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OPTIMAL NUMBER OF DIGITS TO REPORT

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Abstract Instrumentation, computation, and documentation often involve the task of reporting a number with an indicator of its possible error. For quick decision making, reporting only one number with reliable digits is more desirable than reporting two or more numbers with insignificant lower digits. This paper proposes an optimal method to determine the number of digits to report. The method is free of arbitrary thresholds unlike the customary representation by confidence interval and other existing methods. The model used to derive the method is based on the observation that the amount of ignorance regarding the value of a number's lower digits may be expressed by a uniform distribution. If the number to report is accompanied by its standard error, the problem of deciding the number of digits to report reduces to matching the level of ignorance by approximating a normal distribution by the uniform distribution. The goodness of the approximation is measured in terms of the Kullback-Leibler information, which is minimized.

1 Introduction

Numerical readouts from measurement instruments, computation or simulation are often accompanied by indicators of their accuracy. Presenting such information in a quickly understandable format is important especially for instrumentation in which human operators must interpret the numbers in order to make decisions. Misinterpretations may lead to misjudgments, which possibly result in uneconomical or even dangerous operations.

A common practice in interpreting a figure with accuracy indicator is to assume that the true measure follows a normal distribution with given mean and known or estimated standard deviation (SD). This is a Bayesian approach [2, 4] which the operator takes often unconsciously. Throughout this paper the parameter X which represents the true measure is assumed to follow the normal posterior distribution with given mean μ and SD σ. Only the posterior distribution will be used herein; no mention will be made of the prior distribution. Thus X is a random variable whereas μ and σ are constants.

Song and Schmeiser [7] discuss the inadequacy of reporting the information content in the form of a confidence interval (CI) and propose an alternative which is to report only the reliable digits of μ. A method to determine the number of reliable digits is called a significant-digit procedure (SDP). The SDPs they propose reduce to comparing the probability that the least significant digit is correct against a threshold. The threshold is an arbitrary constant determined by the user of the method, in the same way as the coverage probability of a CI is determined.

The involvement of arbitrary thresholds in SDP is undesirable for various reasons. If, for instance, two instruments adopt the same SDP but diferent thresholds, their readouts are not directly comparable.

This paper proposes the use of an optimal SDP rather than SDP with arbitrary threshold. The problem is formulated into matching the level of ignorance by approximating the normal distribution by a uniform distribution representing a rounded number. The goodness of approximation is measured by the Kullback-Leibler (K-L) information [3], which is then minimized. The resulting rule is as simple as those SDPs suggested in [7].

The remainder of this paper is organized as follows. Section 2 briefly reviews Song and Schmeiser [7], on the inadequacy of CI and on the SDP alternatives to CI. Section 3 points out the undesirability of arbitraty thresholds which are inherent in the SDPs proposed in [7]. Section 4 introduces a new K-L information model on approximating a normal distribution by a uniform distribution, which requires no arbitrary threshold. Section 5 solves the model to derive an optimal SDP. Section 6 gives a simpler SDP by approximating the optimum. Section 7 compares the result with [7] through an example. Section 8 discusses the inverse procedure of converting a rounded number to a normal distribution. The paper concludes in Section 9 with a remark on the arbitrariness in the selection of the measure of discrepancy.

2 Previous works

This section reviews Song and Schmeiser [7]. They contend that CI is unsuitable for quick interpretation of simulation results. Although CIs are widely used in textbooks and academic papers, their use is limited in practical situations. This motivates their search for SDPs.

The CI has been deemed inadequate because:

1. A CI is difficult to interpret.
2. A CI presents an incorrect coverage probability when normality and independence assumptions are violated.
3. The coverage probability is arbitrary.
4. Calculation of degrees of freedom can be complex.
5. μ, which is of utmost interest to the reader, has to be calculated from the two extreme points of the CI.
6. Stating a CI is cumbersome.

Here the coverage probability of a CI is the probability that X falls within the CI.

As an alternative to the CI they advocate displaying only the reliable digits. Let μ_k be μ represented down to the k-th digit. For instance, if $\mu_k = 123.46$, then $k = -2$. (With the notation of [7], $k = -r$.) The SDPs proposed in [7] are:

SDP-1 Round the SD to the nearest power of ten: $k := [\log_{10} \sigma]$, where the bracket $[\cdot]$ is for rounding to the nearest integer.

SDP-2 Compare each digit to the SD: $k := \lfloor \log_{10}(\sigma/c) \rfloor$, where c is a constant; $\lfloor \cdot \rfloor$ is the floor function.

SDP-3 Compare the effect of dropping digits to the SD: $k := \arg\min_i \{ | \mu_i - \mu | < \sigma/c \}$, where c is a constant.

Further details on the background may be found in [7] and the bibliography therein.

3 On arbitrary thresholds

The SDPs introduced in the previous section reduce to specifying a probability threshold to judge whether the last digit should be reported or not, which is analogous to specifying the level of confidence for a CI. Thus their SDPs still retain the third of the inadequacies listed against the CIs. This section points out that the involvement of arbitrary threshold in SDP should be avoided.

The standard arguments against the use of tests of hypotheses in model selection as those found in [1] and Section 1.2.1 of [5] apply here as well. One such point is that decisions involving multiple SDP tend to lead to inconsistencies. For instance, if the readouts from two different instruments are 123 and 124, it is desirable that such statements as $123 \leq 124$ be meaningful. This is not always the case if the instruments employ the same SDP but different thresholds.

If a function of the readout is to be used for decision making, it is desirable that the original normal distribution be recoverable since a function of a normal variate is, as a rule, relatively easy to calculate as can be seen in Chapters 5 to 7 of [6]. Recovering the normal distribution setting from a readout means transforming the readout μ_k into the pair of numbers corresponding to μ and σ. Although the mean can be represented by the readout itself, the SD can only be estimated given the threshold. Thus the readout does not stand by itself to represent the original normal distribution but requires the threshold value as a supplement. In this sense, the SDPs with thresholds transform a pair of numbers (μ, σ) into another less informative pair (μ_k, c) .

The present paper aims to remove the arbitrary thresholds from SDP. A work that may serve as a guide to this end is Akaike's information criteria (AIC) [1], in which the statistical model selection procedure switches from the testing framework to the estimation framework thus dismissing the arbitrary level of significance. The present paper loosely follows this approach.

4 Problem formulation

A new model to derive the optimal number of digits to report is introduced in this section.

One who reads a rounded number μ_k of the observation μ will have no knowledge regarding the omitted digits, meaning that μ can be anywhere between $\mu_k - b_k$ and $\mu_k + b_k$ in which

$$b_k := (1/2) \times 10^k .$$

This ignorance can be modeled by considering μ_k as a representation of the uniform distribution in the above interval.

Extending this observation slightly, consider the uniform distribution represented by μ_k as approximating the distribution of the true value X, which is normal determined by the observed value μ and the SD σ. Figure 1 illustrates a typical situation.

Let $f()$ and $g()$ be probability distribution functions for the uniform and the normal distributions, respectively.

$$\begin{aligned} f(x) &:= 1/(2b_k) \qquad && x \in [\mu_k - b_k, \mu_k + b_k[\\ &\quad\; 0 && \text{otherwise} \end{aligned}$$

$$g(x) := \frac{1}{\sqrt{2\pi}\sigma} \exp -\frac{1}{2}(\frac{x-\mu}{\sigma})^2$$

The discrepancy between $f()$ and $g()$ is measured by the K-L discrimination information

$$h(k) := \int_{-\infty}^{\infty} f(x) \log\{f(x)/g(x)\} dx .$$

$g()$ has to be the numerator since while $\forall x \;\; 0 < g(x)$, for the most part $f(x) = 0$.

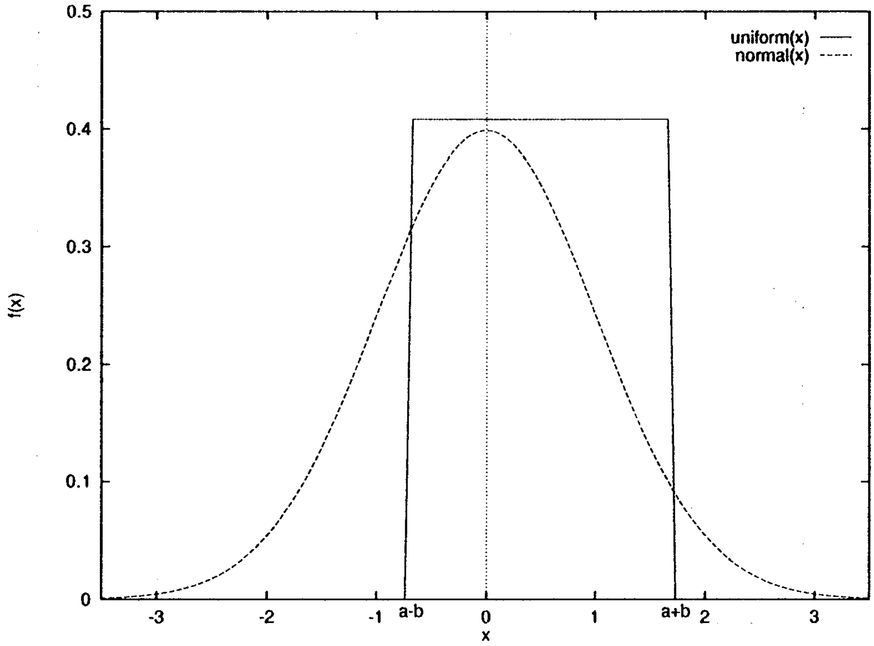

Figure 1: A uniform approximation of the normal

5 Optimal solution

The K-L information to minimize is

$$(1) \qquad h(k) = \int_{\mu_k - b_k}^{\mu_k + b_k} \frac{1}{2b_k} \log \frac{1}{2b_k} - \frac{1}{2b_k} \log\{\frac{1}{\sqrt{2\pi}\sigma} \exp -\frac{1}{2}(\frac{x-\mu}{\sigma})^2\} dx$$

$$= \frac{1}{2}(\frac{a_k}{\sigma})^2 + \frac{1}{6}(\frac{b_k}{\sigma})^2 - \log \frac{b_k}{\sigma} + \frac{1}{2} \log \frac{\pi}{2} ,$$

where

$$a_k := \mu_k - \mu .$$

Figure 2, which corresponds to Figure 1, shows $h()$ as a function of b_k. In both figures $a_k = \sigma^2/2$, which is typical as will be explained in the next section.

The optimal value is

$$(2) \qquad k = \arg\min_i h(i) ,$$

which is an integer. Finding this is computationally trivial since k determines all the subscripted parameters in (1).

6 An approximate solution

Computing $h(k)$ for several k's around the optimum is trivial, but tedious to do manually. This section derives a simpler rule by approximating the optimal k.

Consider $a_k = \mu_k - \mu$ as though it were a random variate following the uniform distribution $f()$. Taking the expectation,

$$(3) \qquad E[a_k^2] = b_k^2/3 .$$

so that, substituting for the original term in (1),

$$h(k) \approx \frac{1}{3}(\frac{b_k}{\sigma})^2 - \log \frac{b_k}{\sigma} + \frac{1}{2} \log \frac{\pi}{2} .$$

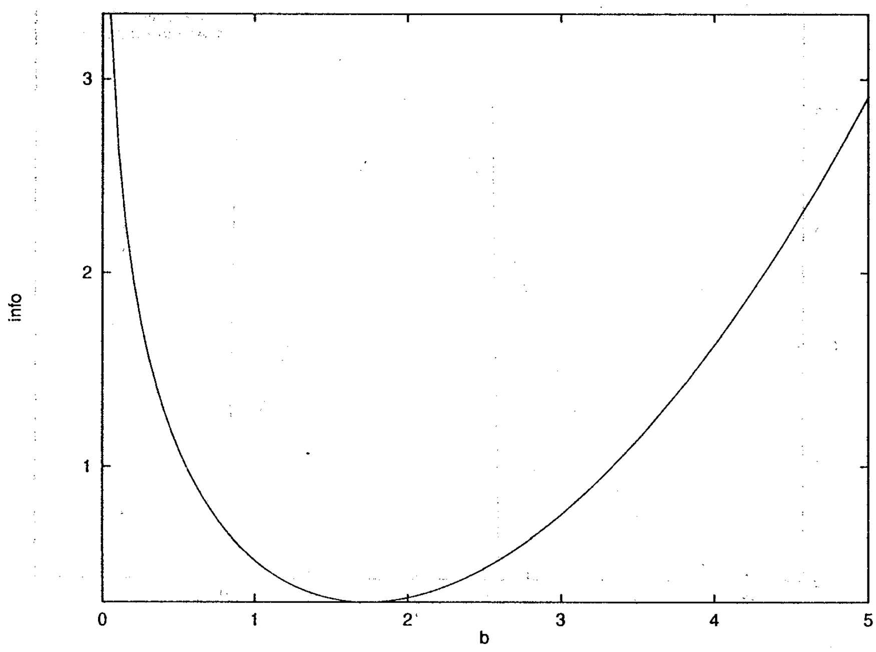

Figure 2: The K-L information against b_k

Solving $\partial h(k)/\partial b_k = 0$ yields $b_k = \sqrt{3/2}\,\sigma$. Substituting in (3), the typical situation mentioned in the previous section and illustrated in Figures 1 and 2 is found to be $a_k = \sigma^2/2$. Writing $b_k = (1/2) \times 10^r$ brings $(1/2) \times 10^r = \sqrt{3/2}\,\sigma$ or $r = \log_{10}(\sqrt{6}\,\sigma)$.

Thus a near-optimal uniform distribution that approximates the normal distribution with parameters μ and σ is over the interval $[\mu - (1/2) \times 10^r, \mu + (1/2) \times 10^r[$, where r is not necessarily an integer. Although Figure 2 suggests that a wider b_k is safer than a narrower b_k, the picture changes when considered in terms of r. Figure 3 shows the K-L information against r rather than b_k, for the same parameters as in Figures 1 and 2. Figure 3 suggests that, when in doubt, displaying more digits is safer by far. Hence the near-optimal interval is $[\mu_k - (1/2) \times 10^k, \mu_k + (1/2) \times 10^k[$, where

$$k = \lfloor \log_{10}(\sqrt{6}\,\sigma) \rfloor \ . \tag{4}$$

To summarize, the simpler rule is to round μ to the digits k as above.

7 Example

Suppose the number to represent is $\mu = 123.46$ with SD $\sigma = 0.32$ which is the example used in [7]. Figure 4 illustrates k against the K-L information, from which the optimum is, according to (1), $k = \arg\min_i h(i) = 0$, $\mu_0 = 123$. The approximate optimum is, from (4), $k = \lfloor \log_{10}(\sqrt{6} \times 0.32) \rfloor = -1$, $\mu_{-1} = 123.5$. The criteria in [7], viz. SDP-1, SDP-2 with $c = 4$, and SDP-3 with $c = 1$, all result also in 123.5 . The probability that the final digit of $\mu_0 = 123$, obtained as the optimum, is correct is 0.55, whereas the same with $\mu_{-1} = 123.5$, obtained by the other criteria, is 0.12, so that the last digit is more likely to be wrong.

8 The inverse problem

As was mentioned in Section 3, a single truncated number μ_k is often desired to be transformed back to the two numbers, the mean and the SD. This is useful when a function of truncated numbers is used for decision making, since a function of normal variates is easier to calculate than that of uniform variates[6].

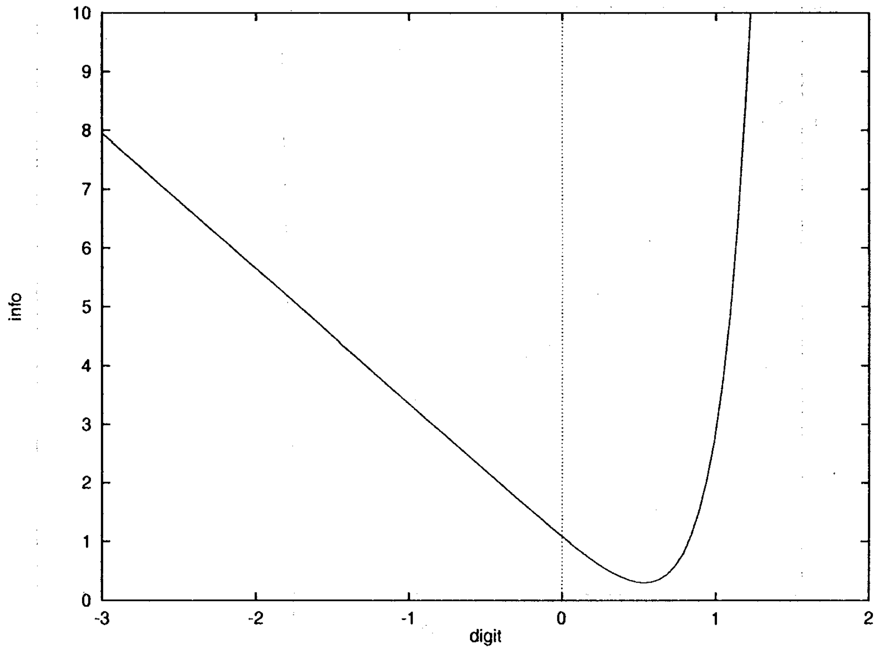

Figure 3: The K-L information against r

The same principle used to convert a normal distribution to a number may be used for the inverse conversion of determining μ and σ, given μ_k. In this case, obviously μ has to be set equal to μ_k. Then the first term of (1) equals zero. Solving $\partial h/\partial\sigma = 0$ yields $\sigma = b_k/\sqrt{3}$. Thus the inverse conversion of

$$(\mu, \sigma) \mapsto \mu_k$$

is

$$\mu_k \mapsto (\mu_k, b_k/\sqrt{3}) \ .$$

A related question of interest is what happens when (μ, σ) and μ_k are repeatedly converted back and forth. If the optimal solution (2) is used, the process is stable

$$(\mu, \sigma) \mapsto \mu_k \mapsto (\mu_k, \sigma) \mapsto \mu_k \mapsto (\mu_k, \sigma) \mapsto \mu_k \mapsto \cdots$$

because $a_k = 0$.

However, if the approximately optimal procedure is employed, k is decreased by one each time (4) is applied, appending zeros to the end of μ_k:

$$(\mu, \sigma) \mapsto \mu_k \mapsto (\mu_k, \sigma_1) \mapsto \mu_{k-1} \mapsto (\mu_{k-1}, \sigma_2) \mapsto \mu_{k-2} \mapsto \cdots$$

where $\sigma_1 > \sigma_2 > \ldots$.

9 Conclusion

A rule for reporting an optimal number of digits has been derived, which is free of arbitrary thresholds. The method is based on the observation that a rounded number represents a uniform distribution expressing the ignorance of the omitted digits. The uniform distribution is adjusted to match, in terms of the K-L information, the level of ignorance expressed by the given normal distribution. A simplified rule has also been proposed as an approximation to the optimal rule.

Many functions other than the K-L information, e.g. the χ^2, can measure the discrepancy between two probability distributions, implying that the proposed SDP still has elements

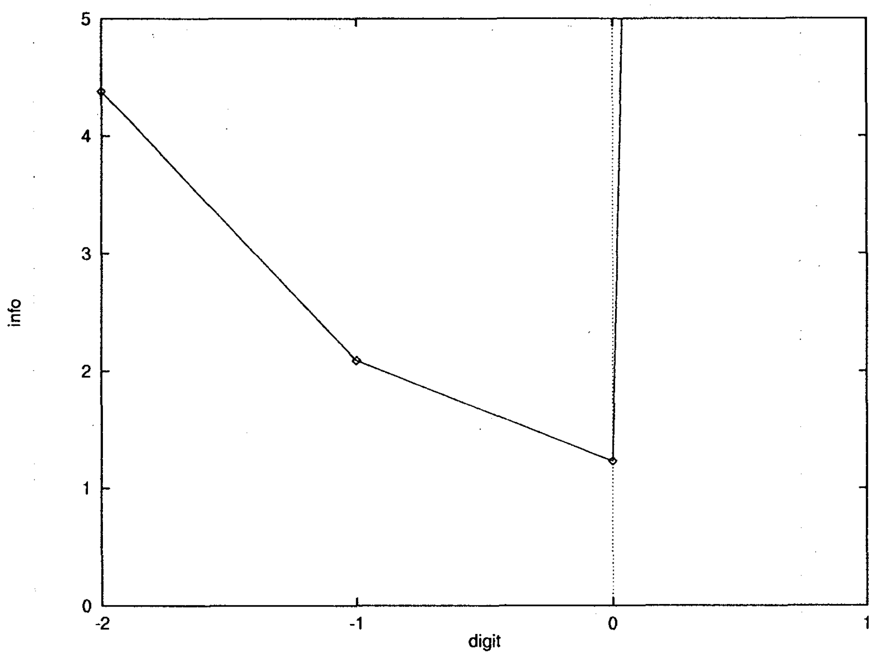

Figure 4: The K-L information against k

of arbitrariness. Assessing the amount of differences resulting from the choice of measures of discrepancy is an open problem. The validity of the approach will be ensured if they produce close results.

Acknowledgments

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