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LOCAL AND SUPERLINEAR CONVERGENCE OF STRUCTURED QUASI-NEWTON METHODS FOR NONLINEAR OPTIMIZATION

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Abstract This paper is concerned with local and q-superlinear convergence of structured quasi-Newton methods for solving unconstrained and constrained optimization problems. These methods have been developed for solving optimization problems in which the Hessian matrix has a special structure. For example, Dennis, Gay and Welsch (1981) proposed the structured DFP update for nonlinear least squares problems and Tapia (1988) derived the structured BFGS update for equality constrained problems within the framework of the SQP method with the augmented Lagrangian function. Recently, Engels and Martinez (1991) unified these methods and showed local and q-superlinear convergence of the convex class of the structured Broyden family. In this paper, we extend the results of Engels and Martinez to a wider class of the structured Broyden family. We prove local and q-superlinear convergence of the method in a way different from the proof by Engels and Martinez. Our proof for convergence is based on the result by Stachurski (1981). Finally, we apply the convergence results to unconstrained nonlinear least squares problems and equality constrained minimization problems.

1. Introduction

In this paper, we consider numerical methods for solving the minimization problem of a nonlinear function:

$$\min_{x\in R^n} f(x), \quad f: R^n \to R. \tag{1.1}$$

Assume that f is twice continuously differentiable and that there exists a local minimum of the problem, say x_*. The problem of interest in this paper is the case where the Hessian matrix of the function $f(x)$ has special structures given by

$$\nabla^2 f(x) = C(x) + G(x), \tag{1.2}$$

where $C(x)$ is a computed part, while it is expensive to calculate the part $G(x)$.

For the problem (1.1), Newton's method constructs a sequence $\{x_k\}$ such that

$$x_{k+1} = x_k + s_k, \tag{1.3}$$

where s_k satisfies the Newton equation

$$(C(x_k) + G(x_k))s = -\nabla f(x_k), \tag{1.4}$$

where ∇f denotes the gradient vector of f.

Standard quasi-Newton methods approximate the whole Hessian of $f(x)$ and many kinds of updating formulae were proposed, and convergence of these methods were discussed. On the other hand, if the information of the Hessian matrix is partially known, it is desiable to use the information to obtain more efficient methods. Quasi-Newton approximations to only the second part $G(x)$ of the Hessian matrix (1.2) have been developed [5]. These strategies are called structured quasi-Newton methods. In this case, the step s_k can be computed by solving

$$(C(x_k) + A_k)s = -\nabla f(x_k), \tag{1.5}$$

where the matrix A_k is the k-th approximation to the second part $G(x_k)$ of $\nabla^2 f(x_k)$ so that

$$\nabla^2 f(x_k) \approx C(x_k) + A_k.$$

Thus, we have a condition a new matrix A_{k+1} should satisfy as the form

$$(C(x_{k+1}) + A_{k+1})s_k = z_k, \tag{1.6}$$

where

$$s_k = x_{k+1} - x_k \tag{1.7}$$

and z_k is imposed to be a good approximation to $\nabla^2 f(x_*)s_k$ (or $\nabla^2 f(x_{k+1})s_k$). In structured quasi-Newton methods, z_k is set to be as follows;

$$z_k = C(x_{k+1})s_k + y_k^\sharp, \tag{1.8}$$

where $y_k^\sharp$ is a good approximation to $G(x_*)s_k$ (or $G(x_{k+1})s_k$). Thus the matrix A_k is updated such that the new matrix A_{k+1} satisfies the secant condition

$$A_{k+1}s_k = y_k^\sharp. \tag{1.9}$$

Problems whose Hessian matrices have special structures like (1.2) arise in several optimization problems. The first example is an unconstrained nonlinear least squares problem

$$\min_{x\in R^n} f(x) = \frac{1}{2}r(x)^T r(x), \quad r : R^n \to R^l, \quad l \geq n, \tag{1.10}$$

where $r(x) = (r_1(x), ..., r_l(x))^T$. In this case, the Hessian matrix is of the form

$$\nabla^2 f(x) = C(x) + G(x),$$

with

$$C(x) = \nabla r(x)\nabla r(x)^T \quad \text{and} \quad G(x) = \sum_{i=1}^{l} r_i(x)\nabla^2 r_i(x), \tag{1.11}$$

where $\nabla r(x) = (\nabla r_1(x), ..., \nabla r_l(x)) \in R^{n\times l}$.

The second example is the equality constrained minimization problem

$$\min_{x\in R^n} F(x) \text{ subject to } h(x) = 0, \; F : R^n \to R, \; h : R^n \to R^m, \; m \leq n. \tag{1.12}$$

Within the framework of the sequential quadratic programming (SQP) method, Tapia [11] dealt with the augmented Lagrangian function

$$L(x, \lambda; \rho) = F(x) + \lambda^T h(x) + \frac{1}{2}\rho h(x)^T h(x), \tag{1.13}$$

where $\lambda \in R^m$ is a Lagrange multiplier vector associated with the equality constraint and ρ is a positive penalty parameter. The Hessian matrix of this function is formed by

$$\nabla_x^2 L(x, \lambda; \rho) = C(x, \lambda) + G(x, \lambda)$$

with

$$C(x,\lambda) = \rho\nabla h(x)\nabla h(x)^T,$$

$$G(x,\lambda) = \nabla^2 F(x) + \sum_{i=1}^{m}\lambda_i\nabla^2 h_i(x) + \rho\sum_{i=1}^{m} h_i(x)\nabla^2 h_i(x). \tag{1.14}$$

For solving the problem (1.10), Dennis, Gay and Welsch [6] proposed the structured DFP update, and Dennis and Walker [5] showed local and q-superlinear convergence of this method. Dennis, Martinez and Tapia [7] later derived the structure principle, and proved local and q-superlinear convergence of the structured BFGS update, which was proposed by Al-Baali and Fletcher [1]. On the other hand, for solving the problem (1.12), Tapia [11] derived the structured BFGS update and showed local and q-superlinear convergence of the SQP method. Recently, Engels and Martinez [8] unified these methods, and proposed the structured Broyden family and showed local and q-superlinear convergence for the convex class of the structured Broyden family.

In this paper, we will extend the results of Engels and Martinez to a wider class of the structured Broyden family. We prove local and q-superlinear convergence of the method in a way different from the proof by them. Our proof for convergence is based on the result by Stachurski [10]. In Section 2, we briefly review the structured Broyden family that was originally given by Engels and Martinez, but we deal with a wider class of the family. In Section 3, we present some useful lemmas, and in Section 4, we show local and q-superlinear convergence. Finally, in Section 5, we apply our convergence results to the unconstrained nonlinear least squares problem (1.10) and the equality constrained minimization problem (1.12).

Throughout this paper, $\|\ \|$ denotes the l_2 norm for vectors or matrices, and $\|\ \|_F$ and $\|\ \|_{F,M}$ denote the Frobenius norm and the weighted Frobenius norm for some nonsingular matrix M, which are defined by

$$\|Q\|_F = \sqrt{Trace(QQ^T)} \quad \text{and} \quad \|Q\|_{F,M} = \|M^{-1}QM^{-1}\|_F,$$

respectively.

2. Structured Broyden Family

Based on the structure principle given by Dennis et al. [7], Engels and Martinez [8] derived the structured Broyden family:

$$B_{k+1} = B_k^\sharp - \frac{B_k^\sharp s_k s_k^T B_k^\sharp}{s_k^T B_k^\sharp s_k} + \frac{z_k z_k^T}{s_k^T z_k} + \phi_k(s_k^T B_k^\sharp s_k)v_k v_k^T, \tag{2.1}$$

where ϕ_k is a scalar parameter and

$$B_k = C(x_k) + A_k, \quad B_k^\sharp = C(x_{k+1}) + A_k \quad \text{and} \quad v_k = \frac{z_k}{s_k^T z_k} - \frac{B_k^\sharp s_k}{s_k^T B_k^\sharp s_k}. \tag{2.2}$$

From this, an A-update can be obtained as follows:

$$A_{k+1} = A_k - \frac{B_k^\sharp s_k s_k^T B_k^\sharp}{s_k^T B_k^\sharp s_k} + \frac{z_k z_k^T}{s_k^T z_k} + \phi_k(s_k^T B_k^\sharp s_k)v_k v_k^T. \tag{2.3}$$

They dealt with the convex class, i.e. $0 \le \phi_k \le 1$, of the family.

Recent researches on standard quasi-Newton methods pay attention to the Broyden family that allows for negative values of ϕ_k. For example, Zhang and Tewarson [12] studied the preconvex class of the Broyden family and presented encouraging numerical results. Hence, in structured quasi-Newton methods, it is interesting to consider a class of updates that is wider than the convex class of the structured Broyden family and it is desirable to discuss a convergence property of such a wider class of updates. In this paper, we will deal with a wider class of the family than that of Engels and Martinez. Specifically, we only impose a boundedness condition on the parameter ϕ_k.

We should note that the family can be rewritten by

$$B_{k+1} = B_{k+1}^{DFP} + (\phi_k - 1)\Delta B_k, \tag{2.4}$$

where

$$B_{k+1}^{DFP} = B_k^\sharp + \frac{(z_k - B_k^\sharp s_k)z_k^T + z_k(z_k - B_k^\sharp s_k)^T}{s_k^T z_k} - \frac{s_k^T(z_k - B_k^\sharp s_k)}{(s_k^T z_k)^2} z_k z_k^T, \tag{2.5}$$

$$\Delta B_k = (s_k^T B_k^\sharp s_k)\left(\frac{z_k}{s_k^T z_k} - \frac{B_k^\sharp s_k}{s_k^T B_k^\sharp s_k}\right)\left(\frac{z_k}{s_k^T z_k} - \frac{B_k^\sharp s_k}{s_k^T B_k^\sharp s_k}\right)^T. \tag{2.6}$$

This is a useful form to analyse a convergence property in the following sections.

3. Basic Preliminaries

In this section, we give assumptions and useful lemmas to show a local convergence property. Since we use the form (2.4) in order to show local convergence of the structured Broyden family, most of the lemmas in [10] can be applied to our proof. The significant difference between our proof and that in [10] is that we must deal with an intermediate matrix $B_k^\sharp$.

Let D be an open convex subset of R^n, which contains a local minimizer x_*. We assume the following standard conditions.

(A1) There exist positive constants ξ and p such that

$$\|\nabla^2 f(x) - \nabla^2 f(x_*)\| \leq \xi\|x - x_*\|^p, \tag{3.1}$$

$$\|C(x) - C(x')\| \leq \xi\|x - x'\|^p \tag{3.2}$$

for any x and x' in D.

(A2) $\nabla^2 f$ is symmetric positive definite at x_*.

It follows easily from assumption (A1) that, for $x, x' \in D$,

$$\begin{aligned}\|\nabla f(x) - \nabla f(x') - \nabla^2 f(x_*)(x - x')\| \\ \leq \xi(\max(\|x - x_*\|,\ \|x' - x_*\|))^p\|x - x'\|\end{aligned} \tag{3.3}$$

(see Lemma 4.1.15 in [4]).

In what follows, define

$$M = \nabla^2 f(x_*)^{\frac{1}{2}} \tag{3.4}$$

and

$$\sigma_k = \max(\|x_{k+1} - x_*\|\ ,\ \|x_k - x_*\|). \tag{3.5}$$

Set

$$\widehat{B}_k^\sharp = M^{-1}B_k^\sharp M^{-1}, \quad \widehat{z}_k = M^{-1}z_k, \quad \widehat{s}_k = Ms_k \quad \text{and} \quad \widehat{B}_{k+1} = M^{-1}B_{k+1}M^{-1}. \tag{3.6}$$

Note that by the equivalence of norms, for any $n \times n$ matrix Q, there exists a positive constant η such that

$$\frac{1}{\eta}\|Q\|_{F,M} \leq \|Q\| \leq \eta\|Q\|_{F,M}. \tag{3.7}$$

Now we have the following three lemmas. These lemmas will play a fundamental role in the analysis presented in Section 4.

Lemma 1 *Suppose that assumptions* (A1) *and* (A2) *hold. Assume that* $x_k, x_{k+1} \in D$. *If* $y_k^\sharp$ *in (1.8) satisfies*

$$\|y_k^\sharp - G(x_*)s_k\| \leq \zeta^\sharp \sigma_k^p \|s_k\| \tag{3.8}$$

for some positive constant $\zeta^\sharp$, *then there exists a positive constant* ζ *such that*

$$\|\widehat{z}_k - \widehat{s}_k\| \leq \zeta \sigma_k^p \|\widehat{s}_k\|. \tag{3.9}$$

Furthermore, assume that

$$\|x_k - x_*\| \leq \varepsilon \qquad \text{and} \qquad \|x_{k+1} - x_*\| \leq \varepsilon$$

for ε *sufficiently small. Then there exist positive constants* β_1 *and* β_2 *such that*

$$\beta_1\|\widehat{s}_k\|^2 \leq s_k^T z_k \leq \beta_2\|\widehat{s}_k\|^2 \tag{3.10}$$

and

$$\beta_1\|\widehat{s}_k\| \leq \|\widehat{z}_k\| \leq \beta_2\|\widehat{s}_k\|. \tag{3.11}$$

Proof. Assumption (A1) and (1.8) yield

$$\begin{aligned}
\|\widehat{z}_k - \widehat{s}_k\| &\leq \|M^{-1}\|\|z_k - \nabla^2 f(x_*)s_k\| \\
&\leq \|M^{-1}\|(\|C(x_{k+1}) - C(x_*)\|\|s_k\| + \|y_k^\sharp - G(x_*)s_k\|) \\
&\leq \|M^{-1}\|^2(\xi + \zeta^\sharp)\sigma_k^p\|\widehat{s}_k\|.
\end{aligned}$$

Setting $\zeta = \|M^{-1}\|^2(\xi + \zeta^\sharp)$, we obtain the first result (3.9).

Since expression (3.9) gives

$$\begin{aligned}
|s_k^T z_k - \|\widehat{s}_k\|^2| &= |\widehat{s}_k^T(\widehat{z}_k - \widehat{s}_k)| \\
&\leq \|\widehat{s}_k\|\|\widehat{z}_k - \widehat{s}_k\| \\
&\leq \zeta\|\widehat{s}_k\|^2\sigma_k^p \\
&\leq \zeta\|\widehat{s}_k\|^2\varepsilon^p,
\end{aligned}$$

we have

$$(1 - \zeta\varepsilon^p)\|\widehat{s}_k\|^2 \leq s_k^T z_k \leq (1 + \zeta\varepsilon^p)\|\widehat{s}_k\|^2.$$

Since expression (3.9) yields

$$|\|\widehat{z}_k\| - \|\widehat{s}_k\|| \leq \|\widehat{z}_k - \widehat{s}_k\| \leq \zeta\sigma_k^p\|\widehat{s}_k\|,$$

we have

$$(1 - \zeta\varepsilon^p)\|\widehat{s}_k\| \leq \|\widehat{z}_k\| \leq (1 + \zeta\varepsilon^p)\|\widehat{s}_k\|.$$

Thus, for ε sufficiently small, the second results (3.10) and (3.11) are proved. Therefore the proof is complete. ∎

Lemma 2 *Assume that, for some positive constants $\delta^\sharp$ and $\tau^\sharp$,*

$$0 < \|B_k^\sharp - \nabla^2 f(x_*)\|_{F,M} \le \delta^\sharp \qquad \textit{and} \qquad \|\widehat{B}_k^\sharp\| \le \tau^\sharp.$$

Suppose that the assumptions of Lemma 1 hold. Then

$$\|B_{k+1}^{DFP} - \nabla^2 f(x_*)\|_{F,M} \le \|B_k^\sharp - \nabla^2 f(x_*)\|_{F,M} - \frac{\|\widehat{z}_k - \widehat{B}_k^\sharp \widehat{z}_k\|^2}{2\delta^\sharp \|\widehat{z}_k\|^2} + \omega\sigma_k^p,$$

where

$$\omega = \frac{\zeta}{\beta_1}\left(\beta_2 + 2\frac{\tau^\sharp(1+\beta_2)}{\beta_1} + 4\tau^\sharp\right).$$

Proof. We use the same estimate as Lemma 3.5 in [10]. It follows from (2.5) and (3.6) that

$$M^{-1}(B_{k+1}^{DFP} - \nabla^2 f(x_*))M^{-1} = \left(I - \frac{\widehat{z}_k\widehat{z}_k^T}{\|\widehat{z}_k\|^2}\right)(\widehat{B}_k^\sharp - I)\left(I - \frac{\widehat{z}_k\widehat{z}_k^T}{\|\widehat{z}_k\|^2}\right) + T_k,$$

where

$$\begin{aligned}
T_k &= -\frac{\widehat{z}_k\widehat{s}_k^T\widehat{B}_k^\sharp + \widehat{B}_k^\sharp\widehat{s}_k\widehat{z}_k^T}{\widehat{s}_k^T\widehat{z}_k} + \frac{\widehat{z}_k\widehat{z}_k^T\widehat{B}_k^\sharp + \widehat{B}_k^\sharp\widehat{z}_k\widehat{z}_k^T}{\|\widehat{z}_k\|^2} \\
&\quad + \left(1 + \frac{\widehat{s}_k^T\widehat{B}_k^\sharp\widehat{s}_k}{\widehat{s}_k^T\widehat{z}_k}\right)\frac{\widehat{z}_k\widehat{z}_k^T}{\widehat{s}_k^T\widehat{z}_k} - \left(1 + \frac{\widehat{z}_k^T\widehat{B}_k^\sharp\widehat{z}_k}{\|\widehat{z}_k\|^2}\right)\frac{\widehat{z}_k\widehat{z}_k^T}{\|\widehat{z}_k\|^2} \\
&= T_{k1} + T_{k2} + T_{k3} + T_{k4}, \\
T_{k1} &= \frac{\widehat{z}_k\widehat{z}_k^T}{\widehat{s}_k^T\widehat{z}_k} - \frac{\widehat{z}_k\widehat{z}_k^T}{\widehat{z}_k^T\widehat{z}_k}, \\
T_{k2} &= \frac{\widehat{s}_k^T\widehat{B}_k^\sharp\widehat{s}_k}{(\widehat{s}_k^T\widehat{z}_k)^2}\widehat{z}_k\widehat{z}_k^T - \frac{\widehat{z}_k^T\widehat{B}_k^\sharp\widehat{z}_k}{(\widehat{z}_k^T\widehat{z}_k)^2}\widehat{z}_k\widehat{z}_k^T, \\
T_{k3} &= \frac{\widehat{B}_k^\sharp\widehat{z}_k\widehat{z}_k^T}{\widehat{z}_k^T\widehat{z}_k} - \frac{\widehat{B}_k^\sharp\widehat{s}_k\widehat{z}_k^T}{\widehat{s}_k^T\widehat{z}_k}, \\
T_{k4} &= T_{k3}^T.
\end{aligned}$$

Since Lemma 1 yields

$$\begin{aligned}
\|T_{k1}\|_F &\le \frac{\|\widehat{z}_k\|\|\widehat{z}_k - \widehat{s}_k\|}{\widehat{s}_k^T\widehat{z}_k} \\
&\le \frac{\zeta\beta_2}{\beta_1}\sigma_k^p, \\
\|T_{k2}\|_F &= \left|\frac{\widehat{s}_k^T\widehat{B}_k^\sharp\widehat{s}_k}{(\widehat{s}_k^T\widehat{z}_k)^2} - \frac{\widehat{z}_k^T\widehat{B}_k^\sharp\widehat{z}_k}{(\widehat{z}_k^T\widehat{z}_k)^2}\right|\widehat{z}_k^T\widehat{z}_k \\
&= \frac{1}{(\widehat{s}_k^T\widehat{z}_k)^2\|\widehat{z}_k\|^2}|(\widehat{s}_k^T\widehat{B}_k^\sharp\widehat{s}_k)(\widehat{z}_k^T\widehat{z}_k + \widehat{s}_k^T\widehat{z}_k)(\widehat{z}_k^T\widehat{z}_k - \widehat{s}_k^T\widehat{z}_k) \\
&\quad + (\widehat{s}_k^T\widehat{z}_k)^2(\widehat{s}_k^T\widehat{B}_k^\sharp(\widehat{s}_k - \widehat{z}_k) + \widehat{z}_k^T\widehat{B}_k^\sharp(\widehat{s}_k - \widehat{z}_k))| \\
&\le \frac{2}{(\widehat{s}_k^T\widehat{z}_k)^2}\|\widehat{B}_k^\sharp\|\|\widehat{s}_k\|^2\|\widehat{z}_k - \widehat{s}_k\|(\|\widehat{s}_k\| + \|\widehat{z}_k\|)
\end{aligned}$$

$$
\begin{aligned}
&\leq \frac{2}{\beta_1^2}\tau^\sharp\zeta(1+\beta_2)\sigma_k^p, \\
\|T_{k3}\|_F &= \|T_{k4}\|_F \\
&\leq \|\widehat{z}_k\|\|\widehat{B}_k^\sharp\|\left\|\frac{\widehat{z}_k}{\widehat{z}_k^T\widehat{z}_k} - \frac{\widehat{s}_k}{\widehat{s}_k^T\widehat{z}_k}\right\| \\
&\leq \frac{\|\widehat{B}_k^\sharp\|\|\widehat{z}_k\|}{\|\widehat{z}_k\|^2(\widehat{s}_k^T\widehat{z}_k)}((\widehat{s}_k^T\widehat{z}_k)\|\widehat{z}_k-\widehat{s}_k\| + \|\widehat{s}_k\|\|\widehat{z}_k\|\|\widehat{z}_k-\widehat{s}_k\|) \\
&\leq \frac{2}{\widehat{s}_k^T\widehat{z}_k}\|\widehat{z}_k-\widehat{s}_k\|\|\widehat{B}_k^\sharp\|\|\widehat{s}_k\| \\
&\leq \frac{2\zeta\tau^\sharp}{\beta_1}\sigma_k^p,
\end{aligned}
$$

we have

$$
\begin{aligned}
\|T_k\|_F &\leq \|T_{k1}\|_F + \|T_{k2}\|_F + \|T_{k3}\|_F + \|T_{k4}\|_F \\
&\leq \omega\sigma_k^p,
\end{aligned}
$$

where

$$
\omega = \frac{\zeta}{\beta_1}\left(\beta_2 + 2\frac{\tau^\sharp(1+\beta_2)}{\beta_1} + 4\tau^\sharp\right).
$$

Denoting

$$
D_k = \left(I - \frac{\widehat{z}_k\widehat{z}_k^T}{\|\widehat{z}_k\|^2}\right)(\widehat{B}_k^\sharp - I)\left(I - \frac{\widehat{z}_k\widehat{z}_k^T}{\|\widehat{z}_k\|^2}\right),
$$

we have

$$
\begin{aligned}
\|D_k\|_F^2 &= Trace(D_kD_k^T) \\
&= Trace\left[\left((\widehat{B}_k^\sharp - I) - \frac{(\widehat{B}_k^\sharp - I)\widehat{z}_k\widehat{z}_k^T}{\|\widehat{z}_k\|^2}\right)^2\right] \\
&= \|\widehat{B}_k^\sharp - I\|_F^2 - 2\frac{\|(\widehat{B}_k^\sharp - I)\widehat{z}_k\|^2}{\|\widehat{z}_k\|^2} + \frac{(\widehat{z}_k^T(\widehat{B}_k^\sharp - I)\widehat{z}_k)^2}{\|\widehat{z}_k\|^4} \\
&\leq \|\widehat{B}_k^\sharp - I\|_F^2 - \frac{\|(\widehat{B}_k^\sharp - I)\widehat{z}_k\|^2}{\|\widehat{z}_k\|^2} \\
&= \left(\|\widehat{B}_k^\sharp - I\|_F - \frac{1}{2}\frac{\|(\widehat{B}_k^\sharp - I)\widehat{z}_k\|^2}{\|\widehat{B}_k^\sharp - I\|_F\|\widehat{z}_k\|^2}\right)^2 - \left(\frac{1}{2}\frac{\|(\widehat{B}_k^\sharp - I)\widehat{z}_k\|^2}{\|\widehat{B}_k^\sharp - I\|_F\|\widehat{z}_k\|^2}\right)^2 \\
&\leq \left(\|\widehat{B}_k^\sharp - I\|_F - \frac{1}{2}\frac{\|(\widehat{B}_k^\sharp - I)\widehat{z}_k\|^2}{\|\widehat{B}_k^\sharp - I\|_F\|\widehat{z}_k\|^2}\right)^2.
\end{aligned}
$$

Since

$$
\|B_k^\sharp - \nabla^2 f(x_*)\|_{F,M} \leq \delta^\sharp
$$

and

$$
\begin{aligned}
\|\widehat{B}_k^\sharp - I\|_F - \frac{1}{2}\frac{\|(\widehat{B}_k^\sharp - I)\widehat{z}_k\|^2}{\|\widehat{B}_k^\sharp - I\|_F\|\widehat{z}_k\|^2} &\geq \|\widehat{B}_k^\sharp - I\|_F - \frac{1}{2}\frac{\|\widehat{B}_k^\sharp - I\|_F^2\|\widehat{z}_k\|^2}{\|\widehat{B}_k^\sharp - I\|_F\|\widehat{z}_k\|^2} \\
&= \frac{1}{2}\|\widehat{B}_k^\sharp - I\|_F > 0,
\end{aligned}
$$

we have

$$\|D_k\|_F \;\le\; \|B_k^\sharp - \nabla^2 f(x_*)\|_{F,M} - \frac{1}{2\delta^\sharp}\frac{\|(\widehat{B}_k^\sharp - I)\widehat{z}_k\|^2}{\|\widehat{z}_k\|^2}.$$

Then

$$\begin{aligned}\|B_{k+1}^{DFP} - \nabla^2 f(x_*)\|_{F,M} \;&\le\; \|D_k\|_F + \|T_k\|_F \\ &\le\; \|B_k^\sharp - \nabla^2 f(x_*)\|_{F,M} - \frac{\|\widehat{z}_k - \widehat{B}_k^\sharp \widehat{z}_k\|^2}{2\delta^\sharp\|\widehat{z}_k\|^2} + \omega\sigma_k^p.\end{aligned}$$

Therefore we obtain the result. ∎

Lemma 3 *Suppose that the assumptions of Lemma 2 hold and that* $\delta^\sharp < 1$. *Then*

$$\|\Delta B_k\|_{F,M} \le \frac{4\beta_2^4}{\beta_1^2(1-\delta^\sharp)}\left(\frac{\|\widehat{z}_k - \widehat{B}_k^\sharp \widehat{z}_k\|}{\|\widehat{z}_k\|} + \|\widehat{B}_k^\sharp\|\frac{\|\widehat{z}_k - \widehat{s}_k\|}{\|\widehat{z}_k\|}\right)^2.$$

Proof. We use the same estimate as Lemma 3.4 in [10]. Using the inequality

$$\begin{aligned}\widehat{s}_k^T \widehat{B}_k^\sharp \widehat{s}_k \;&=\; \widehat{s}_k^T(\widehat{B}_k^\sharp - I)\widehat{s}_k + \|\widehat{s}_k\|^2 \\ &\ge\; \|\widehat{s}_k\|^2 - |\widehat{s}_k^T(\widehat{B}_k^\sharp - I)\widehat{s}_k| \\ &\ge\; \|\widehat{s}_k\|^2 - \|\widehat{s}_k\|^2\|\widehat{B}_k^\sharp - I\|_F \\ &\ge\; (1-\delta^\sharp)\|\widehat{s}_k\|^2,\end{aligned}$$

we have

$$\frac{1}{\widehat{s}_k^T \widehat{B}_k^\sharp \widehat{s}_k} \le \frac{1}{(1-\delta^\sharp)\|\widehat{s}_k\|^2}.$$

Inequalities (3.10) and (3.11) yield

$$\frac{1}{\widehat{s}_k^T \widehat{z}_k} \le \frac{1}{\beta_1\|\widehat{s}_k\|^2} \le \frac{\beta_2^2}{\beta_1\|\widehat{z}_k\|^2}.$$

Since

$$\begin{aligned}&\|(\widehat{s}_k^T \widehat{B}_k^\sharp \widehat{s}_k)\widehat{z}_k - (\widehat{s}_k^T\widehat{z}_k)\widehat{B}_k^\sharp \widehat{s}_k\| \\ &= \|\widehat{s}_k^T \widehat{B}_k^\sharp(\widehat{s}_k - \widehat{z}_k)\widehat{z}_k + \widehat{s}_k^T(\widehat{B}_k^\sharp \widehat{z}_k - \widehat{z}_k)\widehat{z}_k + \widehat{s}_k^T\widehat{z}_k(\widehat{z}_k - \widehat{B}_k^\sharp \widehat{z}_k + \widehat{B}_k^\sharp(\widehat{z}_k - \widehat{s}_k))\| \\ &\le 2(\|\widehat{z}_k - \widehat{B}_k^\sharp \widehat{z}_k\| + \|\widehat{B}_k^\sharp\|\|\widehat{z}_k - \widehat{s}_k\|)\|\widehat{s}_k\|\|\widehat{z}_k\|,\end{aligned}$$

we obtain

$$\begin{aligned}\|\Delta B_k\|_{F,M} \;&=\; \|M^{-1}\Delta B_k M^{-1}\|_F \\ &=\; \frac{\|(\widehat{s}_k^T \widehat{B}_k^\sharp \widehat{s}_k)\widehat{z}_k - (\widehat{s}_k^T\widehat{z}_k)\widehat{B}_k^\sharp \widehat{s}_k\|^2}{(\widehat{s}_k^T \widehat{B}_k^\sharp \widehat{s}_k)(\widehat{s}_k^T\widehat{z}_k)^2} \\ &\le\; \frac{4\beta_2^4\|\widehat{s}_k\|^2\|\widehat{z}_k\|^2(\|\widehat{z}_k - \widehat{B}_k^\sharp \widehat{z}_k\| + \|\widehat{B}_k^\sharp\|\|\widehat{z}_k - \widehat{s}_k\|)^2}{\beta_1^2\|\widehat{z}_k\|^4(1-\delta^\sharp)\|\widehat{s}_k\|^2} \\ &=\; \frac{4\beta_2^4}{\beta_1^2(1-\delta^\sharp)}\left(\frac{\|\widehat{z}_k - \widehat{B}_k^\sharp \widehat{z}_k\|}{\|\widehat{z}_k\|} + \|\widehat{B}_k^\sharp\|\frac{\|\widehat{z}_k - \widehat{s}_k\|}{\|\widehat{z}_k\|}\right)^2.\end{aligned}$$ ∎

4. Local and Q-Superlinear Convergence

This section is devoted to the study of the convergence property of our method. We first prove local and linear convergence of our method.

Theorem 1 *Suppose that the standard assumptions* (A1) *and* (A2) *are satisfied and that there exists a positive constant* $\zeta^\sharp$ *such that*

$$\|y_k^\sharp - G(x_*)s_k\| \leq \zeta^\sharp \sigma_k^p \|s_k\| \tag{4.1}$$

for each k*. Let the matrix* A_k *be updated by (2.3). Assume that there exist positive constants* ϕ_{min} *and* ϕ_{max} *such that* $-\phi_{min} \leq \phi_k \leq \phi_{max}$*. Let the sequence* $\{x_k\}$ *be generated by*

$$x_{k+1} = x_k + s_k \quad and \quad (C(x_k) + A_k)s_k = -\nabla f(x_k). \tag{4.2}$$

Then, for any $\nu \in (0,1)$*, there exist positive constants* ε *and* δ *such that if*

$$\|x_0 - x_*\| \leq \varepsilon, \qquad x_0 \in D$$

and

$$\|(C(x_0) + A_0) - \nabla^2 f(x_*)\|_{F,M} \leq \delta,$$

the sequence $\{x_k\}$ *is well defined and converges linearly to the local minimizer* x_**, i.e.*

$$\|x_{k+1} - x_*\| \leq \nu \|x_k - x_*\|.$$

Proof. Set

$$N_1 = \{x \in R^n | \ \|x - x_*\| \leq \varepsilon\}, \tag{4.3}$$

$$N_2 = \{B \in R^{n \times n} | \ \|B - \nabla^2 f(x_*)\|_{F,M} \leq 2\delta\}. \tag{4.4}$$

Since D is an open set, we can choose ε such that $N_1 \subset D$. The boundedness condition on ϕ_k implies that there exists a positive constant ϕ' such that $|\phi_k| \leq \phi'$.

Now we prove, by mathematical induction, that the following expressions (E1;k) through (E4;k) hold for all $k \geq 0$:

(E1; k) $$B_k \in N_2, \qquad \|B_k\| \leq \tau_1, \qquad \text{and} \qquad \|B_k^{-1}\| \leq \tau_2,$$

(E2; k) $$\|x_{k+1} - x_*\| \leq \nu \|x_k - x_*\|, \quad x_{k+1} \in N_1,$$

(E3; k) $$\|B_k^\sharp - B_k\| \leq 2^p \xi \sigma_k^p, \qquad \|B_k^\sharp\| \leq \tau_1^\sharp, \qquad \text{and} \qquad \|(B_k^\sharp)^{-1}\| \leq \tau_2^\sharp,$$

(E4; k) $$\|B_{k+1} - \nabla^2 f(x_*)\|_{F,M} \leq \|B_k - \nabla^2 f(x_*)\|_{F,M} + \mu \sigma_k^p + \left((\phi' + 1)\tau - \frac{1}{6\delta}\right) \frac{\|\widehat{z}_k - \widehat{B}_k^\sharp \widehat{z}_k\|^2}{\|\widehat{z}_k\|^2},$$

where $\tau_1, \tau_2, \tau_1^\sharp, \tau_2^\sharp, \mu$ and τ are positive constants defined below.

We first consider the case of $k = 0$.

(E1;0) The first and second results follow directly from the choice of the initial matrix. If we choose δ such that

$$2\eta \|\nabla^2 f(x_*)^{-1}\| \delta \leq \frac{\nu}{1+\nu}, \tag{4.5}$$

the first and second results of (E1;0) yield

$$\|\nabla^2 f(x_*)^{-1}\| \, \|B_0 - \nabla^2 f(x_*)\| \leq 2\eta\delta \|\nabla^2 f(x_*)^{-1}\| \leq \frac{\nu}{1+\nu} < 1.$$

By the Banach perturbation lemma (see Theorem 3.1.4 in [4]), B_0 is nonsingular and

$$\|B_0^{-1}\| \leq (1+\nu)\|\nabla^2 f(x_*)^{-1}\| = \tau_2.$$

(E2;0) It follows easily from (3.3) and (4.2) that

$$\begin{aligned} \|x_1 - x_*\| &\leq \|B_0^{-1}\|(\|\nabla f(x_0) - \nabla f(x_*) - \nabla^2 f(x_*)(x_0 - x_*)\| \\ &\qquad + \|B_0 - \nabla^2 f(x_*)\|\|x_0 - x_*\|) \\ &\leq \tau_2(\xi\varepsilon^p + 2\eta\delta) \, \|x_0 - x_*\|. \end{aligned}$$

If we choose ε and δ such that

$$\tau_2(\xi\varepsilon^p + 2\eta\delta) \leq \nu, \tag{4.6}$$

we have

$$\|x_1 - x_*\| \leq \nu \|x_0 - x_*\| \leq \varepsilon.$$

Thus $x_1 \in N_1$.

(E3;0) The above result $x_1 \in N_1 \subset D$ implies that the matrix $C(x_1)$ is available, so $B_0^\sharp$ is well defined. It follows from assumption (A1) that

$$\begin{aligned} \|B_0^\sharp - B_0\| &= \|(C(x_1) + A_0) - (C(x_0) + A_0)\| = \|C(x_1) - C(x_0)\| \\ &\leq \xi(2\sigma_0)^p \leq 2^p \xi \varepsilon^p. \end{aligned} \tag{4.7}$$

Then we see that

$$\|B_0^\sharp\| \leq \|B_0^\sharp - B_0\| + \|B_0\| \leq 2^p\xi\varepsilon^p + \tau_1$$

and

$$\|B_0^{-1}\|\|B_0^\sharp - B_0\| \leq 2^p \xi \tau_2 \varepsilon^p.$$

If we choose ε such that

$$2^p \xi \tau_2 \varepsilon^p < 1, \tag{4.8}$$

then by the Banach perturbation lemma, the matrix $B_0^\sharp$ is nonsingular and

$$\|(B_0^\sharp)^{-1}\| \leq \frac{\|B_0^{-1}\|}{1 - 2^p \xi \tau_2 \varepsilon^p}.$$

Thus there exist positive constants $\tau_1^\sharp$ and $\tau_2^\sharp$ such that $\|B_0^\sharp\| \leq \tau_1^\sharp$ and $\|(B_0^\sharp)^{-1}\| \leq \tau_2^\sharp$.

(E4;0) Recall that

$$\|B_0^\sharp - B_0\|_{F,M} \leq 2^p \eta \xi \varepsilon^p.$$

If we choose ε and δ such that

$$2^p \eta \xi \varepsilon^p < \delta \qquad \text{and} \qquad 3\delta < 1, \tag{4.9}$$

then

$$\|B_0^\sharp - \nabla^2 f(x_*)\|_{F,M} \leq \|B_0^\sharp - B_0\|_{F,M} + \|B_0 - \nabla^2 f(x_*)\|_{F,M} \leq 3\delta < 1. \tag{4.10}$$

Therefore, by setting

$$\delta^\sharp = 3\delta \qquad \text{and} \qquad \tau^\sharp = \|M^{-1}\|^2 \tau_1^\sharp,$$

Lemmas 2 and 3 yield

$$\|B_1^{DFP} - \nabla^2 f(x_*)\|_{F,M} \le \|B_0^\sharp - \nabla^2 f(x_*)\|_{F,M} - \frac{\|\widehat{z}_0 - \widehat{B}_0^\sharp \widehat{z}_0\|^2}{6\delta\|\widehat{z}_0\|^2} + \omega\sigma_0^p \tag{4.11}$$

and

$$\|\Delta B_0\|_{F,M} \le \frac{4\beta_2^4}{\beta_1^2(1-3\delta)} \left(\frac{\|\widehat{z}_0 - \widehat{B}_0^\sharp \widehat{z}_0\|}{\|\widehat{z}_0\|} + \|\widehat{B}_0^\sharp\| \frac{\|\widehat{z}_0 - \widehat{s}_0\|}{\|\widehat{z}_0\|} \right)^2.$$

Recalling that, by (3.6), (4.10) and Lemma 1,

$$\begin{aligned} \|\widehat{z}_0 - \widehat{B}_0^\sharp \widehat{z}_0\| &\le \|M^{-1}(B_0^\sharp - \nabla^2 f(x_*))M^{-1}\|_F \|\widehat{z}_0\| \\ &\le 3\|\widehat{z}_0\|\delta, \end{aligned}$$

$$\|\widehat{z}_0 - \widehat{s}_0\| \le \zeta\sigma_0^p\|\widehat{s}_0\| \le \frac{\zeta}{\beta_1}\sigma_0^p\|\widehat{z}_0\| \qquad \text{and} \qquad \|\widehat{B}_0^\sharp\| \le \|M^{-1}\|^2\tau_1^\sharp,$$

we have

$$\begin{aligned} \|\Delta B_0\|_{F,M} &\le \frac{4\beta_2^4}{\beta_1^2(1-3\delta)} \left[\frac{\|\widehat{z}_0 - \widehat{B}_0^\sharp \widehat{z}_0\|^2}{\|\widehat{z}_0\|^2} \right. \\ &\qquad \left. + \left(2\frac{\|\widehat{z}_0 - \widehat{B}_0^\sharp \widehat{z}_0\|}{\|\widehat{z}_0\|} + \frac{\|\widehat{B}_0^\sharp\|\|\widehat{z}_0 - \widehat{s}_0\|}{\|\widehat{z}_0\|} \right) \frac{\|\widehat{B}_0^\sharp\|\|\widehat{z}_0 - \widehat{s}_0\|}{\|\widehat{z}_0\|} \right] \\ &\le \frac{4\beta_2^4}{\beta_1^2(1-3\delta)} \left(\frac{\|\widehat{z}_0 - \widehat{B}_0^\sharp \widehat{z}_0\|^2}{\|\widehat{z}_0\|^2} + \left(6\delta + \frac{\|M^{-1}\|^2\tau_1^\sharp\zeta}{\beta_1}\varepsilon^p \right) \frac{\|M^{-1}\|^2\tau_1^\sharp\zeta}{\beta_1}\sigma_0^p \right). \end{aligned}$$

For ε and δ, there exist positive constants τ and γ such that

$$\frac{4\beta_2^4}{\beta_1^2(1-3\delta)} \le \tau \quad \text{and} \quad \left(6\delta + \frac{\|M^{-1}\|^2\tau_1^\sharp\zeta}{\beta_1}\varepsilon^p \right) \frac{\|M^{-1}\|^2\tau_1^\sharp\zeta}{\beta_1} \le \gamma. \tag{4.12}$$

Thus we have

$$\|\Delta B_0\|_{F,M} \le \tau \left(\frac{\|\widehat{z}_0 - \widehat{B}_0^\sharp \widehat{z}_0\|^2}{\|\widehat{z}_0\|^2} + \gamma\sigma_0^p \right).$$

Therefore, the update formula (2.4) for B_1 yields

$$\begin{aligned} \|B_1 - \nabla^2 f(x_*)\|_{F,M} &\le \|B_1^{DFP} - \nabla^2 f(x_*)\|_{F,M} + |\phi_0 - 1|\,\|\Delta B_0\|_{F,M} \\ &\le \|B_1^{DFP} - \nabla^2 f(x_*)\|_{F,M} + (\phi' + 1)\tau \left(\frac{\|\widehat{z}_0 - \widehat{B}_0^\sharp \widehat{z}_0\|^2}{\|\widehat{z}_0\|^2} + \gamma\sigma_0^p \right). \end{aligned}$$

We also see by (4.7) that

$$\begin{aligned} \|B_0^\sharp - \nabla^2 f(x_*)\|_{F,M} &\le \|B_0^\sharp - B_0\|_{F,M} + \|B_0 - \nabla^2 f(x_*)\|_{F,M} \\ &\le 2^p\eta\xi\sigma_0^p + \|B_0 - \nabla^2 f(x_*)\|_{F,M}. \end{aligned}$$

Finally, a straightforward calculation using the preceding expression and (4.11) gives

$$\begin{aligned}
\|B_1 - \nabla^2 f(x_*)\|_{F,M} &\le \|B_0^\sharp - \nabla^2 f(x_*)\|_{F,M} - \frac{1}{6\delta}\frac{\|\hat{z}_0 - \hat{B}_0^\sharp \hat{z}_0\|^2}{\|\hat{z}_0\|^2} \\
&\quad + \omega\sigma_0^p + (\phi' + 1)\tau\left(\frac{\|\hat{z}_0 - \hat{B}_0^\sharp \hat{z}_0\|^2}{\|\hat{z}_0\|^2} + \gamma\sigma_0^p\right) \\
&= \|B_0^\sharp - \nabla^2 f(x_*)\|_{F,M} + \left((\phi'+1)\tau - \frac{1}{6\delta}\right)\frac{\|\hat{z}_0 - \hat{B}_0^\sharp \hat{z}_0\|^2}{\|\hat{z}_0\|^2} \\
&\quad + (\omega + (\phi'+1)\tau\gamma)\sigma_0^p \\
&\le \|B_0 - \nabla^2 f(x_*)\|_{F,M} + \left((\phi'+1)\tau - \frac{1}{6\delta}\right)\frac{\|\hat{z}_0 - \hat{B}_0^\sharp \hat{z}_0\|^2}{\|\hat{z}_0\|^2} + \mu\sigma_0^p,
\end{aligned}$$

where $\mu = 2^p\eta\xi + \omega + (\phi'+1)\tau\gamma$. Therefore the case of $k = 0$ is proved.

We assume as an induction hypotheses that expressions (E1;k) through (E4;k) hold for $k = 0, \ldots, t-1$. Then we have

$$\|B_{k+1} - \nabla^2 f(x_*)\|_{F,M} \le \|B_k - \nabla^2 f(x_*)\|_{F,M} + \mu\sigma_k^p + \left((\phi'+1)\tau - \frac{1}{6\delta}\right)\frac{\|\hat{z}_k - \hat{B}_k^\sharp \hat{z}_k\|^2}{\|\hat{z}_k\|^2}$$

for $k = 0, \ldots, t-1$, and by summing both sides from $k = 0$ to $t-1$, it follows that

$$\|B_t - \nabla^2 f(x_*)\|_{F,M} \le \|B_0 - \nabla^2 f(x_*)\|_{F,M} + \mu\sum_{k=0}^{t-1}\sigma_k^p + \left((\phi'+1)\tau - \frac{1}{6\delta}\right)\sum_{k=0}^{t-1}\frac{\|\hat{z}_k - \hat{B}_k^\sharp \hat{z}_k\|^2}{\|\hat{z}_k\|^2}.$$

Noting that

$$\sigma_k = \max(\|x_{k+1} - x_*\|, \|x_k - x_*\|) = \|x_k - x_*\| \le \nu^k\varepsilon$$

and choosing ε and δ such that

$$\text{(4.13)} \qquad \frac{\mu}{1-\nu^p}\varepsilon^p \le \delta \qquad \text{and} \qquad (\phi'+1)\tau - \frac{1}{6\delta} < 0,$$

we have

$$\begin{aligned}
\|B_t - \nabla^2 f(x_*)\|_{F,M} &\le \|B_0 - \nabla^2 f(x_*)\|_{F,M} + \mu\varepsilon^p\sum_{k=0}^{t-1}(\nu^p)^k \\
&\quad + \left((\phi'+1)\tau - \frac{1}{6\delta}\right)\sum_{k=0}^{t-1}\frac{\|\hat{z}_k - \hat{B}_k^\sharp \hat{z}_k\|^2}{\|\hat{z}_k\|^2} \\
&\le \|B_0 - \nabla^2 f(x_*)\|_{F,M} + \frac{\mu}{1-\nu^p}\varepsilon^p \qquad \text{(4.14)} \\
&\quad + \left((\phi'+1)\tau - \frac{1}{6\delta}\right)\sum_{k=0}^{t-1}\frac{\|\hat{z}_k - \hat{B}_k^\sharp \hat{z}_k\|^2}{\|\hat{z}_k\|^2} \\
&\le \|B_0 - \nabla^2 f(x_*)\|_{F,M} + \frac{\mu}{1-\nu^p}\varepsilon^p \\
&\le 2\delta.
\end{aligned}$$

This implies $B_t \in N_2$. We can prove (E1;t) through (E4;t) in the same way as the case of $k = 0$. This concludes the induction, and the proof. ∎

Note that conditions (4.5), (4.6), (4.8), (4.9) and (4.13) are compatible. Also note that $\tau_2\xi\varepsilon^p \leq \nu/2$ and $2\tau_2\eta\delta \leq \nu/2$ implies (4.6). Specifically speaking, we first choose δ such that

$$0<\delta<\min\left(\frac{1}{3}\ ,\ \frac{\nu}{2\eta\|\nabla^2 f(x_*)^{-1}\|(1+\nu)}\ ,\ \frac{\nu}{4\tau_2\eta}\ ,\ \frac{1}{6(\phi'+1)\tau}\right).$$

Next, for this δ, we can choose a positive ε such that $N_1 \subset D$ and

$$\varepsilon^p<\min\left(\frac{1}{2^p\xi\tau_2}\ ,\ \frac{\nu}{2\tau_2\xi}\ ,\ \frac{\delta}{2^p\eta\xi}\ ,\ \frac{1-\nu^p}{\mu}\delta\right).$$

The following theorem shows q-superlinear convergence of our method.

Theorem 2 *Suppose that all conditions of Theorem 1 hold. Then the sequence $\{x_k\}$ generated by the scheme* (4.2) *with the structured Broyden family* (2.3) *converges q-superlinearly to x_*.*

Proof. It follows directly from (4.14) that, for all $t \geq 0$,

$$\|B_t-\nabla^2 f(x_*)\|_{F,M} \leq \|B_0-\nabla^2 f(x_*)\|_{F,M}+\frac{\mu}{1-\nu^p}\varepsilon^p+\left((\phi'+1)\tau-\frac{1}{6\delta}\right)\sum_{k=0}^{t-1}\frac{\|\widehat{z}_k-\widehat{B}_k^{\sharp}\widehat{z}_k\|^2}{\|\widehat{z}_k\|^2}.$$

Then, by (4.13), we have

$$\begin{aligned}\left(\frac{1}{6\delta}-(\phi'+1)\tau\right)\sum_{k=0}^{t-1}\frac{\|\widehat{z}_k-\widehat{B}_k^{\sharp}\widehat{z}_k\|^2}{\|\widehat{z}_k\|^2} &\leq \|B_0-\nabla^2 f(x_*)\|_{F,M}+\frac{\mu}{1-\nu^p}\varepsilon^p\\ &\leq 2\delta,\end{aligned}$$

which guarantees the convergence of the infinite series

$$\sum_{k=0}^{\infty}\frac{\|\widehat{z}_k-\widehat{B}_k^{\sharp}\widehat{z}_k\|^2}{\|\widehat{z}_k\|^2}.$$

Thus

$$\lim_{k\to\infty}\frac{\|\widehat{z}_k-\widehat{B}_k^{\sharp}\widehat{z}_k\|}{\|\widehat{z}_k\|}=0. \tag{4.15}$$

Since Lemma 1, (E3;k) in the proof of Theorem 1 and (4.15) yield

$$\begin{aligned}\|\widehat{B}_k\widehat{s}_k-\widehat{s}_k\| &= \|(\widehat{B}_k-\widehat{B}_k^{\sharp})\widehat{s}_k+\widehat{B}_k^{\sharp}(\widehat{s}_k-\widehat{z}_k)+(\widehat{B}_k^{\sharp}\widehat{z}_k-\widehat{z}_k)+(\widehat{z}_k-\widehat{s}_k)\|\\ &\leq \|\widehat{B}_k-\widehat{B}_k^{\sharp}\|\|\widehat{s}_k\|+(1+\|\widehat{B}_k^{\sharp}\|)\|\widehat{z}_k-\widehat{s}_k\|+\|\widehat{B}_k^{\sharp}\widehat{z}_k-\widehat{z}_k\|\\ &= \sigma_k^p O(\|\widehat{s}_k\|)+o(\|\widehat{z}_k\|)\\ &= o(\|\widehat{s}_k\|),\end{aligned}$$

we have

$$\begin{aligned}\frac{\|(B_k-\nabla^2 f(x_*))s_k\|}{\|s_k\|} &= \frac{\|M(M^{-1}B_kM^{-1}-I)Ms_k\|}{\|Ms_k\|}\frac{\|Ms_k\|}{\|s_k\|}\\ &\leq \|M\|^2\frac{\|\widehat{B}_k\widehat{s}_k-\widehat{s}_k\|}{\|\widehat{s}_k\|}=o(1).\end{aligned}$$

Therefore, this relation implies

$$\lim_{k\to\infty}\frac{\|(B_k-\nabla^2 f(x_*))s_k\|}{\|s_k\|}=0.$$

This is the necessary and sufficient condition that the sequence $\{x_k\}$ converges q-superlinearly to x_* [3]. ∎

5. Applications

The previous section presented a local and q-superlinear convergence property of a wider class of the structured Broyden family than that of Engels and Martinez [8]. They applied the convex class of the structured Broyden family to the problems (1.10) and (1.12), and obtained local and q-superlinear convergence of their method. In this section, we apply our convergence results to these problems in the same way as Engels and Martinez. Specifically, we define a vector $y_k^\sharp$ for each problem and we only investigate that the conditon (4.1) is satisfied.

First we consider the unconstrained nonlinear least squares problem (1.10). Following Dennis [2], the vector $y_k^\sharp$ is defined by

$$y_k^\sharp = (\nabla r(x_{k+1}) - \nabla r(x_k))r(x_{k+1}). \tag{5.1}$$

Then Dennis, Martinez and Tapia showed the following lemma (see Lemma 4.1 in [7]).

Lemma 4 *Let D be an open convex subset of R^n that contains a local minimizer x_*. Let $G(x)$ and $y_k^\sharp$ be defined by (1.11) and (5.1), respectively. Suppose that the functions $r_i, i = 1, \ldots, l$ are twice continuously differentiable and that there exist positive constants ξ and p such that*

$$\|\nabla^2 f(x) - \nabla^2 f(x_*)\| \le \xi \|x - x_*\|^p \qquad \text{and} \qquad \|\nabla r(x) - \nabla r(x')\| \le \xi \|x - x'\|^p$$

for $x, x' \in D$. Assume that $\nabla^2 f(x_*)$ is symmetric positive definite. Let x_k and x_{k+1} be very close to x_*. Then there exists a positive constant $\zeta^\sharp$ such that the condition (4.1) is satisfied.

By using Theorems 1 and 2, and Lemma 4, the following theorem is a straightforward result.

Theorem 3 *Suppose that the assumptions of Lemma 4 hold. Let the matrix A_k be updated by (2.3). Assume that there exist positive constants ϕ_{min} and ϕ_{max} such that $-\phi_{min} \le \phi_k \le \phi_{max}$. Let the sequence $\{x_k\}$ be generated by*

$$x_{k+1} = x_k + s_k \quad \text{and} \quad (\nabla r(x_k)\nabla r(x_k)^T + A_k)s_k = -\nabla r(x_k) r(x_k). \tag{5.2}$$

Then the sequence $\{x_k\}$ converges locally and q-superlinearly to the local minimizer x_*.

Next we consider the equality constrained minimization problem (1.12). The SQP method based on Tapia's idea generates $(x_{k+1}, \lambda_{k+1}) = (x_k + s_k, \lambda_k + \Delta\lambda_k)$ by choosing s_k and $\Delta\lambda_k$ to be the solution and the associated multiplier of the quadratic programming problem

$$\min_{s \in R^n} \quad \frac{1}{2} s^T B_k s + \nabla_x L(x_k, \lambda_k; \rho)^T s$$
$$\text{subject to} \quad \nabla h(x_k)^T s + h(x_k) = 0,$$

where $L(x, \lambda; \rho)$ is an augmented Lagrangian function given in (1.13) and B_k is intended to be an approximation to the Hessian matrix of the augmented Lagrangian function (see [11]). By applying the first-order necessary conditions to the preceding quadratic programming problem, we have the linear system of equations:

$$\begin{pmatrix} B_k & \nabla h(x_k) \\ \nabla h(x_k)^T & 0 \end{pmatrix} \begin{pmatrix} s_k \\ \lambda_{k+1} \end{pmatrix} = - \begin{pmatrix} \nabla F(x_k) + \rho \nabla h(x_k) h(x_k) \\ h(x_k) \end{pmatrix}.$$

Then this system of equations yields the new iterate (x_{k+1}, λ_{k+1}) as follows:

$$(5.3) \qquad \lambda_{k+1} = (\nabla h(x_k)^T B_k^{-1} \nabla h(x_k))^{-1}(h(x_k) - \nabla h(x_k)^T B_k^{-1}(\nabla F(x_k) + \rho \nabla h(x_k) h(x_k)))$$

and

$$(5.4) \qquad x_{k+1} = x_k + s_k, \qquad B_k s_k = -\nabla_x L(x_k, \lambda_{k+1}; \rho).$$

Note that the multiplier (5.3) is equivalent to the Newton multiplier update of the diagonalized multiplier method (see [9]). The Lagrangian function of the problem (1.12) is defined by

$$(5.5) \qquad l(x, \lambda) = F(x) + \lambda^T h(x).$$

Since, at the solution (x_*, λ_*), the matrix $G(x, \lambda)$ in (1.14) becomes

$$G(x_*, \lambda_*) = \nabla_x^2 l(x_*, \lambda_*),$$

Tapia [11] proposed that a matrix A_k should approximate the Hessian matrix of the Lagrangian function (5.5). Then the vector $y_k^\sharp$ is defined by

$$(5.6) \qquad y_k^\sharp = \nabla_x l(x_{k+1}, \lambda_{k+1}) - \nabla_x l(x_k, \lambda_{k+1})$$

and we have the lemma.

Lemma 5 *Let D be an open convex subset of R^n that contains a local minimizer x_*. Let λ_* be a multiplier associated with x_*. Let $G(x, \lambda)$ and $y_k^\sharp$ be defined by (1.14) and (5.6), respectively. Suppose that the functions F and $h_i, i = 1, \ldots, m$ are twice continuously differentiable and that there exist positive constants ξ and p such that*

$$\|\nabla^2 F(x) - \nabla^2 F(x_*)\| \le \xi \|x - x_*\|^p$$

and

$$\|\nabla^2 h_i(x) - \nabla^2 h_i(x')\| \le \xi \|x - x'\|^p, \; i = 1, \ldots, m$$

for $x, x' \in D$. Assume that the matrix $\nabla h(x_*)$ has full rank and that the Hessian matrix $\nabla_x^2 l(x_*, \lambda_*)$ is positive definite on the subspace $\{v \in R^n \mid \nabla h(x_*)^T v = 0\}$. If x_k, x_{k+1} are very close to x_*, then there exist positive constants γ and $\zeta^\sharp$ such that

$$(5.7) \qquad \|\lambda_{k+1} - \lambda_*\| \le \gamma \|x_k - x_*\|$$

and

$$(5.8) \qquad \|y_k^\sharp - \nabla_x^2 l(x_*, \lambda_*) s_k\| \le \zeta^\sharp \sigma_k^p \|s_k\|.$$

In this lemma, the results (5.7) and (5.8) followed from Proposition 4.2 in [9] and from Lemma 11 in [8], respectively. Note that the assumptions of Lemma 5 guarantees that there exists a penalty parameter $\rho \ge 0$ such that $\nabla_x^2 L(x_*, \lambda_*; \rho)$ is positive definite. By using Theorems 1 and 2, and Lemma 5 and by the same discussion as the proof of Theorem 12 in [8], we obtain the following theorem.

Theorem 4 *Suppose that the assumptions of Lemma 5 hold and that the penalty parameter* $\rho \geq 0$ *is chosen so that* $\nabla_x^2 L(x_*, \lambda_*; \rho)$ *is positive definite. Let the matrix* A_k *be updated by (2.3). Assume that there exist positive constants* ϕ_{min} *and* ϕ_{max} *such that* $-\phi_{min} \leq \phi_k \leq \phi_{max}$*. Let the sequence* $\{(x_k, \lambda_k)\}$ *be generated by (5.3) and (5.4) with*

$$B_k = \rho \nabla h(x_k) \nabla h(x_k)^T + A_k.$$

Then the sequence $\{x_k\}$ *generated by the SQP method based on Tapia's idea converges locally and q-superlinearly to the local minimizer* x_*.

6. Conclusions

This paper has presented local and q-superlinear convergence of the quasi-Newton method with the structured Broyden family. We have extended the results of Engels and Martinez [8] and have shown the convergence in a way different from their proof. The family we have dealt with is wider than that dealt by Engels and Martinez. Note that, in the boundedness condition on ϕ_k, ϕ_{min} must satisfy

$$\phi_{min} < \left(\frac{(s_k^T B_k^\sharp s_k)(z_k^T (B_k^\sharp)^{-1} z_k)}{(s_k^T z_k)^2} - 1 \right)^{-1}$$

to guarantee the nonsingularity of the matrix B_{k+1}. Finally we have applied our convergence theory to the unconstrained nonlinear least squares problem (1.10) and the equality constrained minimization problem (1.12).

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