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FEASIBLE REGION REDUCTION CUTS FOR THE SIMPLE PLANT LOCATION PROBLEM

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Abstract We introduce a conceptually new method of generating strong cuts for the simple plant location problem. Based on the simple structure of the problem, we generate inequalities which even cut off part of the integer feasible region but still provide valid and sharp lower bounds of the problem when added to its linear programming relaxation. Also shown is the relation between such inequalities and the conventional valid inequalities of the problem.

1. Introduction

Consider the following simple plant location problem(SPLP)

$$
\begin{aligned}
(P) \qquad & \min \quad \sum_{i \in I}\sum_{j \in J} c_{ij}x_{ij} + \sum_{i \in I} f_i y_i & & \\
& \text{s.t.} \quad \sum_{i \in I} x_{ij} = 1, & j \in J, & \qquad (1)\\
& \qquad y_i - x_{ij} \geq 0, & i \in I,\ j \in J, & \qquad (2)\\
& \qquad x_{ij} \geq 0, & i \in I,\ j \in J, & \qquad (3)\\
& \qquad y_i = 0 \text{ or } 1, & i \in I, & \qquad (4)
\end{aligned}
$$

where $I = \{1, \cdots, m\}$ is the set of potential locations; $J = \{1, \cdots, n\}$ is the set of customers; y_i is 1 if facility i is established and 0 otherwise; x_{ij} is the fraction of customer j's demand supplied from facility i; f_i is the nonnegative fixed cost for establishing facility i, and c_{ij} is the nonnegative variable cost for supplying all of customer j's demand from facility i. The SPLP has received a great deal of attention due to its practical and theoretical significance. An excellent survey on this subject can be found in Krarup and Pruzan[11].

Recently, several researchers have studied on valid inequalities and facets for (P): see Guignard[9], Cornuejols and Thizy[6], and Cho et al.[4,5]. This kind of approach is based on the belief that any linear programming(LP) based algorithm can be sped up by incorporating valid inequalities in it. In fact, a number of researches show that integer programming problems can be solved efficiently using valid inequalities[2,7,13]. Valid inequalities, despite their redundancy to the integer feasible region, are useful in that they can cut off fractional points of the LP relaxation. Thus, the quality of a given valid inequality depends on how deeply it chops off part of the feasible region of the LP relaxation.

The purpose of this paper is to introduce a new type of inequality by extending the concept of the valid inequality. Consider an inequality which doesn't cut off the optimal solution but some integer feasible solutions of (P). The LP relaxation of (P) with those inequalities which we will call *feasible region reduction cuts* (FRRC's), still can provide a lower bound for (P). An FRRC can be thought stronger than a conventional valid inequality in that it can cut off even part of the integer feasible region as well as the feasible region of the linear programming relaxation. Now we extend the concept of the valid inequality to the one which includes FRRC's. An inequality is called *quasi-valid* for (P), if the optimal solution of (P) satisfies that inequality. Thus, both FRRC's and valid inequalities for (P) are quasi-valid for (P). It is obvious that this extended concept of the valid inequality can be applied to any integer programming problem.

For the SPLP, we can easily characterize the specific subset of the feasible region of (P) which contains an optimal solution of (P). We will call this specific region the *efficient region* of (P). In this paper, we derive quasi-valid inequalities and FRRC's for (P) using its efficient region. Barany et al.[1] also consider the convex hull of a subset of the feasible solutions, however, the new region derived by them is not the reduced one compared with the feasible region of the original formulation. Martin et al.[12] and Guignard and Spielberg[10] introduced some constraints, generated during the branch and bound procedure, which cut off a certain subset of integer feasible solutions. Beasley[3] also derived a cut which eliminates some integer feasible solutions found during the solution process. Their constraints might be thought similar to FRRC's. However the FRRC's derived in this paper are basically different from theirs in that the former can be generated without enumerating any integer feasible solution eliminated by the cut. Moreover, we can show that even their constraints can be strengthened using the underlying idea of an FRRC, i.e., the efficient region of (P).

In the next section, we briefly describe valid inequalities and facets of (P) presented by Guignard[9], Cornuejols and Thizy[6], and Cho et al.[4,5], since those inequalities can be used for generating FRRC's of (P). In Section 3, we describe the efficient region of (P) and introduce some results on its polyhedral structure. Section 4 presents the procedure generating FRRC's using the efficient region. Also is shown that the constraints developed by Martin et al.[12], Guignard and Spielberg[10], and Beasley[3], are strengthened using the efficient region of the SPLP. Finally, some concluding remarks are given in the last section.

2. Valid inequalities for the SPLP

Cho et al.[4,5] and Cornuejols and Thizy[6] derived several facets and valid inequalities of (P) by using the fact that (P) can be transformed into a set packing problem. Let $G = (N, E)$ be the graph whose node is associated with each variable x_{ij} for $i \in I$ and $j \in J$ and y_i for $i \in I$. We will use the same notation for a node and its associated variable. E contains the following arcs: For all $i \in I$ and $j \in J$, the arcs joining x_{ij} to y_i, and the arcs joining x_{ij} to every x_{kj} for $k \neq i$. For $I^S \subseteq I$ and $J^S \subseteq J$, let $S = (s_{ij})$ be $|I^S| \times |J^S|$ 0-1 matrix with no zero column and no zero row and G^S be the subgraph of G induced by the vertices y_i, for $i \in I^S$, and x_{ij}, for $i \in I^S$ and $j \in J^S$ such that $s_{ij} = 1$. Let $\beta(G^S)$ be the minimum number of plants $i \in I^S$ necessary to cover all destinations $j \in J^S$ using arcs of G^S.

Let F be the convex hull of all the integer feasible solutions of (P). Then the inequality

$$\sum_{i \in I^S} \sum_{j \in J^S} s_{ij} x_{ij} - \sum_{i \in I^S} y_i \leq |J^S| - \beta(G^S) \tag{5}$$

is a valid inequality of (P). Cho et al.[5] derived the necessary and sufficient condition for (5) to be a facet of F.

Theorem 1 (Cho et al.[5]) *Let $I^S \subseteq I$ and $J^S \subseteq J$, then the inequality* (5) *is a facet of*

F iff S is an $|I^S| \times |J^S|$ $0-1$ matrix with no zero column and no zero row which satisfies the following conditions: (i) *G^S is connected,* (ii) *there exists at least one zero element in each column of S,* (iii) *$|I^S| \geq 3$ and $|J^S| \geq 3$, and* (iv) *changing a zero element of S to one decreases $\beta(G^S)$ by one.*

Cornuejols and Thizy[6] and Guignard[9] derived two particular families of facets of F which are special cases of Theorem 1.

Theorem 2 (Cornuejols and Thizy[6]) *Consider any integers q and t such that $2 \leq t < q \leq m$ and $\binom{q}{t} \leq n$, and any subsets $I^S \subseteq I$ and $J^S \subseteq J$ such that $|J^S| = \binom{q}{t}$ and $|I^S| = q$. Let S be a matrix with $|I^S|$ rows and $|J^S|$ columns whose columns are all the different 0-1 vectors with t ones and $q-t$ zeros. Then the corresponding inequality* (5) *is a facet of F, and $\beta(G^S) = q - t + 1$.*

Theorem 3 (Guignard[9]) *Let S be a $k \times k$ cyclic matrix whose rows are 0-1 vectors in which $k-1$ consecutive ones are successively moved one position to the right. Then the corresponding inequality* (5) *is a facet of F, and $\beta(G^S) = 2$.*

3. The efficient region of (P)

In this section, we show that the optimal solution of (P) exists in the specific subset of the integer feasible region of (P) which depends on the objective function. For any given binary y-vector, an optimal set of x_{ij}'s is easily determined. Let $I^o = \{i : y_i = 1\}$ be a nonempty subset of open facilities, then an optimal set of x_{ij}'s is as follows:

$$x_{ij} = \begin{cases} 1, & \text{if } c_{ij} = \min_{k \in I^o} c_{kj} \text{ and } j \in J, \\ 0, & \text{otherwise.} \end{cases} \tag{6}$$

Define $\widehat{F} = \text{conv}\{(x, y) : y \in Y, \text{ and } x \text{ satisfies (6) for all } y \in Y\}$ where conv means convex hull and Y is the set of all the 0-1 vectors of dimension m except the zero vector.

$\widehat{F}$ is the efficient region of (P) as already defined since the optimal solution of (P) exists in $\widehat{F}$. Consider an inequality that every $(x, y) \in \widehat{F}$ satisfies. Even though some $(x, y) \in F - \widehat{F}$ violates that inequality, the inequality is quasi-valid with respect to F, and becomes an FRRC of (P). Therefore, if we are able to know the polyhedral structure of $\widehat{F}$, we can derive strong cutting planes for F by using the above extended concept of the valid inequality. Since $\widehat{F}$ depends on a given cost coefficient vector and doesn't show a nice structure, it isn't easy to describe the facial structure of $\widehat{F}$ which we will discuss shortly in the remaining section. Nevertheless, we still use the concept of the quasi-valid inequality to derive the strong cutting planes for (P), which will be shown in the next section.

Here we introduce an integer programming representation of $\widehat{F}$ and the following partial results for the polyhedral structure of $\widehat{F}$: the dimension of $\widehat{F}$ and a necessary and sufficient condition for the nonnegativity constraint to define a facet of $\widehat{F}$. Throughout the paper, we assume that for each $j \in J$, c_{ij}'s are strictly ordered, that is, ties are resolved arbitrarily. Without this assumption, $\widehat{F}$ may cut off an alternative optimal solution of (P). However, this assumption is made only for the exposition brevity. This can be relaxed and most of the results in this paper will remain valid in a slightly modified form. Moreover, this restriction doesn't affect the optimal selection of open facilities, i.e., y-vector. And since the determination of x_{ij}'s for a given y-vector is a relatively easy job, our assumption doesn't bring in any noticeable difficulties when solving the real-life problems having the same values of c_{ij}'s for some j. For each j we define the following notation to represent the order of c_{ij}: $c_{1(j),j} < c_{2(j),j} < \cdots < c_{m(j),j}$. Let $I^<(i,j) = \{k \in I | c_{kj} < c_{ij}\}$ and $I^>(i,j) = \{k \in I | c_{kj} > c_{ij}\}$ for $i \in I$ and $j \in J$. And let $A = \{(i,j) | i \in I, j \in J\}$.

One possible way to represent $\widehat{F}$ is to append the following constraint set to (P).

$$y_i + \sum_{k \in I^>(i,j)} x_{kj} \leq 1, \qquad i \in I,\, j \in J. \tag{7}$$

If $i = 1(j)$, (7) is satisfied as an equality by all the points in $\widehat{F}$. In other words, every $(x,y) \in \widehat{F}$ satisfies the following equations.

$$y_{1(j)} + \sum_{k \in I^>(1(j),j)} x_{kj} = 1, \qquad j \in J. \tag{8}$$

Whether (7) defines a facet of $\widehat{F}$ or not depends on the structure of c_{ij}'s. Later, we will show an example which illustrates a case where (7) can't define a facet.

If $I^>(i,j) = I^>(i',j') = \hat{I}$ for any pair of distinct elements (i,j) and (i',j') of A, all points in $\widehat{F}$ satisfies

$$\sum_{k \in \hat{I}} x_{kj} = \sum_{k \in \hat{I}} x_{kj'}. \tag{9}$$

If $i = 1(j)$ (i.e., $i = i' = 1(j')$), (9) is linearly dependant on the linear equation system (8). Now consider the case where $i \in \{2(j), \cdots, (m-1)(j)\}$. For some nonempty strict subset $\hat{I}$ of I, let $\widehat{A} = \{(i,j) \in A | I^>(i,j) = \hat{I}\}$. If $|\widehat{A}| \geq 2$, then for any pair of distinct elements (i,j) and (i',j') of $\widehat{A}$, (9) holds and $|\widehat{A}| - 1$ linearly independent equations exist corresponding $\widehat{A}$. Let $\mathcal{I} = \{I^>(i,j) | i = 2(j), \cdots, (m-1)(j) \text{ and } j \in J\}$. Then the total number of linearly independent equations (9) is $n(m-2) - |\mathcal{I}|$.

Based on the above observations, we can figure out the dimension of $\widehat{F}$.

Proposition 4 $\quad \dim \widehat{F} = m + |\mathcal{I}|$.

Proof:
Let $l = m + |\mathcal{I}|$. It is not difficult to show that the total number of independent equations (1), (8) and (9) is equal to $2n + n(m-2) - |\mathcal{I}|$. So $\dim \widehat{F} \leq l$. Thus it suffices to identify $l+1$ affinely independent points in $\widehat{F}$. We only exhibit y_i's since x_{ij}'s are determined uniquely as in (6). Consider the following three types of points: one vector which has only 1's as its elements; m vectors, each of which has a 0 in ith position and 1's elsewhere; $\mathcal{I}$ vectors, each of which is constructed such that for each distinct $I' \in \mathcal{I}$, $y_i = 1$ for $i \in I'$ and 0 otherwise. Then those $l+1$ points are affinely independent as required. □

Theorem 5 $x_{ij} \geq 0$ *defines a facet of* $\widehat{F}$ *if and only if* (a) $i \neq 1(j)$, *and* (b) *there exists no* $j' \in J$ *such that* $I^<(i,j) \subset I^<(i,j')$.

Proof:
($\Rightarrow$) It is obvious from the following two facts. If $i = 1(j)$

$$\{(x,y) \in \widehat{F} : x_{i,j} = 0\} \subseteq \{(x,y) \in \widehat{F} : y_{2(j)} + \sum_{k \in I^>(2(j),j)} x_{kj} = 1\} \subset \widehat{F},$$

and if there exists some $j' \in J$ such that $I^<(i,j) \subset I^<(i,j')$,

$$\{(x,y) \in \widehat{F} : x_{ij} = 0\} \subset \{(x,y) \in \widehat{F} : x_{ij'} = 0\} \subset \widehat{F}.$$

($\Leftarrow$) Consider the inequality $x_{ij} \geq 0$ which satisfies (a) and (b), then it suffices to show that there exist l (as defined in the proof of Proposition 4) affinely independent points. All the points exhibited in the proof of Proposition 4 satisfy $x_{ij} = 0$ except one point $(\hat{x}, \hat{y}) \in \widehat{F}$ such that $\hat{y}_k = 0$ for each $k \in I^<(i,j)$ and 1 otherwise. Note that such $(\hat{x}, \hat{y})$ corresponds to a point of the second type when $i = 2(j)$ and that of the third type when $i \in \{3(j), \cdots, m(j)\}$. □

Example 3.1. We illustrate the above results via an example presented by Erlenkotter[8]. In this example, $m = 5$ and $n = 8$. The total demand costs c_{ij} are given in Table 1, and the facility fixed charge vector $(f_i) = (200, 200, 200, 400, 300)$. We arbitrarily resolve ties as $c_{21} > c_{41}$, $c_{32} > c_{42}$, $c_{23} > c_{43}$, $c_{53} > c_{33}$, $c_{16} > c_{36}$, and $c_{18} > c_{58}$.

Table 1.
total demand cost c_{ij}

i\j	1	2	3	4	5	6	7	8
1	120	180	100	$+\infty$	60	$+\infty$	180	$+\infty$
2	210	$+\infty$	150	240	55	210	110	165
3	180	190	110	195	50	$+\infty$	$+\infty$	195
4	210	190	150	180	65	120	160	120
5	170	150	110	150	70	195	200	$+\infty$

$|\mathcal{I}| = 16$ and thus $\dim \widehat{F} = 21$. In this example, any inequality (7) with $i \neq 1(j)$ doesn't define a facet of $\widehat{F}$. For example, $y_5 + x_{31} + x_{41} + x_{21} \leq 1$ doesn't define a facet of $\widehat{F}$ since all the feasible solutions of $\widehat{F}$ satisfying $y_5 + x_{31} + x_{41} + x_{21} \leq 1$ as an equality also satisfy $x_{14} \geq 0$ as an equality. By Theorem 5, $x_{14} \geq 0$ is a facet but $x_{15} \geq 0$ is not.

4. Generating FRRC's using $\widehat{F}$

In this section, we show how quasi-valid inequalities and FRRC's for (P) can be derived using its efficient region. We first show that some classes of valid inequalities including those defining facets of F can be strengthened while being kept valid with respect to $\widehat{F}$. Second, we derive some particular classes of FRRC's for (P).

4.1. Lifting valid inequalities of (P) using $\widehat{F}$

As already shown in Section 2, several classes of valid inequalities for (P) are derived using its set packing formulation. Here we show that those inequalities can be more strengthened using the concept of the quasi-valid inequality. For a given $|I^S| \times |J^S|$ 0-1 matrix S, let $I_j^S = \{i \in I^S | s_{ij} = 1\}$ for $j \in J^S$ and $A^S = \{(i,j) | i \in I^S, j \in J^S, \text{ and } s_{ij} = 1\}$. And also let $J^S(i',j') = \{j \in J^S |\, I_j^S \subseteq I^>(i',j),\ I_j^S \subseteq I^<(i',j'), \text{ or } j = j'\}$ for each $(i',j') \in A \setminus A^S$.

Theorem 6 *Suppose that* (5) *is valid with respect to* F. *Then for any* $(i',j') \in A \setminus A^S$, *the following inequality*

$$\sum_{i \in I^S} \sum_{j \in J^S} s_{ij} x_{ij} - \sum_{i \in I^S} y_i + (\rho - 1) x_{i'j'} \leq |J^S| - \beta(G^S) \qquad (10)$$

is valid for $\widehat{F}$ *where* $\rho = \max\left(1, \max_{i \in I^S} |\{j \in J^S(i',j') | s_{ij} = 1\}|\right)$.

Proof:
If $\rho = 1$, the theorem trivially holds. Suppose $\rho \geq 2$. Consider any integer vector $(\hat{x}, \hat{y}) \in \widehat{F}$. If $\hat{x}_{i'j'} = 0$, then the inequality is trivially satisfied since $\widehat{F} \subseteq F$. If $\hat{x}_{i'j'} = 1$, then $\hat{x}_{ij} = 0$

for all $i \in I_j^S$ and $j \in J^S(i',j')$. Let S' be a submatrix of S obtained by removing all the columns of $J^S(i',j')$. Then

$$\sum_{i \in I^{S'}} \sum_{j \in J^{S'}} s'_{ij} x_{ij} - \sum_{i \in I^{S'}} y_i \leq |J^{S'}| - \beta(G^{S'}) \tag{11}$$

is also valid with respect to F and $(\hat{x}, \hat{y})$ satisfies (11) And by the definition, $\beta(G^S) \leq \beta(G^{S'}) + (|J^S(i',j')| - \rho + 1)$, $I^{S'} \subseteq I^S$, and $|J^S| = |J^{S'}| + |J^S(i',j')|$. Therefore, $(\hat{x}, \hat{y})$ satifies (10). □

Moreover, the procedure of Theorem 6 can be performed sequentially by the following procedure:

Initialization. Set $t = 1$, $A^0 = A^S$, $J^0 = \emptyset$, and $\rho_{ij} = 1$ for all $(i,j) \in A \setminus A^S$.

Iterative Step. For some $(i_t, j_t) \in A \setminus A^{t-1}$,
set $\rho_{i_t j_t} = \max\left(1, \max_{i \in I^S} |\{j \in J^S(i_t, j_t) \setminus J^{t-1} | s_{ij} = 1\}|\right)$,
$A^t = A^{t-1} \cup \{(i_t, j_t)\}$ and $J^t = J^{t-1} \cup J^S(i_t, j_t)$.

Termination. If either $t = |A \setminus A^S|$ or $|J^S \setminus J^t| \leq 1$, stop. Otherwise, increase t by 1 and go back to the iterative step.

Then we obtain the following inequality

$$\sum_{i \in I^S} \sum_{j \in J^S} s_{ij} x_{ij} - \sum_{i \in I^S} y_i + \sum_{(i,j) \in A \setminus A^S} (\rho_{ij} - 1) x_{ij} \leq |J^S| - \beta(G^S)$$

which is valid with respect to $\widehat{F}$.

Now consider why this new lifting procedure can be any help when implementing valid inequalities to solve the SPLP. The usual procedure of implementing valid inequalities takes the following steps: (i) solve the LP relaxation of the current problem; (ii) find a valid inequality which cuts off the LP solution; and (iii) add this inequality to the current problem and return to Step (i). Any algorithm which directly finds a cut of Step (ii) has not been reported as yet. So one tractable method is to generate a candidate inequality and then to check whether this inequality cuts off the current fractional solution. In this case, if the current fractional solution is not cut off by an inequality generated, an attempt is usually made to further strengthen that inequality by a lifting procedure. Theorem 6 and its generalization can be used as an effective vehicle for this purpose. Recall that for the conventional way of lifting (5), it requires to solve an NP-hard set covering problem for obtaining the associated covering number[5,6]. If priority is given to the tightening effect, we can apply the simple procedure developed here in addition to the conventional lifting procedure. Otherwise, only our procedure, bypassing the conventional one, can be run to save the associated computation. Another point to note is that even the facets of F can be strengthened, as illustrated by the following example.

Example 4.1. Refer to Example 3.1. The optimal solution to the LP relaxation is fractional. $y_1 = y_3 = y_5 = 1/3$, $y_2 = 2/3$ and $y_4 = 0$. The corresponding values of x_{ij} are indicated in Table 2.

Table 2.
x_{ij}'s of the LP optimal solution

i\j	1	2	3	4	5	6	7	8
1	1/3	1/3	1/3	0	0	0	1/3	0
2	0	0	0	1/3	2/3	2/3	2/3	2/3
3	1/3	1/3	1/3	1/3	1/3	0	0	1/3
4	0	0	0	0	0	0	0	0
5	1/3	1/3	1/3	1/3	0	1/3	0	0

Consider the following two valid inequalities of F.

$$x_{12}+x_{17}+x_{26}+x_{27}+x_{32}+x_{36}-y_1-y_2-y_3 \leq 1, \tag{12}$$

$$\begin{aligned} x_{11}+x_{13}+x_{15}+x_{25}+x_{26}+x_{28}+x_{31}+x_{32}+x_{38} \\ +x_{52}+x_{53}+x_{56}-y_1-y_2-y_3-y_5 \leq 3. \end{aligned} \tag{13}$$

(12) and (13) define facets of F. Our new lifting procedure can be applied to strengthen both (12) and (13) as follows:

$$x_{12}+x_{17}+x_{26}+x_{27}+x_{32}+x_{36}-y_1-y_2-y_3+x_{5j} \leq 1, j=1,\cdots,8 \tag{14}$$

$$\begin{aligned} x_{11}+x_{13}+x_{15}+x_{25}+x_{26}+x_{28}+x_{31}+x_{32}+x_{38} \\ +x_{52}+x_{53}+x_{56}-y_1-y_2-y_3-y_5+x_{33}+x_{46} \leq 3. \end{aligned} \tag{15}$$

(14) and (15) cut off some integer feasible solutions of (P), but they are valid for $\widehat{F}$. Furthermore, (14) with $j=1,2,3,4,$ and 6 and (15) cut off the LP optimal solution while (12) and (13) don't.

Now we show another example of strengthening some valid inequalities based on (P) by using the concept of the quasi-valid inequality. Martin et al.[12], Guignard and Spielberg[10] and Beasley[3] also developed some specially constructed constraints which cut off the integer feasible solutions of integer programming problems. However their constraints, when applying to the SPLP, are quite different from FRRC's derived in this paper, because their constraints can be used only after the complete information about the integer feasible solutions eliminated by the constraint, is known through any implicit or explicit enumeration process. Moreover, their constraints can even be strengthened using the concept of FRRC's.

Let I^o and I^c be two mutually exclusive sets of indices of y_i variables whose values are fixed as 1 or 0, respectively. To eliminate a set of feasible solutions, $\{(x,y) \in F : y_i = 1 \text{ for } i \in I^o \text{ and } y_i = 0 \text{ for } i \in I^c\}$ from the integer feasible region of (P), we add the constraint

$$\sum_{i\in I^c} y_i + \sum_{i \in I^o}(1-y_i) \geq 1. \tag{16}$$

Guignard and Spielberg[10] called (16) a 'preferred variable inequality' and showed how to derive and use it. When $I^o \cup I^c = I$, (16) eliminates a single integer y vector. Beasley[3] tries to obtain sharp lower bounds of the capacitated plant location problem using that constraint. Martin et al.[12] also uses (16) with $I^o = \emptyset$, through the specially designed branch and bound procedure.

Suppose that we obtain an inequality (16) during the solution process of (P). One possible way of using that inequality is to append it to the LP relaxation of (P) as in [3,12].

In this case, we strengthen the inequality using the same way as used in deriving FRRC's of (P). (16) can be converted into the following form

$$\sum_{i\in I^o} y_i + \sum_{i\in I^c}(1-y_i) \;\le\; |I^o| + |I^c| - 1, \tag{17}$$

and (17) can be lifted by extending it to a valid inequality for $\widehat{F}$.

Proposition 7 *Any integer feasible solution of* $\widehat{F}$ *satisfying* (17) *also satisfies the following inequality:*

$$\sum_{i\in I^o} y_i + \sum_{i\in I^c}(1-y_i) + \gamma_{ij}x_{ij} \;\le\; |I^o| + |I^c| - 1, \;\; \text{for } i \in I,\ j \in J$$

where,

$$\gamma_{ij} = \begin{cases} |I^o - I^>(i,j)|, & \text{for } i \in I^c, j \in J, \\ \max(0, |I^o - I^>(i,j)| - 1), & \text{otherwise.} \end{cases}$$

The proof is straightforward and thus omitted.

Example 4.2. Refer to Example 3.1. Suppose that all the integer feasible solutions with $y_1 = 1$ and $y_3 = 0$ are known and we want to know lower bounds of the remaining integer feasible solutions. Then, we can add $y_1 + (1-y_3) \le 1$ to the LP relaxation of (P). However, $y_1 + (1-y_3) + x_{31} \le 1$ is stronger than $y_1 + (1-y_3) \le 1$. Moreover, the former cuts off the LP optimal solution while the latter doesn't.

4.2. Valid inequalities for $\widehat{F}$

We can also directly derive the following particular classes of valid inequalities of $\widehat{F}$ without using a valid inequality for (P).

Theorem 8 *For any* $\widehat{A} \subseteq A$, *suppose that any pair of elements,* (i,j) *and* (i',j') *of* $\widehat{A}$ *satisfy* $i \in I^<(i',j')$, $i' \in I^<(i,j)$, *or* $j' = j$. *Then the inequality*

$$\sum_{(i,j)\in\widehat{A}} x_{ij} \le 1 \tag{18}$$

is valid with respect to $\widehat{F}$.

Proof:
Consider any integer vector $(\hat{x},\hat{y}) \in \widehat{F}$. If $\hat{x}_{i'j'} = 1$ for some $(i',j') \in \widehat{A}$, then $\hat{x}_{ij} = 0$ for all $(i,j) \in \widehat{A}$ with $(i,j) \ne (i',j')$. □

Theorem 9 *For some* $i' \in I$, $j' \in J$, *and* $J' \subseteq J \setminus \{j'\}$ *with* $|J'| \ge 2$, *the inequality*

$$\sum_{j\in J'} x_{i'j} + \sum_{i\in I\setminus\{i'\}} x_{ij'} - y_{i'} \le |J'| - 1 \tag{19}$$

is valid for $\widehat{F}$ *if* $I^<(i',j') \subseteq \bigcup_{j\in J} I^<(i',j)$.

Proof:
Consider an integer vector $(\hat{x},\hat{y}) \in \widehat{F}$. If $\hat{y}_{i'} = 0$, $\hat{x}_{i'j} = 0$ for all $j \in J$. Since $|J| \ge 2$ and $\sum_{i\in P}\hat{x}_{ij'} \le 1$, (19) is satisfied. Suppose $\hat{y}_{i'} = 1$. Since $I^l(i',j') \subseteq \bigcup_{j\in J} I^l(i',j)$, if some

$\hat{x}_{ij'}$, $i \in P \setminus \{i'\}$ equals 1, then at least one $\hat{x}_{i'j}$ for some $j \in J$ equals 0. Therefore, (19) is satisfied. □

Example 4.3. Refer to Example 3.1. The inequalities $x_{53} + x_{43} + x_{14} + x_{26} + x_{38} \leq 1$ and $x_{53} + x_{26} + x_{38} \leq 1$ are those of the form of (18). $x_{25} + x_{26} + x_{54} + x_{44} + x_{34} + x_{14} - y_2 \leq 1$ is an inequality of the form (19) where $i' = 2, j' = 4$, and $J = \{5, 6\}$. Moreover, all the three inequalities cut off the LP optimal solution.

5. Conclusions

We have proposed a method of generating valid inequalities for the SPLP by considering the objective function as well as its integer feasible region. The cutting planes thus generated even cut off a part of the feasible region, which is not possible with the conventional valid inequalities. This way of widening the boundary of valid inequalities gives us flexibility in deriving valid inequalities which are computationally expensive to generate.

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