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A SIMPLE CHARACTERIZATION OF RETURNS TO SCALE IN DEA

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Abstract In this paper, we will present a simple method for deciding the local returns-to-scale characteristics of DMUs (Decision Making Units) in Data Envelopment Analysis. This method proceeds as follows: first, we solve the BCC (Banker-Charnes-Cooper) model and find the returns-to-scale of BCC-efficient DMUs and a reference set to each BCC-inefficient DMU. We can then decide the local returns-to-scale characteristics of each BCC-inefficient DMU by observing only the returns-to-scale characteristics of DMUs in their respective reference sets. No extra computation is required. We can apply this method to the output-oriented model and to the additive model as well.

1. Introduction

The standard medel for analyzing returns-to-scale in DEA was first proposed by Banker (1980) and subsequently, Banker, Charnes and Cooper (1984), Banker (1984) and Färe, Grosskopf and Lovell (FGL, 1985) have extended its contents and analysis substantially.

Then, Banker and Thrall (BT, 1992) presented extensive research with respect to returns-to-scale in DEA. Recently, Banker, Bardham and Cooper (BBC, 1995) added computational convenience and efficiency to the work of Banker and Thrall (BT, 1992).

Although the concept of returns-to-scale, i.e. *increasing*, *constant* and *decreasing*, is unambiguous only at points on the efficient sections of the production frontier, several pieces of research extended this concept to inefficient DMUs by moving them to the efficient frontiers. Naturally, in this case, returns-to-scale depend on the method used to bring such DMUs to efficient frontiers. FGL (1985) and Banker, Chang and Cooper (BCC, 1995) address this subject. Both methods employ a two-step approach to estimate returns-to-scale. Specifically, FGL (1985) solve, in step 1, the BCC and CCR models and in step 2 solve a linear program for each nonconstant returns-to-scale DMU. Therefore, they need to solve three LPs for nonconstant returns-to-scale DMUs. On the other hand, BCC (1995) solve the CCR model in step 1 and then an LP further for each DMU with nonconstant returns-to-scale characteristics in step 2.

The method that we will propose in this paper solves the BCC model and then determines returns-to-scale of BCC-efficient DMUs. It will be demonstrated that returns-to-scale of projected BCC-inefficient DMUs can be determined automatically from their reference set. From the computational point of view, one pass of BCC computation and BT (1992) process for BCC-efficient DMUs are all that is needed for estimating returns-to-scale.

We would like to add some comments on the reason why we address returns-to-scale for inefficient DMUs. Usually, the number of BCC-inefficient DMUs is larger than that of efficient ones which have definite characteristics in returns-to-scale. So, in some cases, the majority of DMUs remain unanswered for returns-to-scale characteristics. While for BCC-efficient DMUs, we can decide the returns-to-scale characteristics by such processes as one in BT (1992), there is no way of deciding the characteristics for BCC-inefficient DMUs directly. For this purpose, first we must project such DMUs on the efficient frontier and then try to decide their characteristics. Thus, the characteristics depend on the method of projection. It may occur that a projected DMU by the input-oriented BCC model exhibits *increasing* characteristics, while one, projected by the output-oriented BCC model, shows *decreasing* ones.

The rest of the paper is organized as follows. Sections 2 and 3 offer preliminary information, concerned with definitions and known theorems that will be used in the succeeding section. Section 4 is the main part of the paper, in which an alternative method will be presented. In Section 5, we will exhibit a numerical example of our method and then a similar analysis will be presented for the output-oriented case and for the additive model. Finally, comparisons with other methods will be discussed in Section 7.

2. The CCR and BCC Models

We will deal with n DMUs (Decision Making Units) with the input and output matrices $X = (\boldsymbol{x}_j) \in R^{m\times n}$ and $Y = (\boldsymbol{y}_j) \in R^{s\times n}$, respectively. We assume that $\boldsymbol{x}_j$ and $\boldsymbol{y}_j$ are *semipositive*, *i.e.* $\boldsymbol{x}_j \geq \mathbf{0}$, $\boldsymbol{x}_j \neq \mathbf{0}$ and $\boldsymbol{y}_j \geq \mathbf{0}$, $\boldsymbol{y}_j \neq \mathbf{0}$ for $j = 1,\cdots,n$.

2.1 The CCR Model

The production possibility set P_C of the CCR (Charnes, Cooper and Rhodes, 1978) model is defined as a set of semipositive $(\boldsymbol{x}, \boldsymbol{y})$ as follows:

$$P_C = \{(\boldsymbol{x}, \boldsymbol{y}) \mid \boldsymbol{x} \geq X\boldsymbol{\lambda},\ \boldsymbol{y} \leq Y\boldsymbol{\lambda},\ \boldsymbol{\lambda} \geq \mathbf{0}\}, \tag{1}$$

where $\boldsymbol{\lambda}$ is a semipositive vector in R^n.

The CCR model evaluates the efficiency of each DMU_o $(\boldsymbol{x}_o, \boldsymbol{y}_o)$ $(o = 1,\cdots,n)$ by solving the following linear program:

$$\begin{array}{llrcl}
(2) & & (CCR_o) \quad \min & \theta_C & \\
(3) & \text{subject to} & \theta_C \boldsymbol{x}_o - X\boldsymbol{\lambda} - \boldsymbol{s}_x & = & \mathbf{0} \\
(4) & & Y\boldsymbol{\lambda} - \boldsymbol{s}_y & = & \boldsymbol{y}_o \\
(5) & & \boldsymbol{\lambda} \geq \mathbf{0},\quad \boldsymbol{s}_x & \geq & \mathbf{0},\quad \boldsymbol{s}_y \geq \mathbf{0}.
\end{array}$$

The dual problem of (CCR_o) is described by:

$$\begin{array}{lrcl}
(6) & (DCCR_o) \quad \max & \boldsymbol{u}\boldsymbol{y}_o & \\
(7) & \text{subject to} \quad \boldsymbol{v}\boldsymbol{x}_o & = & 1 \\
(8) & -\boldsymbol{v}X + \boldsymbol{u}Y & \leq & \mathbf{0} \\
(9) & \boldsymbol{v} \geq \mathbf{0},\quad \boldsymbol{u} & \geq & \mathbf{0},
\end{array}$$

where $\boldsymbol{v} \in R^m$ and $\boldsymbol{u} \in R^s$ are row vectors and represent dual variables corresponding to (3) and (4), respectively.

In every optimal solution for (CCR_o) and $(DCCR_o)$, the pairs $(\boldsymbol{s}_x, \boldsymbol{v})$ and $(\boldsymbol{s}_y, \boldsymbol{u})$ are complementary each other, *i.e.*, it holds

$$\boldsymbol{v}\boldsymbol{s}_x = 0 \qquad \text{and} \qquad \boldsymbol{u}\boldsymbol{s}_y = 0. \tag{10}$$

We use the following two-phase process with the purpose of solving (CCR_o) and determining the input surplus $\boldsymbol{s}_x$ and output shortage $\boldsymbol{s}_y$. In Phase I , we solve (CCR_o), and in

Phase II we maximize $\boldsymbol{e}\boldsymbol{s}_x + \boldsymbol{e}\boldsymbol{s}_y$ (the sum of input surplus and output shortage) under the added condition $\theta_C = \theta_C^*$ (the Phase I optimal objective value), where $\boldsymbol{e}$ is a row vector in which all elements are equal to 1.

Let an optimal solution in Phase II be $(\theta_C^*, \boldsymbol{\lambda}^*, \boldsymbol{s}_x^*, \boldsymbol{s}_y^*)$, based on which we define CCR-efficiency as follows:

Definition 1 (CCR-Efficiency). *A DMU_o is called* CCR-efficient *if it has* $\theta_C^* = 1$, $\boldsymbol{s}_x^* = \boldsymbol{0}$ *and* $\boldsymbol{s}_y^* = \boldsymbol{0}$. *Otherwise, it is called* CCR-inefficient.

If a DMU_o is CCR-efficient, it holds that $\boldsymbol{s}_x^* = \boldsymbol{0}$ and $\boldsymbol{s}_y^* = \boldsymbol{0}$ for every optimal solution for (CCR_o). Thus, the strong theorem of complementary slackness ensures the existence of a positive optimal solution $(\boldsymbol{v}^*, \boldsymbol{u}^*)$ for the dual problem $(DCCR_o)$.

Lemma 1. *If a DMU_o is CCR-efficient, then $(DCCR_o)$ has an optimal solution $(\boldsymbol{v}^*, \boldsymbol{u}^*)$ such that*

$$\boldsymbol{v}^* > \boldsymbol{0} \quad \text{and} \quad \boldsymbol{u}^* > \boldsymbol{0}. \tag{11}$$

Suppose that $DMU_{j_1}, \cdots, DMU_{j_k}$ are CCR-efficient. Let a semipositive activity $(\boldsymbol{x}, \boldsymbol{y})$ be a nonnegative combination of them:

$$\boldsymbol{x} = \sum_{l=1}^{k} \lambda_l \boldsymbol{x}_{j_l} \quad \text{and} \quad \boldsymbol{y} = \sum_{l=1}^{k} \lambda_l \boldsymbol{y}_{j_l}. \tag{12}$$

Then, we have the following well established theorem, which will be utilized later in demonstrating our main theorems.

Theorem 1. *If $DMU_{j_1}, \cdots, DMU_{j_k}$ are CCR-efficient, then the activity expressed by* (12) *is also CCR-efficient.*

If a DMU_o is CCR-inefficient, a *reference set* to the DMU_o is defined by

$$E_o^C = \{j \mid \lambda_j^* > 0, \quad j = 1, \cdots, n\}. \tag{13}$$

The optimal solution satisfies the following relations:

$$\theta_C^* \boldsymbol{x}_o = \sum_{j \in E_o^C} \lambda_j^* \boldsymbol{x}_j + \boldsymbol{s}_x^* \tag{14}$$

$$\boldsymbol{y}_o = \sum_{j \in E_o^C} \lambda_j^* \boldsymbol{y}_j - \boldsymbol{s}_y^*. \tag{15}$$

The *CCR-projection* based on the optimal solution is defined by:

$$\boldsymbol{x}_e^C = \theta_C^* \boldsymbol{x}_o - \boldsymbol{s}_x^* = \sum_{j \in E_o^C} \lambda_j^* \boldsymbol{x}_j \tag{16}$$

$$\boldsymbol{y}_e^C = \boldsymbol{y}_o + \boldsymbol{s}_y^* = \sum_{j \in E_o^C} \lambda_j^* \boldsymbol{y}_j. \tag{17}$$

Lemma 2. *Every DMU in the reference set E_o^C is CCR-efficient.*

Lemma 3. *The CCR-projected activity $(\boldsymbol{x}_e^C, \boldsymbol{y}_e^C)$ is CCR-efficient.*

2.2 The BCC Model

The production possibility set of the BCC (Banker, Charnes and Cooper, 1984) model is described as:

(18) $$P_B = \{(\boldsymbol{x}, \boldsymbol{y}) \mid \boldsymbol{x} \geq X\boldsymbol{\lambda},\ \boldsymbol{y} \leq Y\boldsymbol{\lambda},\ \boldsymbol{e\lambda} = 1,\ \boldsymbol{\lambda} \geq \boldsymbol{0}\}.$$

The BCC model evaluates the efficiency of each $DMU_o(\boldsymbol{x}_o, \boldsymbol{y}_o)$ $(o = 1, \cdots, n)$ by solving the following linear program:

$$
\begin{array}{llrcl}
(19) & (BCC_o) & \min & & \theta_B \\
(20) & \text{subject to} & \theta_B x_o - X\boldsymbol{\lambda} - \boldsymbol{s}_x & = & \boldsymbol{0} \\
(21) & & Y\boldsymbol{\lambda} - \boldsymbol{s}_y & = & \boldsymbol{y}_o \\
(22) & & \boldsymbol{e\lambda} & = & 1 \\
(23) & & \boldsymbol{\lambda} \geq \boldsymbol{0}, \quad \boldsymbol{s}_x & \geq & \boldsymbol{0}, \quad \boldsymbol{s}_y \geq \boldsymbol{0}.
\end{array}
$$

We express the dual program of (BCC_o) as:

$$
\begin{array}{llrcl}
(24) & (DBCC_o) & \max \quad z & = & \boldsymbol{u}\boldsymbol{y}_o - u_o \\
(25) & & \text{subject to} \quad \boldsymbol{v}\boldsymbol{x}_o & = & 1 \\
(26) & & -\boldsymbol{v}X + \boldsymbol{u}Y - u_o\boldsymbol{e} & \leq & \boldsymbol{0} \\
(27) & & \boldsymbol{v} \geq \boldsymbol{0}, \quad \boldsymbol{u} & \geq & \boldsymbol{0},
\end{array}
$$

where u_0 is free in sign.

As with the CCR case, we employ the two phase process for solving (BCC_o). Let an optimal solution in Phase II be $(\theta_B^*, \boldsymbol{\lambda}^*, \boldsymbol{s}_x^*, \boldsymbol{s}_y^*)$, based on which we define BCC-efficiency as follows:

Definition 2 (BCC-Efficiency). *A DMU_o is called* BCC-efficient, *if it has $\theta_B^* = 1$, $\boldsymbol{s}_x^* = \boldsymbol{0}$ and $\boldsymbol{s}_y^* = \boldsymbol{0}$. Otherwise, it is called* BCC-inefficient.

If a DMU_o is BCC-efficient, then there exists an optimal solution $(\boldsymbol{v}^*, \boldsymbol{u}^*, u_0^*)$ with $\boldsymbol{v}^* > \boldsymbol{0}$ and $\boldsymbol{u}^* > \boldsymbol{0}$.

If a DMU_o is BCC-inefficient, the *reference set* E_o^B and the *BCC-projection* $(\boldsymbol{x}_e^B, \boldsymbol{y}_e^B)$ based on the reference set, are defined as:

$$
\begin{array}{lrcl}
(28) & E_o^B & = & \{j \mid \lambda_j^* > 0, \quad j = 1, \cdots, n\} \\
(29) & \boldsymbol{x}_e^B & = & \theta_B^* \boldsymbol{x}_o - \boldsymbol{s}_x^* = \sum_{j \in E_o^B} \lambda_j^* \boldsymbol{x}_j \\
(30) & \boldsymbol{y}_e^B & = & \boldsymbol{y}_o + \boldsymbol{s}_y^* = \sum_{j \in E_o^B} \lambda_j^* \boldsymbol{y}_j.
\end{array}
$$

Corresponding to Lemmas 2 and 3, we have:

Lemma 4. *Every DMU in the reference set E_o^B is BCC-efficient.*

Lemma 5. *The BCC projected activity (x_e^B, y_e^B) is BCC-efficient.*

3. Returns to Scale of BCC-Efficient DMUs

Banker and Thrall (1992) demonstrated the following theorem on returns-to-scale of BCC efficient DMUs.

Theorem 2 (Returns-to-Scale). *Suppose DMU_o is BCC-efficient and let the* sup *and* inf *of u_0 in the optimal solution for $(DBCC_o)$ be $\bar{u}_0$ and $\underline{u}_0$, respectively. Then, we have:*

1. *If* $0 > \bar{u}_0$, *then increasing returns-to-scale prevail in the* DMU_o.
2. *If* $\bar{u}_0 \geq 0 \geq \underline{u}_0$, *then constant returns-to-scale prevail in the* DMU_o.
3. *If* $\underline{u}_0 > 0$, *then decreasing returns-to-scale prevail in the* DMU_o.

We will denote *increasing*, *constant* and *decreasing* returns-to-scale by IRS, CRS and DRS, respectively.

Corollary 1. *DMU_o is CCR-efficient if and only if it is BCC-efficient and displays CRS.*

Usually, we solve (BCC_o) by the simplex method of linear programming and obtain an optimal dual solution $(\boldsymbol{v}^*, \boldsymbol{u}^*, u_0^*)$ as the simplex multiplier of the Phase I optimal tableau. Thus, if $u_0^* > 0$, then we need to solve the lower bound $\underline{u}_0$, and if $u_0^* < 0$, then we need to solve the upper bound $\bar{u}_0$ for deciding the returns-to-scale characteristics of the DMU. If $u_0^* = 0$, the DMU shows CRS. The computation of $\underline{u}_0$ (or $\bar{u}_0$) is carried out in the primal side of LP.

4. Characterization of Returns to Scale

Theorem 3. *If a $DMU(\boldsymbol{x}_o, \boldsymbol{y}_o)$ is BCC-inefficient, the reference set E_o^B to $(\boldsymbol{x}_o, \boldsymbol{y}_o)$ defined by* (28), *does not include both IRS and DRS DMUs.*

Proof. Let an optimal solution for $(DBCC_o)$ to the BCC-projected activity $(\boldsymbol{x}_e^B, \boldsymbol{y}_e^B)$ be $(\boldsymbol{v}_e, \boldsymbol{u}_e, u_{0e})$, with $\boldsymbol{v}_e > \mathbf{0}$ and $\boldsymbol{u}_e > \mathbf{0}$. Since $(\boldsymbol{x}_e^B, \boldsymbol{y}_e^B)$ is BCC-efficient, we obtain the following relations:

$$\begin{aligned}
(31) \qquad & \boldsymbol{u}_e \boldsymbol{y}_e^B - u_{0e} &=& \ 1 \\
(32) \qquad & \boldsymbol{v}_e \boldsymbol{x}_e^B &=& \ 1 \\
(33) \qquad & -\boldsymbol{v}_e X + \boldsymbol{u}_e Y - u_{0e}\boldsymbol{e} &\leq& \ \mathbf{0} \\
(34) \qquad & \boldsymbol{v}_e \geq \mathbf{0}, \quad \boldsymbol{u}_e &\geq& \ \mathbf{0}.
\end{aligned}$$

Hence, for $j \in E_o^B$, we have:

$$(35) \qquad -\boldsymbol{v}_e \boldsymbol{x}_j + \boldsymbol{u}_e \boldsymbol{y}_j - u_{0e} \leq 0.$$

From (31) and (32), it holds:

$$(36) \qquad -\boldsymbol{v}_e \boldsymbol{x}_e^B + \boldsymbol{u}_e \boldsymbol{y}_e^B - u_{0e} = 0.$$

By substituting the righthand side of equations (29) and (30) for (36), we obtain:

$$(37) \qquad -\boldsymbol{v}_e \left(\sum_{j \in E_o^B} \lambda_j^* \boldsymbol{x}_j \right) + \boldsymbol{u}_e \left(\sum_{j \in E_o}^{B} \lambda_j^* \boldsymbol{y}_j \right) - u_{0e} = 0.$$

This equation can be transformed, using $\sum_{j \in E_o^B} \lambda_j^* = 1$, into:

$$(38) \qquad \sum_{j \in E_o^B} \lambda_j^* (-\boldsymbol{v}_e \boldsymbol{x}_j + \boldsymbol{u}_e \boldsymbol{y}_j - u_{0e}) = 0.$$

From (35), (38) and $\lambda_j^* > 0$ $(j \in E_o^B)$, we have the equation:

$$(39) \qquad -\boldsymbol{v}_e \boldsymbol{x}_j + \boldsymbol{u}_e \boldsymbol{y}_j - u_{0e} = 0. \quad (\forall j \in E_o^B)$$

Thus, $(\boldsymbol{v}_e, \boldsymbol{u}_e, u_{0e})$ is a coefficient of a supporting hyperplane at $(\boldsymbol{x}_j, \boldsymbol{y}_j)$ for every $j \in E_o^B$. Let $t_j = 1/\boldsymbol{v}_e\boldsymbol{x}_j$ (> 0), then $(t_j\boldsymbol{v}_e, t_j\boldsymbol{u}_e, t_ju_{0e})$ is an optimal solution for $(DBCC_j)$ for $j \in E_o^B$.

Suppose that E_o^B contains an IRS DMU α and a DRS DMU β. Then, u_{0e} must be negative, since DMU α shows IRS. At the same time, u_{0e} must be positive, since DMU β has DRS. This leads to a contradiction. □

Corollary 2. *Let a reference set to a BCC-inefficient DMU* $(\boldsymbol{x}_o, \boldsymbol{y}_o)$ *be* E_o^B. *Then,* E_o^B *consists of one of the following combinations of BCC-efficient DMUs.*

- (i) *All DMUs have IRS.*
- (ii) *Mixture of DMUs with IRS and CRS.*
- (iii) *All DMUs have CRS.*
- (iv) *Mixture of DMUs with CRS and DRS.*
- (v) *All DMUs show DRS.*

Theorem 4 (Characterization of Return-to-Scale). *Let the BCC-projected activity of a BCC-inefficient DMU* $(\boldsymbol{x}_o, \boldsymbol{y}_o)$ *be* $(\boldsymbol{x}_e^B, \boldsymbol{y}_e^B)$ *and the reference set to* $(\boldsymbol{x}_o, \boldsymbol{y}_o)$ *be* E_o^B. *Then,* $(\boldsymbol{x}_e^B, \boldsymbol{y}_e^B)$ *belongs to*

1. *IRS, if* E_o^B *consists of DMUs in categories* (i) *or* (ii) *of Corollary* 2,
2. *CRS, if* E_o^B *consists of DMUs in category* (iii), *and*
3. *DRS, if* E_o^B *consists of DMUs in categories* (iv) *or* (v).

Proof. In the case of (i) or (ii), E_o^B contains at least one DMU with IRS and any supporting hyperplane at $(\boldsymbol{x}_e^B, \boldsymbol{y}_e^B)$ is also a supporting hyperplane at the IRS DMU, as shown in the proof of Theorem 3. Thus, the upper bound of u_{0e} must be negative. By the same reasoning, in the case of (iv) or (v), the projected activity is DRS. In the case of (iii), every DMU j $(j \in E_o^B)$ is CCR-efficient by Corollary 1. Since $(\boldsymbol{x}_e^B, \boldsymbol{y}_e^B)$ is a convex combination of CCR-efficient DMUs, it is CCR-efficient, too. Thus, it shows CRS by Corollary 1. □

5. A Numerical Example

Table 1 exhibits the data of 14 general hospitals, each with 2 inputs and 2 outputs. Our method is as follows. First, we solve the BCC model; the results are shown in the columns 'BCC (RTS)' and 'Reference Set'. In the 'BCC (RTS)' column, the returns-to-scale of BCC-efficient DMUs ($\theta_B^* = 1$) are evaluated by BT (1992) and denoted by (I), (C) and (D), which mean IRS, CRS and DRS, respectively. The returns-to-scale characteristics of BCC-inefficient DMUs are decided by those in the reference set, using Theorem 4. For example, H4 has the reference set consisting of H1, which has IRS. Hence, the BCC-projected activity of H4, as exhibited below the H4 data, shows IRS. The reference set of H7 consists of H1(I), H3(C) and H8(C) and we can conclude that the BCC-projected H7 has IRS. Since H13 has H10(C), H12(D) and H14(D) as reference, the BCC-projected H13 has DRS.

6. Applications to Other Models

The characterization described in Section 4 can be applied to other models that have the same production possibility set P_B with the BCC model as follows.

6.1 The Output-Oriented Case

The output-oriented BCC model can be dealt with similarly. This model is described as:

$$\begin{aligned} (BCCO_o) \quad & \max \quad \tau_B \\ \text{subject to} \quad & X\boldsymbol{\lambda} + \boldsymbol{s}_x = \boldsymbol{x}_0 \end{aligned}$$

Table 1: Data of General Hospital and Returns-to-Scale

	Data BCC-projected Data				Returns-to-scale		
DMU	Input		Output		BCC	Reference	RTS
Hospital	Doctor	Nurse	Outpatient	Inpatient	(RTS)	Set	
H1	3008	20980	97775	101225	1(I)		
H2	3985	25643	135871	130580	1(C)		
H3	4324	26978	133655	168473	1(C)		
H4	3534	25361	46243	100407	0.851	H1	I
	3008	20980	97775	101225			
H5	8836	40796	176661	215616	0.845	H2, H3, H10	C
	5763	34487	176661	215616			
H6	5376	37562	182576	217615	1(C)		
H7	4982	33088	98880	167278	0.862	H1, H3, H8	I
	4295	28522	131312	167278			
H8	4775	39122	136701	193393	1(C)		
H9	8046	42958	225138	256575	0.996	H2, H10	C
	7342	42771	225138	264516			
H10	8554	48955	257370	312877	1(C)		
H11	6147	45514	165274	227099	0.919	H6, H12	D
	5649	39165	184529	227099			
H12	8366	55140	203989	321623	1(D)		
H13	13479	68037	174270	341743	0.794	H10, H12, H14	D
	10704	54029	265518	341743			
H14	21808	78302	322990	487539	1(D)		

(40)
$$\begin{aligned} \tau_B \boldsymbol{y}_o - Y\boldsymbol{\lambda} + \boldsymbol{s}_y &= \boldsymbol{0} \\ \boldsymbol{e\lambda} &= 1 \\ \boldsymbol{\lambda} \geq \boldsymbol{0}, \quad \boldsymbol{s}_x &\geq \boldsymbol{0}, \quad \boldsymbol{s}_y \geq \boldsymbol{0}. \end{aligned}$$

Let an optimal solution for $(BCCO_o)$ be $(\tau_B^*, \boldsymbol{\lambda}^*, \boldsymbol{s}_x^*, \boldsymbol{s}_y^*)$. A DMU $(\boldsymbol{x}_o, \boldsymbol{y}_o)$ is defined to be *output-oriented BCC-efficient* (BCCO-efficient) if and only if it holds that $\tau_B^* = 1, \boldsymbol{s}_x^* = \boldsymbol{0}$ and $\boldsymbol{s}_y^* = \boldsymbol{0}$, and hence if and only if it is BCC-efficient.

The returns-to-scale characteristics of a BCCO-efficient DMU are the same as those in the BCC case. For a BCCO-inefficient DMU, the BCCO-projected activity $(\boldsymbol{x}_e, \boldsymbol{y}_e)$ is defined by

$$\begin{aligned} \boldsymbol{x}_e &= \boldsymbol{x}_o - \boldsymbol{s}_x^* \\ \boldsymbol{y}_e &= \tau_B^* \boldsymbol{y}_o + \boldsymbol{s}_y^*. \end{aligned}$$

The returns-to-scale of such activity are generally different from those of the BCC-projected case. However, we can decide the characteristics in a similar way as in the input-oriented BCC case, using the reference set.

Table 2 shows an example of the output-oriented BCC model that deals with the same data as in Table 1. Inefficient DMUs have different projected values from those in Table 1. Also, the reference set is not always the same as that in Table 1. For example, H7 has the reference set composed of H3(C), H6(C) and H10(C), and hence by Theorem 4 its projected

Table 2: Returns-to-Scale by the Output-oriented BCC Model

	Data BCCO-projected Data				Returns-to-scale		
DMU	Input		Output		τ_B^*	Reference	RTS
Hospital	Doctor	Nurse	Outpatient	Inpatient	(RTS)	Set	
H1	3008	20980	97775	101225	1(I)		
H2	3985	25643	135871	130580	1(C)		
H3	4324	26978	133655	168473	1(C)		
H4	3534	25361	46243	100407	1.280	H1	I
	3534	25361	110297	128472			
H5	8836	40796	176661	215616	1.198	H2, H3, H10	C
	6981	40796	211726	258413			
H6	5376	37562	182576	217615	1(C)		
H7	4982	33088	98880	167278	1.183	H3, H6, H10	C
	4982	33088	162445	167892			
H8	4775	39122	136701	193393	1(C)		
H9	8046	42958	225138	256575	1.004	H2, H10	C
	7379	42958	226114	265981			
H10	8554	48955	257370	312877	1(C)		
H11	6147	45514	165274	227099	1.076	H6, H12	D
	6147	42095	188098	244434			
H12	8366	55140	203989	321623	1(D)		
H13	13479	68037	174270	341743	1.126	H12, H14	D
	13479	63950	249254	384733			
H14	21808	78302	322990	487539	1(D)		

activity has *constant* returns-to-scale characteristics, in contrast to the input-oriented case where H7 has H1(I), H3(C) and H10(C) as its reference set, and shows *increasing* returns-to-scale.

6.2 The Additive Model

The efficiency of DMU_o by the additive model is identified by solving the following linear program:

$$
\begin{aligned}
(ADD_o) \quad \max \quad & \boldsymbol{e}\boldsymbol{s}_x + \boldsymbol{e}\boldsymbol{s}_y \\
\text{subject to} \quad & X\boldsymbol{\lambda} + \boldsymbol{s}_x = \boldsymbol{x}_o \\
& Y\boldsymbol{\lambda} - \boldsymbol{s}_y = \boldsymbol{y}_o \\
& \boldsymbol{e}\boldsymbol{\lambda} = 1 \\
& \boldsymbol{\lambda} \geq \boldsymbol{0}, \quad \boldsymbol{s}_x \geq \boldsymbol{0}, \quad \boldsymbol{s}_y \geq \boldsymbol{0}.
\end{aligned}
$$

Let an optimal solution of (ADD_o) be $(\boldsymbol{\lambda}^*, \boldsymbol{s}_x^*, \boldsymbol{s}_y^*)$. Then, DMU_o is on the efficient frontier of P_B, if and only if it holds that $\boldsymbol{s}_x^* = \boldsymbol{0}$ and $\boldsymbol{s}_y^* = \boldsymbol{0}$. The returns-to-scale characteristics of the efficient DMUs are the same as those in the BCC case. For an ADD-inefficient case, *i.e.* $\boldsymbol{s}_x \neq \boldsymbol{0}$ or $\boldsymbol{s}_y \neq \boldsymbol{0}$, the projected activity $(\boldsymbol{x}_e, \boldsymbol{y}_e)$ is defined by $\boldsymbol{x}_e = \boldsymbol{x}_o - \boldsymbol{s}_x^*$ and $\boldsymbol{y}_e = \boldsymbol{y}_o + \boldsymbol{s}_y^*$. We can decide the returns-to-scale characteristics of the projected activity, using those of the reference set to DMU_o.

7. Comparisons with Other Methods

We will briefly survey two representative methods related to returns-to-scale and compare them with our method.

7.1 Färe, Grosskopf and Lovell (1985)

FGL(1985) suggest the following two-step method to estimate returns-to-scale. In step 1, the BCC and the CCR models are solved to determine the optimal objective values θ_B^* and θ_C^* and define the scale efficiency θ_S^* as $\theta_S^* = \theta_C^*/\theta_B^*$. A value $\theta_S^* = 1$ indicates that the DMU has CRS and a value $\theta_S^* < 1$ indicates IRS or DRS. In step 2, when $\theta_S^* < 1$, the following linear program is solved to determine whether the scale inefficiency is associated with IRS or DRS.

$$
\begin{aligned}
(LP_E) \quad \theta_E^* &= \min \theta_E \\
\text{subject to} \quad \theta_E \boldsymbol{x}_o - X\boldsymbol{\lambda} - \boldsymbol{s}_x &= \boldsymbol{0} \\
Y\boldsymbol{\lambda} - \boldsymbol{s}_y &= \boldsymbol{y}_o \\
\boldsymbol{e\lambda} &\leq 1 \\
\boldsymbol{\lambda} \geq \boldsymbol{0}, \quad \boldsymbol{s}_x &\geq \boldsymbol{0} \quad \boldsymbol{s}_y \geq \boldsymbol{0}.
\end{aligned}
$$

FGL (1985) state that if $\theta_E^* < 1$ and $\theta_E^* = \theta_B^*$, then the DMU has IRS and if $\theta_E^* < 1$ and $\theta_E^* < \theta_B^*$, then the DMU shows DRS.

This method requires 3 LP solutions (BCC, CCR and the above (LP_E)) for IRS and DRS DMUs, and is said to be a three-pass method. We can simplify this method by applying Theorem 4 described in this paper, since solving (LP_E) is necessary only for DMUs with $\theta_S^* < 1$ and $\theta_B^* = 1$.

7.2 Banker, Chang and Cooper (1995)

BCC (1995) solve the CCR model in step 1, and if $\boldsymbol{e\lambda}^* = 1$, then the DMU has CRS, and if $\boldsymbol{e}\lambda^* < 1$, they proceed to step 2. In step 2, the following linear program is solved to determine whether the DMU is associated with CRS or IRS.

$$
\begin{aligned}
(LPz) \quad z^* &= \max \boldsymbol{e\lambda} \\
\text{subject to} \quad X\boldsymbol{\lambda} + \boldsymbol{s}_x &= \theta_C^* \boldsymbol{x}_o \\
Y\boldsymbol{\lambda} - \boldsymbol{s}_y &= \boldsymbol{y}_o \\
\boldsymbol{e\lambda} &\leq 1 \\
\boldsymbol{\lambda} \geq \boldsymbol{0}, \quad \boldsymbol{s}_x &\geq \boldsymbol{0}, \quad \boldsymbol{s}_y \geq \boldsymbol{0},
\end{aligned}
$$

where θ_C^* is the optimal objective value for the CCR model. If $z^* = 1$, then the DMU has CRS and if $z^* < 1$, it shows IRS. The case $\boldsymbol{e\lambda}^* > 1$ in the CCR model can be handled in an obvious modification of the above linear program (LP_Z).

This method requires 2 LP solutions for DMUs with $\boldsymbol{e\lambda}^* \neq 1$ and is concerned only with the returns-to-scale characteristics and the information obtained from the CCR model. The BCC-projection is not explicitly described in this method.

8. Conclusion

In this paper we presented a simple alternative method for deciding the returns-to-scale characteristics of BCC (BCCO)-projected activities. This method is 'one' pass in the sense that BCC software equipped with a procedure for deciding returns-to-scale of BCC-efficient DMUs (*e.g.* Banker and Thrall (1992)), is sufficient for this purpose. No other software is needed. Since the number of BCC-efficient DMUs is considerably less than that of BCC-inefficient ones, this method will contribute to save computation time. Also, Theorems 3, 4 and Corollary 2 contribute to the development of theory and algorithms in DEA.

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