boldsymbol{x}^{(r+1)}$ which is an $\boldsymbol{a}_k$-oriented adjacent intersection point to $\boldsymbol{x}^{(r)}$. Next we consider the procedure to determine $\boldsymbol{x}^{(r+1)}$. For each j, let $f_j(x, y)$ be the left side of L_{ij}, i.e.

$$f_j(x,y) = \begin{cases} -x\tan\alpha_j + y & \text{if } \alpha_j \neq \frac{\pi}{2}, \\ x & \text{if } \alpha_j = \frac{\pi}{2}. \end{cases}$$

If $\alpha_j \neq \frac{\pi}{2}$, then $\nabla f_j(x,y) = (-\tan\alpha_j, 1)$, otherwise $\nabla f_j(x,y) = (1,0)$.

We assume that an initial point $\boldsymbol{x}^{(0)} = \boldsymbol{y}_{i_0}$ is given. Set $r = 0$ where r is a counter. We determine

$$\ell^j_{s_r(j)},\ j = 1, 2, \cdots, m; 1 \leq s_r(j) \leq n_j$$

corresponding to $L_{i_0 j}$'s (e.g. binary search). Note that $\boldsymbol{x}^{(r)}$ is an intersection point of $\ell^j_{s_r(j)}$'s, i.e. $\boldsymbol{x}^{(r)}$ can be represented by $s_r(j)$'s. We concentrate on $s_r(j)$'s. We assume that α_k in Step 5 is determined. Set

$$j' = \begin{cases} k & \text{if } 1 \leq k \leq m, \\ k - m & \text{if } m < k \leq 2m. \end{cases}$$

For $j \neq j' (1 \leq j \leq m)$, set

$$t_{kj} = < \nabla f_j(x,y), \boldsymbol{a}_k >$$

where $< \cdot, \cdot >$ is the inner product, and we determine an intersection point of $\ell^{j'}_{s_r(j')}$ and $\ell^j_{s_r(j)+sign(t_{kj})}$ where

$$sign(x) = \begin{cases} +1 & \text{if } x > 0, \\ -1 & \text{if } x < 0, \end{cases}$$

for $x \neq 0$. For $j \neq j'$ $(1 \leq j \leq m)$, let $\boldsymbol{z}_{kj}$ be an intersection point of

$$\ell^{j'}_{s_r(j')} \quad \text{and} \quad \ell^j_{s_r(j)+sign(t_{kj})}. \tag{3.6}$$

$\boldsymbol{z}_{kj}$'s are candidates for $\boldsymbol{x}^{(r+1)}$. Set

$$J^{(r)} = \left\{ j : d_2(\boldsymbol{x}^{(r)}, \boldsymbol{z}_{kj}) = \min_{j \neq j', 1 \leq j \leq m} \left\{ d_2(\boldsymbol{x}^{(r)}, \boldsymbol{z}_{kj}) \right\} \right\}. \tag{3.7}$$

$\boldsymbol{x}^{(r+1)}$ is an intersection point of $\ell^{j'}_{s_r(j')}$ and $\ell^{j}_{s_r(j)+sign(t_{kj})}$, $j \in J^{(r)}$. Set

$$s_{r+1}(j) = \begin{cases} s_r(j) & \text{if } j = j', \\ s_r(j) + sign(t_{kj}) & \text{if } j \in J^{(r)}, \\ s_r(j) + 0.5sign(t_{kj}) & \text{otherwise.} \end{cases} \tag{3.8}$$

Set $r = r + 1$, and go to Step 2.

Now, a point $\boldsymbol{x}^{(r)}$ is an intersection point of $\ell^{j}_{s_r(j)}$'s such that $s_r(j) \in \boldsymbol{N}$ where $\boldsymbol{N}$ is a set of natural numbers. For j such that $s_r(j) \notin \boldsymbol{N}$, it means that $\boldsymbol{x}^{(r)}$ lies between $\ell^{j}_{[s_r(j)]}$ and $\ell^{j}_{[s_r(j)]+1}$ where $[\cdot]$ is Gauss' symbol.

In The Edge Tracing Algorithm, we represent a point $\boldsymbol{x}^{(r)}$ after the rth iteration as

$$\boldsymbol{x}^{(r)},\ \ell^{j}_{s_r(j)},\ j = 1, 2, \cdots, m.$$

The above representation is only the case of $r = 0$. If we consider the case of $r \geq 1$, $s_r(j) + sign(t_{kj})$ in (3.6) and the second equation in (3.8) is replaced by

$$\begin{cases} [s_r(j) - 0.5] & \text{if } sign(t_{kj}) = -1, \\ [s_r(j) + 1] & \text{if } sign(t_{kj}) = 1, \end{cases}$$

and $sign(t_{kj})$ in the third equation in (3.8) is replaced by

$$sign(t_{kj})([s_r(j)] - [s_r(j) - 0.5]).$$

4. Numerical Example

We consider a single facility minisum location problem

$$\min_{\boldsymbol{x} \in \boldsymbol{R}^2} F(\boldsymbol{x})$$

where $F(\boldsymbol{x}) = \sum_{i=1}^{5} d_A(\boldsymbol{x}, \boldsymbol{y}_i)$, $A = \left\{0, \frac{\pi}{4}, \frac{\pi}{2}, \frac{3\pi}{4}\right\}$, $\boldsymbol{y}_1 = (63, 97)$, $\boldsymbol{y}_2 = (102, 7)$, $\boldsymbol{y}_3 = (10, 90)$, $\boldsymbol{y}_4 = (197, 57)$, $\boldsymbol{y}_5 = (73, 20)$. In this case, $n = 5$ and $w_1 = w_2 = w_3 = w_4 = w_5 = 1$. We set $\boldsymbol{x}^{(0)} = \boldsymbol{y}_1$ as an initial point, and apply The Edge Tracing Algorithm to this problem.

1. (Step 1) The initial point is $\boldsymbol{x}^{(0)} = (63, 97)$ with

   $$s_0(1) = 5, s_0(2) = 4, s_0(3) = 2, s_0(4) = 4.$$

   Go to Step 2.
2. (Step 2) We have $u^{(0)} < 0$. Go to Step 5.
3. (step 5) We have $k = 7$, so $\boldsymbol{x}^{(1)}$ is an $\boldsymbol{a}_7$-oriented adjacent intersection point to $\boldsymbol{x}^{(0)}$. Go to Step 2.
4. (Step 2) We have $\boldsymbol{x}^{(1)} = (63, 90)$ with

   $$\begin{aligned} s_1(1) &= s_0(1) - 1 = 4\ , \\ s_1(2) &= s_0(2) - 0.5 = 3.5\ , \\ s_1(3) &= s_0(3) = 2\ , \\ s_1(4) &= s_0(4) - 0.5 = 3.5\ . \end{aligned}$$

Continuing the above procedure, we have $\boldsymbol{x}^{(2)} = (63, 57), \boldsymbol{x}^{(3)} = (73, 57), \boldsymbol{x}^{(4)} = (73, 36)$, and $u^{(4)} > 0$. The optimal solution is $\boldsymbol{x}^{(4)} = (73, 36)$, and the optimal value is $F(\boldsymbol{x}^{(4)}) = 340.22$ (Figure 5).

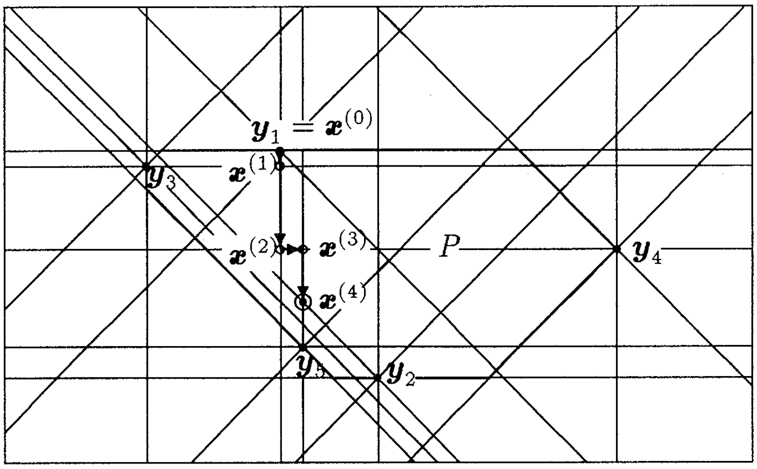

Figure 5. ⊙ : the optimal solution, • : demand points

5. Conclusion

We considered a single facility minisum location problem under the A-distance. This problem can be used when the facility to be located is the public one. In this case, we can regard the A-distance as the approximation to the road distance, and demand points as locations of users of the facility, and each weight as the number of users in the location of the demand point. We showed that at least one optimal solution is an intersection point(Theorem 2) and any optimal solution belongs to P(Theorem 4). A set of optimal solutions is an intersection point or an A-oriented line segment or a region by Theorem 3 and 4. Based on these results, we proposed The Edge Tracing Algorithm to determine all optimal solutions. The Edge Tracing Algorithm is an iterative algorithm using the descent method. Its algorithm generates a finite sequence of intersection points converging to an optimal solution. We also proposed the method of determining the next point efficiently in its algorithm by sorting drawn lines. We chose the demand point with the largest weight as an initial point. But we may need many iterations if the initial point is not near to an optimal solution and there are a lot of demand points near the optimal solution. In this sense, we need further research on the choice of an initial point to find an optimal solution more efficiently.

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