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THE MINIMUM-WEIGHT IDEAL PROBLEM FOR SIGNED POSETS

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Abstract The concept of signed poset has recently been introduced by V. Reiner as a generalization of that of ordinary poset (partially ordered set). We consider the problem of finding a minimum-weight ideal of a signed poset. We show a representation theorem that there exists a bijection between the set of all the ideals of a signed poset and the set of all the "reduced ideals" (defined here) of the associated ordinary poset, which was earlier proved by S. D. Fischer in his Ph.D. thesis. It follows from this representation theorem that the minimum-weight ideal problem for a signed poset can be reduced to a problem of finding a minimum-weight (reduced) ideal of the associated ordinary poset and hence to a minimum-cut problem. We also consider the case when the weight of an ideal is defined in terms of two weight functions. The problem is also reduced to a minimum-cut problem by the same reduction technique as above. Furthermore, the relationship between the minimum-weight ideal problem and a certain bisubmodular function minimization problem is revealed.

1. Introduction

Let $\mathbf{R}$ be the set of reals and $\mathbf{Z}$ the set of integers.

The concept of signed poset has recently been introduced by V. Reiner [17]. A signed poset is a kind of bidirected graph ([9]). A *bidirected graph* $G = (V, A; \partial)$ is a graph with a vertex set V, an arc set A and a boundary operator $\partial :\ A \to \mathbf{Z}^V$, where for each arc $a \in A$ there exist $v, w \in V$ such that one of the following three holds:

(1) $\partial a = v - w$ (arc a has a tail at v and a head at w),
(2) $\partial a = v + w$ (arc a has two tails, one at v and the other at w),
(3) $\partial a = -v - w$ (arc a has two heads, one at v and the other at w).

Here, each $\partial a \in \mathbf{Z}^V$ is represented by an element of a free module with a base V. If $v = w$ in (1)$\sim$(3), the arc a is called a *selfloop*. We do not allow selfloops of type (1). We say an arc a is *incident to* a vertex v (and a vertex w) if $\partial a = \pm v \pm w$, and a is *positively* (or *negatively*) incident to v if the coefficient of v in ∂a is positive (or negative). Two arcs a_1, a_2 are said to be *oppositely incident to* a vertex v if one of the two is positively incident to v and the other is negatively incident to v.

A *signed poset* $\mathcal{P} = (V, A; \partial)$ is a bidirected graph with a vertex set V, an arc set A and a boundary operator ∂ such that

(i) for any two arcs $a_1, a_2 \in A$ we have $\partial a_1 \neq -\partial a_2$,
(ii) for any $a_1, a_2 \in A$ oppositely incident to a common vertex there exists an arc $a_3 \in A$ satisfying $\partial a_3 = \partial a_1 + \partial a_2$,
(iii) for any two selfloops $a_1, a_2 \in A$ incident to distinct vertices there exists an arc $a_3 \in A$ satisfying $2\partial a_3 = \partial a_1 + \partial a_2$.

Figure 1.1 shows an example of a signed poset on $V = \{1, 2, 3, 4\}$.

We can easily see that the concept of signed poset is a generalization of that of (ordinary)

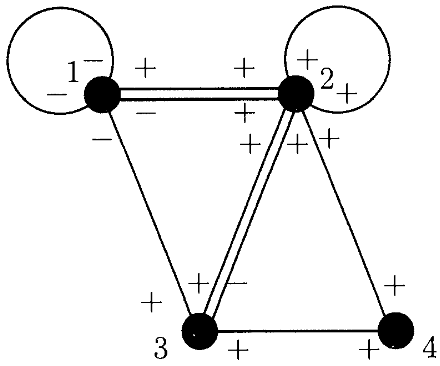

Figure 1.1. An example of a signed poset.

poset. Posets play a fundamental rôle in a lot of practical problems arising in scheduling problems, network optimization etc. Though we have not yet found concrete practical applications of signed posets, we expect that there should be many possible applications of signed posets. For theoretical applications see, e.g., [3] and [12].

Denote by 3^V the set of all the ordered pairs (X, Y) of disjoint subsets X and Y of V. Each element $(X, Y) \in 3^V$ can be made correspond to its characteristic vector $\chi_{(X,Y)} \in \{0, \pm 1\}^V$ defined by $\chi_{(X,Y)}(v) = 1$ if $v \in X$, $= -1$ if $v \in Y$ and $= 0$ otherwise. We call each element of 3^V a *signed subset of* V. An *ideal* of a signed poset $\mathcal{P} = (V, A; \partial)$ is a signed subset $(X, Y) \in 3^V$ such that for any arc $a \in A$

$$\langle \partial a, (X, Y) \rangle \leq 0, \tag{1.1}$$

where $\langle \cdot, \cdot \rangle$ denotes the (canonical) inner product and ∂a and (X, Y) should be regarded as vectors in $\mathbf{R}^V$ under natural correspondences. In Figure 1.1 $(\{1,3\}, \{2,4\})$ is an ideal but $(\{1,3\}, \{4\})$ is not.

Given a signed poset $\mathcal{P} = (V, A; \partial)$ and a weight function $w : V \to \mathbf{R}$, the *minimum-weight ideal problem* is defined as follows:

$$(P) \qquad \text{Minimize} \quad w(X, Y) = \sum_{v \in X} w(v) - \sum_{v \in Y} w(v)$$
$$\text{subject to} \quad (X, Y) \in \mathcal{I}(\mathcal{P}), \tag{1.2}$$

where $\mathcal{I}(\mathcal{P})$ is the set of all the ideals of the signed poset $\mathcal{P}$.

In Section 2 we show a representation theorem of the set of ideals of a signed poset, which was also shown by S. D. Fischer [10]. In Section 3 we show that the minimum-weight ideal problem for signed posets can be reduced to the minimum-weight ideal problem for ordinary posets, so that it can be solved by any minimum-cut algorithm for two-terminal networks (see [15], [16] and [1]). In Section 4 we also consider the relationship between the minimum-weight ideal problem and the problem of minimizing bisubmodular functions and show that the problem of minimizing so-called box-bisubmodular functions can also be reduced to a minimum-cut problem by the same reduction technique.

2. A Representation of a Signed Poset

Given a signed poset $\mathcal{P} = (V, A; \partial)$, construct a directed graph $\hat{G}(\mathcal{P}) = (\hat{V}, \hat{A}; \hat{\partial})$ as follows.

$$\hat{V} = \{(v, +) \mid v \in V, \forall a \in A : \partial a \neq 2v\} \cup \{(v, -) \mid v \in V, \forall a \in A : \partial a \neq -2v\}. \tag{2.1}$$

The arc set $\hat{A}$ consists of the following arcs $\hat{a}$.
(i) $\hat{a} \in \hat{A}$ with $\hat{\partial}\hat{a} = (v,+)-(w,+)$ if and only if $\exists a \in A : \partial a = v-w$ and $(v,+),(w,+) \in \hat{V}$.
(ii) $\hat{a} \in \hat{A}$ with $\hat{\partial}\hat{a} = (v,+)-(w,-)$ if and only if $\exists a \in A : \partial a = v+w, v \neq w$ and $(v,+),(w,-) \in \hat{V}$.
(iii) $\hat{a} \in \hat{A}$ with $\hat{\partial}\hat{a} = (v,-)-(w,+)$ if and only if $\exists a \in A : \partial a = -v-w, v \neq w$ and $(v,-),(w,+) \in \hat{V}$.
(iv) $\hat{a} \in \hat{A}$ with $\hat{\partial}\hat{a} = (v,-)-(w,-)$ if and only if $\exists a \in A : \partial a = -v+w$ and $(v,-),(w,-) \in \hat{V}$.
This construction is almost the same as in [5] and [10]. Here, note that we have no arc $\hat{a} \in \hat{A}$ such that $\hat{\partial}\hat{a} = (v,\pm)-(w,\pm)$ with $v = w$. Also, it should be noted that each arc of $\mathcal{P}$ corresponds to at most two arcs of $\hat{G}(\mathcal{P})$. For example, (ii) means that for an arc $a \in A$ with $\partial a = v+w$ and $v \neq w$, if $(v,+),(w,-) \in \hat{V}$, then there exists an arc $\hat{a} \in \hat{A}$ with $\hat{\partial}\hat{a} = (v,+)-(w,-)$, and if $(w,+),(v,-) \in \hat{V}$, then there exists an arc $\hat{a}' \in \hat{A}$ with $\hat{\partial}\hat{a}' = (w,+)-(v,-)$. See Figure 2.1 for the directed graph $\hat{G}(\mathcal{P})$ corresponding to the signed poset shown in Figure 1.1. Here, the directed graph $\hat{G}(\mathcal{P})$ constructed as above is in general an ordinary poset, which will be shown below as Lemma 2. 1 .

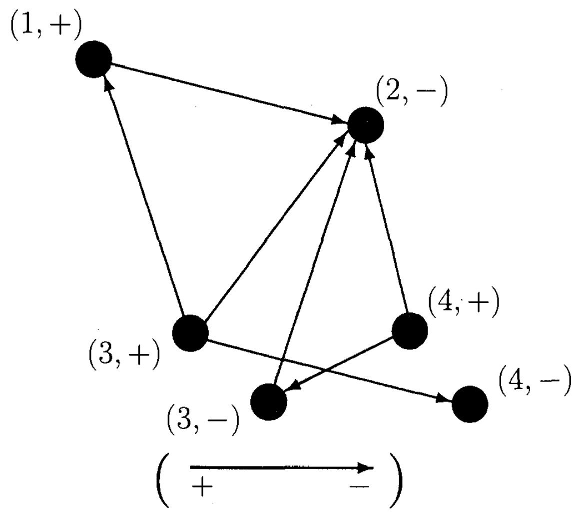

Figure 2.1. The directed graph representing the poset $\hat{G}(\mathcal{P})$.

Lemma 2. 1 : *The (ordinary) directed graph $\hat{G}(\mathcal{P})$ defined above is transitive and acyclic (i.e., $\hat{G}(\mathcal{P})$ represents an ordinary poset).*
(Proof) Suppose that there exist two arcs $\hat{a}_1, \hat{a}_2 \in \hat{A}$ such that

$$\hat{\partial}\hat{a}_1 = (v_0,\sigma_0)-(v_1,\sigma_1), \qquad \hat{\partial}\hat{a}_2 = (v_1,\sigma_1)-(v_2,\sigma_2), \tag{2.2}$$

where $(v_i,\sigma_i) \in \hat{V}$ $(i = 0,1,2)$. Then, by the definition of $\hat{G}(\mathcal{P})$ there exist arcs $a_1, a_2 \in A$ such that

$$\partial a_1 = \sigma_0 v_0 - \sigma_1 v_1, \qquad \partial a_2 = \sigma_1 v_1 - \sigma_2 v_2. \tag{2.3}$$

Since $\mathcal{P}$ is a signed poset, we have $(v_0,\sigma_0) \neq (v_2,\sigma_2)$ and there exists an arc $a_3 \in A$ with $\partial a_3 = \sigma_0 v_0 - \sigma_2 v_2$. If $v_0 = v_2$, then $\sigma_0 = -\sigma_2$ and hence there is a selfloop at v_0 with sign σ_0, which contradicts that $(v_0,\sigma_0) \in \hat{V}$. Hence, $v_0 \neq v_2$ and by the definition of $\hat{G}(\mathcal{P})$ there exists an arc $\hat{a}_3$ such that $\hat{\partial}\hat{a}_3 = (v_0,\sigma_0)-(v_2,\sigma_2)$. Therefore, $\hat{G}(\mathcal{P})$ is transitive.

In the above proof of transitivity we have already shown that $\hat{G}(\mathcal{P})$ does not contain any two arcs $\hat{a}_1$ and $\hat{a}_2$ such that $\hat{\partial}\hat{a}_1 = -\hat{\partial}\hat{a}_2$. From this fact and the transitivity of $\hat{G}(\mathcal{P})$ we immediately see that $\hat{G}(\mathcal{P})$ is acyclic. □

From Lemma 2. 1 we can consider an (ordinary order) ideal $\hat{J}$ of the poset $\hat{G}(\mathcal{P})$. (A subset $\hat{J}$ of $\hat{V}$ is called an ideal of $\hat{G}(\mathcal{P})$ if for each arc $\hat{a} \in \hat{A}$ with $\hat{\partial}\hat{a} = (w,\tau) - (v,\sigma)$ and $(w,\tau) \in \hat{J}$ we have $(v,\sigma) \in \hat{J}$.) It should be noted that, unfortunately, for an ideal $\hat{J}$ of $\hat{G}(\mathcal{P})$ the ordered pair (X,Y), defined by

$$X = \{v \mid (v,+) \in \hat{J}\}, \qquad Y = \{v \mid (v,-) \in \hat{J}\}, \tag{2.4}$$

is not necessarily an ideal of $\mathcal{P}$ since it may not even be a signed subset. However, for an ideal $\hat{J}$ such that (X,Y) defined by (2.4) satisfies $X \cap Y = \emptyset$, (X,Y) is an ideal of $\mathcal{P}$ as the following lemma shows. We call such an ideal $\hat{J}$ a *reduced ideal* of $\hat{G}(\mathcal{P})$.

Lemma 2. 2 : *For any reduced ideal $\hat{J}$ of $\hat{G}(\mathcal{P})$ the signed subset (X,Y) defined by* (2.4) *is an ideal of $\mathcal{P}$.*

(Proof) Suppose that $a \in A$ is an arc such that $\partial a = v + \sigma w$ and $v \in X$ for some $\sigma \in \{+,-\}$. The arc a cannot be a selfloop since $(v,+) \in \hat{V}$. We have $(w,-\sigma) \in \hat{V}$ since otherwise there would exist a positive selfloop at v and this would contradict $(v,+) \in \hat{V}$. Hence, there is an arc $\hat{a} \in \hat{A}$ such that $\hat{\partial}\hat{a} = (v,+) - (w,-\sigma)$. Since $\hat{J}$ is an ideal of $\hat{G}(\mathcal{P})$, we have $(w,-\sigma) \in \hat{J}$ and hence $\langle \partial a, (X,Y) \rangle = 0$. The case when $\partial a = -v + \sigma w$ and $v \in Y$ for some $\sigma \in \{+,-\}$ can be treated similarly. It follows that (X,Y) is an ideal of $\mathcal{P}$. □

We now have the following representation theorem for ideals of a signed poset by means of an ordinary poset.[1]

Theorem 2. 3 : *The correspondence* (2.4) *gives a one-to-one and onto mapping from the set of all the reduced ideals of the poset $\hat{G}(\mathcal{P})$ to the set of all the ideals of the signed poset $\mathcal{P}$.*

(Proof) First, we show that the correspondence is onto. Suppose that $(X,Y) \in 3^V$ is an ideal of $\mathcal{P}$. Then, we must have

$$(v,+) \in \hat{V} \quad (v \in X), \qquad (v,-) \in \hat{V} \quad (v \in Y). \tag{2.5}$$

We will show that $\hat{J} \subseteq \hat{V}$ defined by

$$\hat{J} = \{(v,+) \mid v \in X\} \cup \{(v,-) \mid v \in Y\} \tag{2.6}$$

is an ideal of the poset $\hat{G}(\mathcal{P})$, from which follows the fact that the correspondence is onto. Here, note that an ideal $\hat{J}$ of the form (2.6) is a reduced ideal and that $\hat{J}$ is clearly made correspond to (X,Y).

Now, let $\hat{a}$ be an arc in $\hat{A}$ such that $\hat{\partial}\hat{a} = (v,\sigma) - (w,\tau)$ and $(v,\sigma) \in \hat{J}$. Then there is an arc $a \in A$ such that $\partial a = \sigma v - \tau w$. Suppose that $\sigma = +$. Then, by the definition of $\hat{J}$ we have $v \in X$. Since (X,Y) is an ideal of $\mathcal{P}$, we have

$$w \in X, \qquad \tau = +, \tag{2.7}$$

or

$$w \in Y, \qquad \tau = -. \tag{2.8}$$

[1]Essentially the same theorem was found by S. D. Fischer (see Theorem 1.2 in his Ph.D. thesis [10]). We learned this fact after writing the original version [6] of our present paper. Fischer [10] called a reduced ideal defined here an *isotropic ideal.*

Hence, $(w, \tau) \in \hat{J}$. It follows that $\hat{J}$ is a (reduced) ideal of $\hat{G}(\mathcal{P})$. The case when $\sigma = -$ can be treated similarly.

Next, we show that the correspondence is one to one. Let $\hat{J}_1$ and $\hat{J}_2$ be two distinct reduced ideals of the poset $\hat{G}(\mathcal{P})$ and for each $i = 1, 2$ let (X_i, Y_i) be the ideal of the signed poset $\mathcal{P}$ that corresponds to $\hat{J}_i$. Suppose $\hat{J}_1 - \hat{J}_2 \neq \emptyset$, without loss of generality. Let $(v, \sigma) \in \hat{J}_1 - \hat{J}_2$. If $\sigma = +$, then $v \in X_1$ but $v \notin X_2$, otherwise $v \in Y_1$ but $v \notin Y_2$. Hence, we have $(X_1, Y_1) \neq (X_2, Y_2)$. □

Remark: If we construct an auxiliary graph $\tilde{\mathcal{P}}$ defined in [5, Section 5] for a signed poset $\mathcal{P}$, then the set of all the reduced ideals of $\tilde{\mathcal{P}}$ coincides with that of $\hat{G}(\mathcal{P})$ defined in this section (also see [10]). However, $\hat{G}(\mathcal{P})$ is a more concise representation than $\tilde{\mathcal{P}}$. The auxiliary graph $\tilde{\mathcal{P}}$ is also considered by T. Zaslavsky [18], [19].

3. The Minimum-Weight Ideal Problem

To solve the minimum-weight ideal problem (P) given by (1.2), construct the directed graph $\hat{G}(\mathcal{P}) = (\hat{V}, \hat{A}; \hat{\partial})$ defined in the previous section. First, we can show the following.

Theorem 3. 1 : *Let $\hat{J}$ be an (ordinary order) ideal of the poset $\hat{G}(\mathcal{P})$ and define*

$$\hat{J}_0 = \{(v,+),(v,-) \mid v \in V, \{(v,+),(v,-)\} \subseteq \hat{J}\}. \tag{3.1}$$

Then, $\hat{J} - \hat{J}_0$ is also an ideal of $\hat{G}(\mathcal{P})$.

(Proof) Let $\hat{J}$ be an ideal of the poset $\hat{G}(\mathcal{P})$ and $\hat{J}_0$ be the set defined by (3.1). Suppose, on the contrary, that $\hat{J} - \hat{J}_0$ is not an ideal of the poset $\hat{G}(\mathcal{P})$. Then, there exist $(v, \sigma) \in \hat{J}_0$ and $(w, \tau) \in \hat{J} - \hat{J}_0$ such that

$$\hat{\partial}\hat{a} = (w, \tau) - (v, \sigma) \tag{3.2}$$

for some $\hat{a} \in \hat{A}$. By the definition of $\hat{G}(\mathcal{P})$ there exists an arc $a \in A$ such that $\partial a = \tau w - \sigma v$. If there exists an arc $a' \in A$ with $\partial a' = -2\tau w$, then there exists a selfloop $a'' \in A$ with $\partial a'' = -2\sigma v$, which contradicts $(v, -\sigma) \in \hat{J}_0 \subseteq \hat{V}$. Therefore, there is no selfloop $a' \in A$ with $\partial a' = -2\tau w$, so that we have $(w, -\tau) \in \hat{V}$. It follows from the definition of $\hat{G}(\mathcal{P})$ that there is an arc $\hat{a}' \in \hat{A}$ such that

$$\hat{\partial}\hat{a}' = (v, -\sigma) - (w, -\tau). \tag{3.3}$$

Since $\hat{J}$ is an ideal of $\hat{G}(\mathcal{P})$ and $(v, -\sigma) \in \hat{J}$, we have $(w, -\tau) \in \hat{J}$ and hence, $(w, \pm) \in \hat{J}_0$, a contradiction. □

Now, define a weight function $\hat{w} : \hat{V} \to \mathbf{R}$ by

$$\hat{w}((v,+)) = w(v) \qquad ((v,+) \in \hat{V}), \tag{3.4}$$

$$\hat{w}((v,-)) = -w(v) \qquad ((v,-) \in \hat{V}). \tag{3.5}$$

It follows from Theorems 3. 1 and 2. 3 that Problem (P) is reduced to the problem of finding a minimum-weight (ordinary) ideal of the poset $\hat{G}(\mathcal{P})$ with respect to the weight $\hat{w}$. That is, to find a minimum-weight ideal of the signed poset $\mathcal{P}$ we first compute a minimum-weight (ordinary order) ideal $\hat{J}$ of the poset $\hat{G}(\mathcal{P})$ by any minimum-cut algorithm (see [15], [16] and [1]), then obtain the reduced ideal $\tilde{J} \leftarrow \hat{J} - \hat{J}_0$ by (3.1), and find the ideal (X, Y) of $\mathcal{P}$ corresponding to $\tilde{J}$ by (2.4). Note that the weight of an ideal $\hat{J}$ of $\hat{G}(\mathcal{P})$ is equal to that of its corresponding reduced ideal $\tilde{J} = \hat{J} - \hat{J}_0$. This fact and the representation theorem, Theorem 2. 3 , are essential in the above argument for the problem reduction.

Remark: Suppose that we are given a bidirected subgraph $G' = (V, A'; \partial')$ of a signed poset $\mathcal{P} = (V, A; \partial)$ that misses some redundant arcs of $\mathcal{P}$. (Here, we say an arc $a \in A$ is *redundant* if ∂a can be expressed as a nonnegative linear combination of the other $\partial a'$ $(a' \in A - \{a\})$. If we remove all the redundant arcs from $\mathcal{P}$, we have the Hasse diagram of $\mathcal{P}$.) For such $G' = (V, A'; \partial')$, if we define the auxiliary directed graph $\tilde{G}' = (\tilde{V}, \tilde{A}'; \tilde{\partial}')$ used in [5, Section 5] (see also [18], [19]), then the set of all the reduced ideals of $\tilde{G}'$ is the same as that of $\hat{G}(\mathcal{P})$. Therefore, we can obtain a minimum-weight ideal of the signed poset $\mathcal{P}$ from a minimum-weight ideal of $\tilde{G}'$, without constructing $\mathcal{P}$ from G' by taking the transitive closure of G'.

4. The Problem of Minimizing Bisubmodular Functions

We can apply the technique developed in the previous section to the problem of minimizing certain bisubmodular functions (for bisubmodular functions see [8], [14], [7], [4] and [11]).

Define two binary operations, $\sqcup$ (*reduced union*) and $\sqcap$ (*intersection*), on 3^V as follows. For each $(X_i, Y_i) \in 3^V$ $(i = 1, 2)$

$$(X_1, Y_1) \sqcup (X_2, Y_2) = ((X_1 \cup X_2) - (Y_1 \cup Y_2), (Y_1 \cup Y_2) - (X_1 \cup X_2)), \tag{4.1}$$

$$(X_1, Y_1) \sqcap (X_2, Y_2) = (X_1 \cap X_2, Y_1 \cap Y_2). \tag{4.2}$$

Note that 3^V is closed with respect to $\sqcup$ and $\sqcap$. Also, it is known (see [2]) that the set $\mathcal{I}(\mathcal{P})$ of all the ideals of a signed poset is closed with respect to $\sqcup$ and $\sqcap$. For a $\{\sqcup, \sqcap\}$-closed family $\mathcal{F} \subseteq 3^V$ a function $f : \mathcal{F} \to \mathbf{R}$ is called a *bisubmodular function* if for each $(X_i, Y_i) \in \mathcal{F}$ $(i = 1, 2)$

$$f(X_1, Y_1) + f(X_2, Y_2) \geq f((X_1, Y_1) \sqcup (X_2, Y_2)) + f((X_1, Y_1) \sqcap (X_2, Y_2)). \tag{4.3}$$

For a weight function $w : V \to \mathbf{R}$ define for each ideal (X, Y) of a signed poset $\mathcal{P} = (V, A; \partial)$

$$f(X, Y) = w(X) - w(Y), \tag{4.4}$$

where $w(Z) = \sum_{v \in Z} w(v)$ for any $Z \subseteq V$. Then $f : \mathcal{I}(\mathcal{P}) \to \mathbf{R}$ satisfies (4.3) with equality for each $(X_i, Y_i) \in \mathcal{I}(\mathcal{P})$ $(i = 1, 2)$. Conversely, any bisubmodular function f such that (4.3) holds with equality for each $(X_i, Y_i) \in \mathcal{I}(\mathcal{P})$ $(i = 1, 2)$ can be expressed as (4.4) for some weight function w (see [3]). Such an f is called a *bimodular function*. Hence, the minimum-weight ideal problem for signed posets is equivalent to the problem of minimizing bimodular functions.

Now, consider two weight functions $w^+, w^- : V \to \mathbf{R}$ satisfying $w^+ \geq w^-$ and define

$$f(X, Y) = w^+(X) - w^-(Y) \tag{4.5}$$

for each $(X, Y) \in \mathcal{I}(\mathcal{P})$. We can easily show that $f : \mathcal{I}(\mathcal{P}) \to \mathbf{R}$ is a bisubmodular function but not necessarily a bimodular function. It is called a *box-bisubmodular function* in [13]. The problem of minimizing the box-bisubmodular function, or the problem of finding a minimum-weight ideal of the signed poset with the weight defined by (4.5), is reduced to a minimum-cut problem as follows. Consider the directed graph (poset) $\hat{G}(\mathcal{P}) = (\hat{V}, \hat{A}; \hat{\partial})$ defined in Section 2. Define the weight function $\hat{w} : \hat{V} \to \mathbf{R}$ by

$$\hat{w}((v, +)) = w^+(v) \qquad ((v, +) \in \hat{V}), \tag{4.6}$$

$$\hat{w}((v, -)) = -w^-(v) \qquad ((v, -) \in \hat{V}). \tag{4.7}$$

We can easily see that for an ideal $\hat{J}$ of $\hat{G}(\mathcal{P})$ and its corresponding reduced ideal $\tilde{J}(=\hat{J}-\hat{J}_0)$ we have

$$\hat{w}(\hat{J}) \geq \hat{w}(\tilde{J}). \tag{4.8}$$

Therefore, the problem is reduced to that of finding a minimum-weight (reduced) ideal, which is further reduced to a minimum-cut problem.

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References

[1] R. K. Ahuja, T. L. Magnanti and J. B. Orlin: *Network Flows: Theory, Algorithms, and Applications* (Prentice Hall, New York, 1993).

[2] K. Ando and S. Fujishige: $\sqcup, \sqcap$-closed families and signed posets. Report No. 93813, Forschungsinstitut für Diskrete Mathematik, Universität Bonn (January 1994).

[3] K. Ando and S. Fujishige: On structures of bisubmodular polyhedra. *Mathematical Programming* **74** (1996) 293-318.

[4] K. Ando, S. Fujishige and T. Naitoh: A characterization of bisubmodular functions. *Discrete Mathematics* **148** (1996) 299-303.

[5] K. Ando, S. Fujishige and T. Nemoto: Decomposition of a bidirected graph into strongly connected components and its signed poset structure. *Discrete Applied Mathematics* **68** (1996) 237-248.

[6] K. Ando, S. Fujishige and T. Nemoto: The minimum-weight ideal problem for signed posets. Report No. 94828, Forschungsinstitut für Diskrete Mathematik, Universität Bonn (June 1994).

[7] A. Bouchet and W. H. Cunningham: Delta-matroids, jump systems and bisubmodular polyhedra. *SIAM Journal on Discrete Mathematics* **8** (1995) 17-32.

[8] R. Chandrasekaran and S. N. Kabadi: Pseudomatroids. *Discrete Mathematics* **71** (1988) 205-217.

[9] J. Edmonds and E. L. Johnson: Matching: a well-solved class of linear programs. In: *Combinatorial Structures and Their Applications* (R. Guy, H. Hanani, N. Sauer and J. Schönheim, eds., Gordon and Breach, New York, 1970), pp. 88-92.

[10] S. D. Fischer: *Signed Poset Homology and q-analog Möbius Functions.* Ph.D. Thesis, University of Michigan, 1993.

[11] S. Fujishige: *Submodular Functions and Optimization* (North-Holland, Amsterdam, 1991).

[12] S. Fujishige: A min-max theorem for bisubmodular polyhedra. *SIAM Journal on Discrete Mathematics* (to appear).

[13] S. Fujishige and S. B. Patkar: The box convolution and the Dilworth truncation of bisubmodular functions. Report No. 94823, Forschungsinstitut für Diskrete Mathematik, Universität Bonn (May 1994).

[14] S. N. Kabadi and R. Chandrasekaran: On totally dual integral systems. *Discrete Applied Mathematics* **26** (1990) 87-104.

[15] J. C. Picard: Maximal closure of a graph and applications to combinatorial problems. *Management Science* **22** (1976) 1268-1272.

[16] J. C. Picard and M. Queyranne: Selected applications on minimum cuts in networks. *INFOR* **20** (1982) 394-422.

[17] V. Reiner: Signed posets. *Journal of Combinatorial Theory*, Ser. A **62** (1993) 324-360.

[18] T. Zaslavsky: Signed graphs. *Discrete Applied Mathematics* **4** (1982) 47-74. Erratum, *ibid.*, **5** (1983) 248.

[19] T. Zaslavsky: Orientation of signed graphs. *European Journal of Combinatorics* **12** (1991) 361-375.

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