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RELIABILITY OF A CIRCULAR CONNECTED-(r,s)-OUT-OF-(m,n): F LATTICE SYSTEM

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Abstract A circular connected-(r,s)-out-of-(m,n):F lattice system has m· n components and is consisted of n rays and m circles. The system fails if and only if a connected (r,s)-matrix of components fail. Firstly, this paper proposes a recursive algorithm for the reliability of the system, which requires $O(r^n n^2 sm)$ computing time. In the statistically independent identically distributed case, the computing time is able to be reduced. Secondly, the upper and lower bounds for the system reliability are obtained for the circular connected-(r,s)-out-of-(m,n):F lattice system. Finally, it is proved that the reliability of the large system tends to $\exp[-\mu\lambda^{rs}]$ as $n = \mu n^{\eta-1}$, $n \to \infty$ if every component has failure probability $\lambda n^{-\eta/rs}$, where μ, λ and η are constant, $\mu, \lambda > 0$, $\eta > r$.

1. INTRODUCTION

The linear (circular) consecutive-k-out-of-n:F system has n linearly (circularly) ordered components. Each component either functions or fails. The systems fail if and only if k consecutive components fail (see Figure 1.1). Defining the random variable $\mathbf{Z}_i$ by

$$\mathbf{Z}_i = \begin{cases} 1, & \text{if component i functions} \\ 0, & \text{if component i fails,} \end{cases}$$

for i = 1, 2,···, n, the reliability of the linear consecutive-k-out-of-n:F system can be expressed as $\Pr\{\bigcap_{a=1}^{n-k+1}\{\mathbf{Z}_i = 0, a \le i \le a+k-1\}^c\}$ and the circular consecutive-k-out-of-n:F system as

$$\Pr\{\bigcap_{a=1}^{n}\{\mathbf{Z}_i = 0, a \le i \le a+k-1\}^c\},$$ where $\mathbf{Z}_i \equiv \mathbf{Z}_{i-n}$, for i = n+1, n+2,···, n+k-1. Such systems have been studied by many reseachers including *Bollinger and Salvia* [3], *Derman, Lieberman and Ross* [4], *Fu* [5] and *Hwang* [6].

A linear (circular) connected-(r, s)-out-of-(m, n):F lattice system is a 2-dimensional version of the linear (circular) consecutive-k-out-of-n:F system [2]. Especially, the linear connected-(r, s)-out-of-(m, n):F lattice system is an extended system of 2- dimensional consecutive-k-out-of-n: F system in *Salvia and Lasher* [9]. The linear connected-(r, s)-out-of-(m, n):F lattice system has m· n components and is consisted of n columns and m rows. The system fails if and only if a connected-(r, s)-matrix of components fail (see Figure 1.2). Defining the random variable $\mathbf{Z}_{ij}$ by

$$\mathbf{Z}_{ij} = \begin{cases} 1, & \text{if the component functions on the intersection of row i and column j} \\ 0, & \text{if the component fails on the intersection of row i and column j,} \end{cases}$$

for i = 1, 2,···, m, j = 1, 2,···, n, the reliability of the linear connected-(r, s)-out-of-(m, n):F lattice system can be expressed as

$$\Pr\{\bigcap_{a=1}^{m-r+1}\bigcap_{b=1}^{n-s+1}\{\mathbf{Z}_{ij} = 0, a \le i \le a+r-1, b \le j \le b+s-1\}^c\}.$$

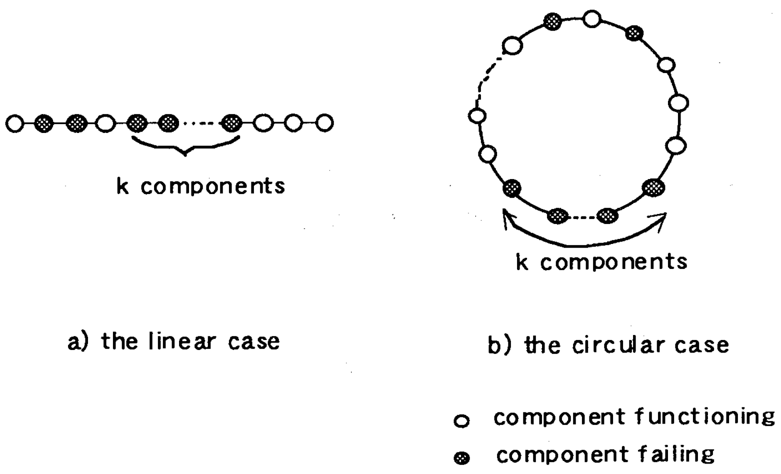


Figure 1.1 Consecutive k-out-of-n:F System failing

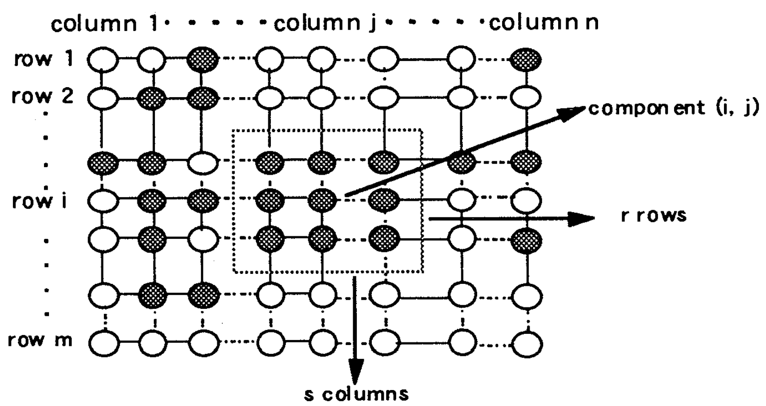


Figure 1.2 Linear Connected-(r, s)-out-of-(m, n):F Lattice System Failing

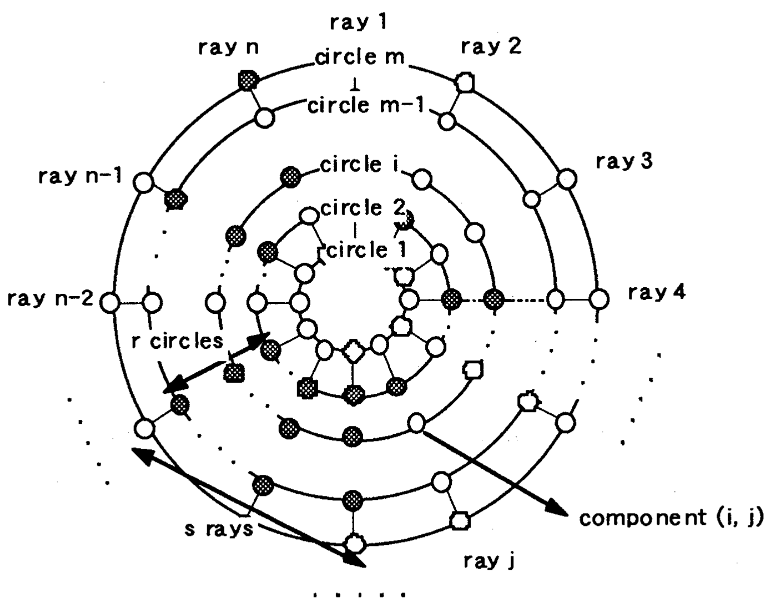


Figure 1.3 Circular Connected-(r, s)-out-of-(m, n):F Lattice System Failing

Yamamoto and Miyakawa [8] proposed the algorithm for obtaining the reliability of the linear connected-(r, s)-out-of-(m, n):F lattice system and considered the limit value of the system reliability.

A circular connected-(r, s)-out-of-(m, n):F lattice system has m· n components and is consisted of n rays and m circles. The intersections of the circles and the rays represent the components, ie, each of the circles includes n components and each of the rays has m components. The system fails if and only if a connected (r, s)-matrix of components fail (see Figure 1.3). Such a system can represent a system for measuring the temperature in a reactor, etc. (see Figure 1.4) [2]. In the case of m = r or n = s, the reliability of this system can be evaluated as the reliability of the linear or circular consecutive-k-out-of-n: F system [2]. Recently, though *Zuo* [11] suggested that the SDP method [1], [7] and [10] was able to be applied for general cases, the procedure of the SDP method is very complex.

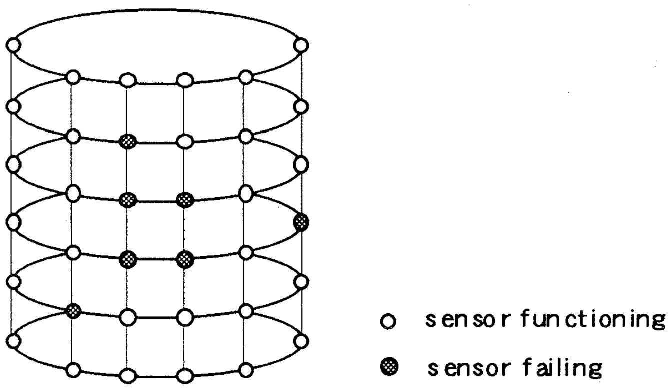


Figure 1.4 The System for Measuring the Temperature in a Reactor

In this paper, we consider the circular connected-(r, s)-out-of-(m, n):F lattice system, without any restrictions about r, s, m and n, where each component either functions or fails and all the components are statistically independent.

This paper is organized as follows. In section 2, system description, their assumptions and notations used throughout this paper are introduced. In section 3, we develop a recursive algorithm to compute the system reliability, with unequal component failure probabilities. The system reliability can be computed in $O(r^n n^2 sm)$ time by our algorithm, that is, polynomial for m but exponential for n. The recursive algorithm is given by generalizing the algorithm for the linear case in *Yamamoto and Miyakawa* [8]. When all the components have an equal failure probability, we can reduce the computing time for the system reliability. In section 4, the upper and lower bounds for the system reliability are obtained for the circular connected-(r, s)-out-of-(m, n):F lattice system. In order to evaluate how these bounds approximate the exact system reliability, some numerical results are investigated. In section 5, it is proved that the reliability of the large system tends to $\exp[-\mu\lambda^{rs}]$ as $m = \mu n^{\eta-1}$, $n \to \infty$ if every component has failure probability $\lambda n^{-\frac{\eta}{rs}}$, where μ, λ and η are constant, $\mu, \lambda > 0$ and $\eta > r$. By the above, we can obtain an approximate value for the reliability of the large circular connected-(r, s)-out-of-(m, n):F lattice system.

2. SYSTEM DESCRIPTION, ASSUMPTIONS AND NOTATIONS

The circular connected-(r, s)-out-of-(m, n):F lattice system is defined in the following. As shown in the Figure 1.3, the circular connected-(r, s)-out-of-(m, n):F lattice system has m· n components and is consisted of n rays and m circles. Each circle is numbered from the center to the outside, and each ray clockwise from any chosen ray. The component on the intersection of circle i and ray j are denoted by component (i, j), for i = 1, 2,···, m and j = 1, 2,···, n. We denotes the random variable $\mathbf{Z}_{ij}$ by

$$\mathbf{Z}_{ij} = \begin{cases} 1, & \text{if component } (i,j) \text{ functions} \\ 0, & \text{if component } (i,j) \text{ fails,} \end{cases}$$

for i = 1, 2,···, m and j = 1, 2,···, n, and $\mathbf{Z}_{ij} \equiv \mathbf{Z}_{i,\,j-n}$, for i = 1, 2,···, m and j = n+1, n+2,⋯, n+s-1. Then, using $\mathbf{Z}_{ij}$, the reliability of the circular connected-(r, s)-out-of-(m, n):F lattice system can be expressed by

$$\Pr\{\bigcap_{a=1}^{m-r+1}\bigcap_{b=1}^{n}\{\mathbf{Z}_{ij}=0,\ a\le i\le a+r-1,\ b\le j\le b+s-1\}^c\}.$$

Assumptions and some notations are defined to be used throughout this paper, as follows.

Assumptions

A. Each component and the system either function or fail.

B. All the components are mutually statistically independent.

Notations

r, s, m, n : parameters of the system considered

p_{ij}, q_{ij} : $\Pr\{\mathbf{Z}_{ij}=1\}$, reliability of component (i, j) ; $q_{ij}\equiv 1-p_{ij}$, for i = 1, 2,⋯, m and j = 1, 2,⋯, n+s-1

p, q : reliability, failure probability of the component in the statistically independent and identically distributed case; $q\equiv 1-p$

$R(s, n; p_{i1}, p_{i2},\cdots, p_{in})$
: $\Pr\{\bigcap_{b=1}^{n}\{\mathbf{Z}_{i\beta}=0,\ b\le\beta\le b+s-1\}^c\}$, that is, reliability of the circular consecutive-s-out-of-n:F system composed of the components forming circle i on the system considered, for i = 1, 2,⋯, m

R(s, n; p) : reliability of the circular consecutive-s-out-of-n:F system with the identical component reliability p

$\mathbf{A}(i)$: the event $\{\bigcap_{a=1}^{i-r+1}\bigcap_{b=1}^{n}\{\mathbf{Z}_{\alpha\beta}=0,\ a\le\alpha\le a+r-1,\ b\le\beta\le b+s-1\}^c\}$ if $i\ge r$, the sample space if i < r, that is, the event that the circular connected-(r, s)-out-of-(i, n):F lattice system functions, which consists of first i circles on the system considered, for i = 1, 2,⋯, m

$R((r, s), (i, n);[p_{ij}])$
: $\Pr\{\mathbf{A}(i)\}$, that is, reliability of the circular connected-(r, s)-out-of-(i, n):F lattice system, which consists of first i circles on the system considered, for i = 1, 2,⋯, m

R((r, s), (i, n); p): reliability of the circular connected-(r, s)-out-of-(i, n):F lattice system, which consists of first i circles on the system considered with the identical component reliability p, for i = 1, 2,⋯, m

We let ϕ be the null event and Ω the sample space, for the convenience's sake.

3. SYSTEM RELIABILITY

3.1 MAIN RESULTS FOR THE SYSTEM RELIABILITY

Our recursive algorithm is given in Theorem 1. Before explaining our main results, We will describe some notations.

Firstly, we consider the states of circle i, for i =1, 2,⋯, m. In order to express the states of circle i, we define n-dimensional binary random variable $\mathbf{Y}_i=(\mathbf{Y}_{i1}, \mathbf{Y}_{i2},\cdots, \mathbf{Y}_{in})$ by

$$\mathbf{Y}_{ij}=\begin{cases}1, & \text{if the event } \{\mathbf{Z}_{i\beta}=0,\ j\le\beta\le j+s-1\} \text{ occurs}\\ 0, & \text{if the event } \{\mathbf{Z}_{i\beta}=0,\ j\le\beta\le j+s-1\}^c \text{ occurs,}\end{cases}$$

for j =1, 2,⋯, n and i =1, 2,⋯, m. For example, as shown in Figure 3.1, in the case of s=2 and n=4, $\mathbf{Y}_i$ =(1, 1, 0, 0) means components (i, 1), (i, 2) and (i, 3) failing and component (i, 4) functioning and $\mathbf{Y}_i$ =(0, 0, 0, 0) means that there does not exist 2 consecutive components failing on circle i.

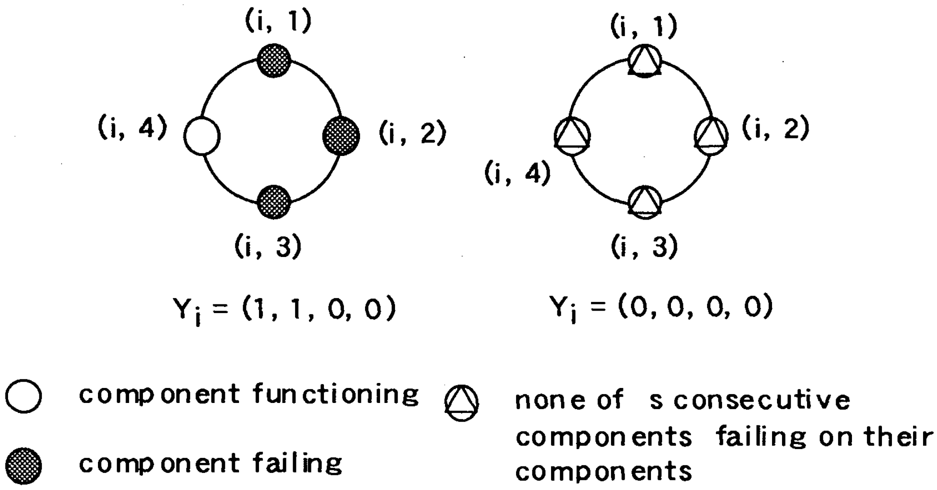


Figure 3.1 Y_i in the case of s = 2 and n = 4

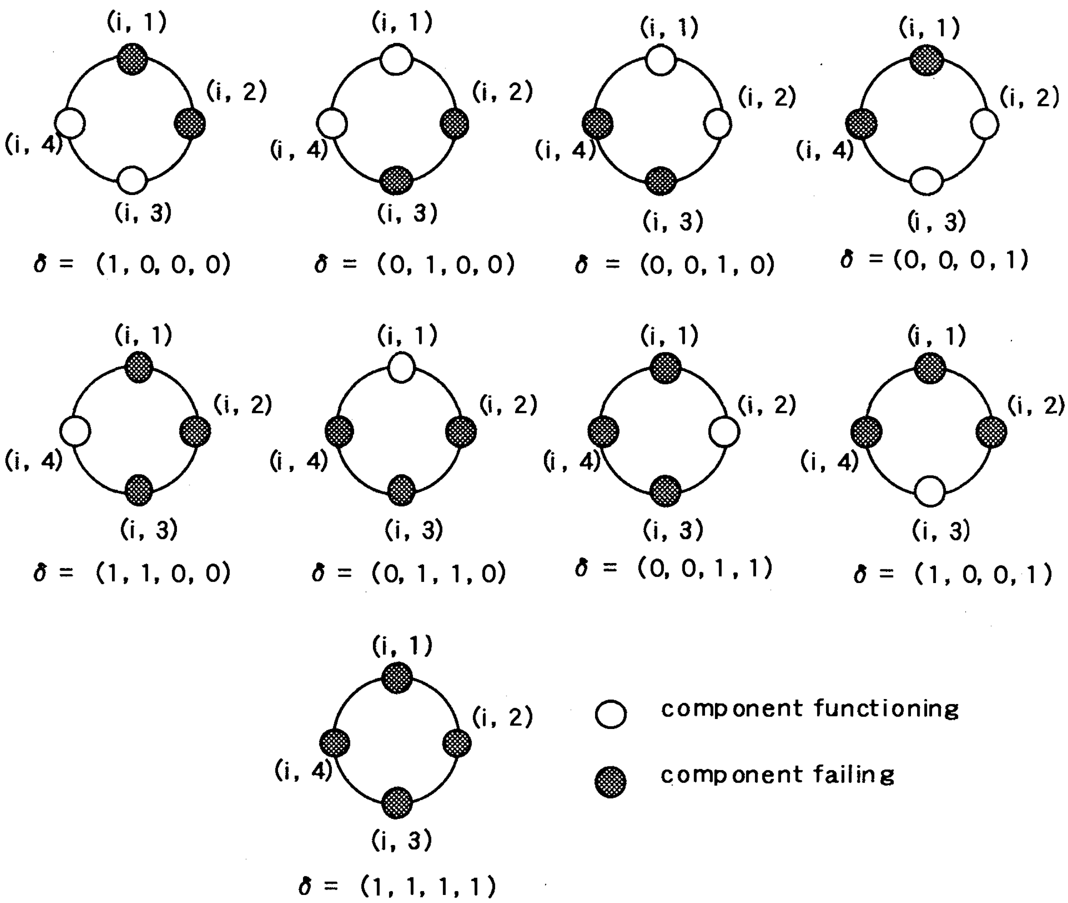


Figure 3.2 Φ in the case of s = 2 and n = 4

Letting n-dimensional binary vector δ be $(\delta_1, \delta_2, \cdots, \delta_n)$, we define the set Φ by

$$\Phi \equiv \{\delta \mid \text{the event } \{\mathbf{Y}_i = \delta\} \neq \phi\} - \{\mathbf{0}\},$$

where $\mathbf{0}$ refers to n-dimensional vector (0, 0,$\cdots$, 0). It follows from the definition of $\mathbf{Y}_i$ that Φ depends only on n and s and not on i, m and r, as each of all the circles has a similar shape. In the case of s = 2 and n = 4, for example, $\mathbf{Y}_i = (\delta_1, \delta_2, \delta_3, \delta_4) = (1, 0, 1, 1)$ is not the element of Φ, as $\delta_1 = 1$ means component (i, 2) failing but $\delta_2 = 0$ and $\delta_3 = 1$ means component (i, 2) functioning. As the result, in the case of s = 2 and n = 4,

$$\Phi = \{(1,0,0,0), (0,1,0,0), (0,0,1,0), (0,0,0,1), (1,1,0,0), (0,1,1,0), (0,0,1,1), (1,0,0,1), (1,1,1,1)\},$$

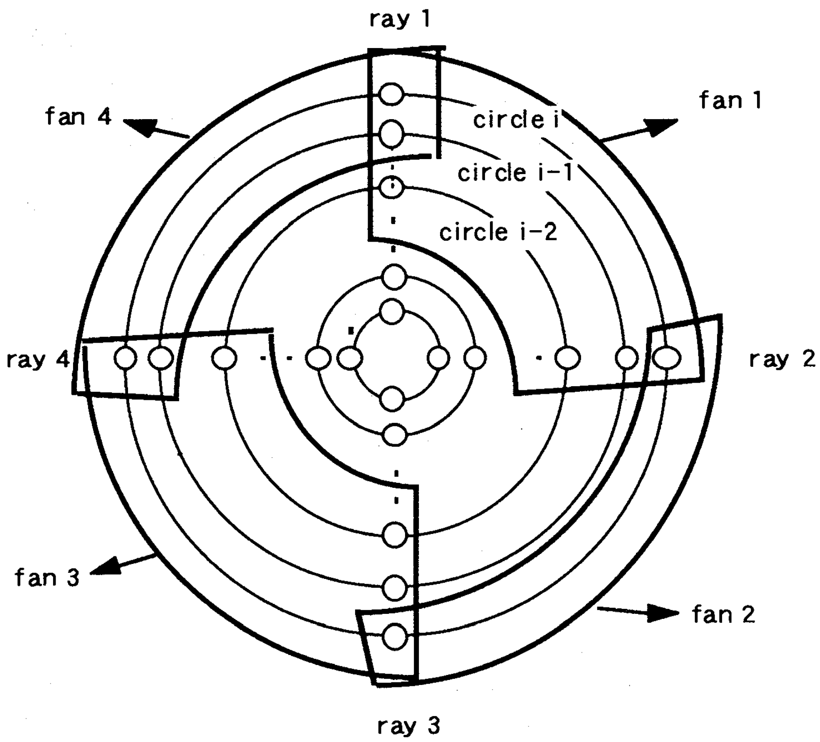
as shown in Figure 3.2.

We let

$$F_i(s, n; \delta) \equiv \Pr\{\mathbf{Y}_i = \delta\} = \Pr\{\bigcap_{j=1}^{n}\{\mathbf{Y}_{ij} = \delta_j\}\}, \tag{3.1}$$

for $\delta \in \Phi$ and i = 1,$\cdots$, m.

Furthermore, we define $R(i; h_1, h_2, \cdots, h_n)$ in the following.



$\Pr\{A(i) \cap \{\bigcap_{j=1}^{4}\{\text{all components in fan } j\}^c\}$

Figure 3.3 $R(i; h_1, h_2, \cdots, h_n)$ in the case of s = 2, n = 4, h_1 = 3, h_2 = 1, h_3 = 3 and h_4 = 2

$$R(i;h_1,h_2,\cdots,h_n) \equiv \begin{cases} \Pr\{A(i)\cap\{\bigcap_{j=1}^{n}\{Y_{\alpha j}=1, i-h_j+1\le\alpha\le i\}^c\}, & \text{if } i\ge 1 \text{ and } h\ge 1 \\ 1, & \text{if } i=0 \text{ and } h\ge 1 \\ 0, & \text{if } h=0, \end{cases}$$

where $h=\min(h_1, h_2,\cdots, h_n)$, for $h_\beta = 0, 1, 2,\cdots, r$, $\beta = 1, 2,\cdots, n$ and $i = 0, 1,\cdots, m$. For example, when $s = 2$ and $n = 4$, $R(i;3,1,3,2)$ is shown in Figure 3.3.

By using the above notations, our recursive algorithm is given in Theorem 1.

Theorem 1.

a) Let ε be a n-dimensional vector $(\varepsilon_1, \varepsilon_2,\cdots, \varepsilon_n)$. Then, the following recursive formula is obtained.

$$R(i;h_1,h_2,\cdots,h_n) \equiv \begin{cases} R(s,n;p_{i1},p_{i2},\cdots,p_{in})\bullet R(i-1;r,r,\cdots,r)+\sum_{\delta\in\Phi}R(i-1;\varepsilon)\bullet F_i(s,n;\delta), & (i\ge h\ge 1) \\ 1, & (h>i\ge 0) \\ 0, & (h=0), \end{cases} \quad (3.2)$$

where $\varepsilon_j = h_j - 1,$ if $\delta_j = 1$

$\varepsilon_j = r,$ if $\delta_j = 0,$

for $j = 1, 2,\cdots, n$ and $h_\beta = 0, 1,\cdots, r$, $\beta = 1, 2,\cdots, n$ and $i = 0, 1,\cdots, m$.

b) The reliability of the circular connected-(r, s)-out-of-(i, n):F lattice system is obtained by

$$R((r,s),(i,n);[p_{ij}]) = R(i;r,r,\cdots,r), \quad (3.3)$$

for $i = 1, 2,\cdots, m$.

In order to prove Theorem 1, we define the following events.

$$V_j(i,d) \equiv \begin{cases} \left[\bigcap_{a=r}^{i-1}\{Y_{\alpha j}=1, a-r+1\le\alpha\le a\}^c\right]\cap\{Y_{\alpha j}=1, a-d+1\le\alpha\le a\}^c & \text{if } i\ge d \\ \Omega, & \text{if } i<d, \end{cases}$$

for $d = 1, 2,\cdots, r$, $j = 1, 2,\cdots, n$ and $i = 1, 2,\cdots, m$.

Some important relations among the events $\{Y_{ij}=e\}$ for $e = 0$ and 1 and the events $V_j(i,d)$ are given in Lemma 3.1.

Lemma 3.1.

$$V_j(i,d)\cap\{Y_{ij}=e\} = \begin{cases} V_j(i-1,d-1)\cap\{Y_{ij}=e\}, & \text{if } e=1, d\ge 2 \text{ and } i\ge 2 \\ \{Y_{ij}=e\}, & \text{if } (e=1, d\ge 2 \text{ and } i=1) \text{ or } \\ & (e=0 \text{ and } i=1) \text{ or } \\ & (e=0, i<d \text{ and } i\ge 2), \\ \phi, & \text{if } e=1 \text{ and } d=1, \\ V_j(i-1,r)\cap\{Y_{ij}=e\}, & \text{if } e=0, i\ge d \text{ and } i\ge 2, \end{cases}$$

for $d = 1, 2,\cdots, r$, $e = 0, 1$, $i = 1, 2,\cdots, m$ and $j = 1, 2,\cdots, n$.

Proof of Lemma 3.1: By considering the intersection of the events, the proof of Lemma 3.1 is given. ***Q.E.D.***

Proof of Theorem 1:

The equation (3.3) holds obviously from the definition of $\mathbf{V}_j(i,d), \mathbf{A}(i)$ and $R(i; h_1, h_2, \cdots, h_n)$.

The proof of the equation (3.2) are given in the following.

(A) The case of $i \geq h \geq 1$

In this case, as

$$\mathbf{A}(i) \bigcap \left\{ \bigcap_{j=1}^{n} \left\{ \mathbf{Y}_{\alpha j} = 1, i - h_j + 1 \leq \alpha \leq i \right\}^c \right\} = \bigcap_{j=1}^{n} \mathbf{V}_j(i, h_j),$$

it is obvious that

$$R(i; h_1, h_2, \cdots, h_n) = \sum_{\delta \in \Phi} \Pr\left\{ \bigcap_{j=1}^{n} \mathbf{V}_j(i,h_j) \mid \mathbf{Y}_i = \delta \right\} \cdot \Pr\{\mathbf{Y}_i = \delta\} + \Pr\left\{ \bigcap_{j=1}^{n} \mathbf{V}_j(i,h_j) \mid \mathbf{Y}_i = \mathbf{0} \right\} \cdot \Pr\{\mathbf{Y}_i = \mathbf{0}\}. \tag{3.4}$$

By Lemma 3.1 and Assumption A, for $\delta \in \Phi \cup \{\mathbf{0}\}$,

$$\Pr\left\{ \bigcap_{j=1}^{n} \mathbf{V}_j(i,h_j) \mid \mathbf{Y}_i = \delta \right\} = \frac{\Pr\left\{ \bigcap_{j=1}^{n} \left[\mathbf{V}_j(i,h_j) \bigcap \{\mathbf{Y}_{ij} = \delta_j\} \right] \right\}}{\Pr\{\mathbf{Y}_i = \delta\}}$$

$$= R(i-1; \varepsilon_1, \varepsilon_2, \cdots, \varepsilon_n), \qquad \text{(by Lemma 3.1)} \tag{3.5}$$

where $\varepsilon_j = h_j - 1$ if $\delta_j = 1$,
$\varepsilon_j = r$ if $\delta_j = 0$

for $j = 1, 2, \cdots, n$ and $h_\beta = 1, 2, \cdots, r$, $\beta = 1, 2, \cdots, n$ and $i = 1, 2, \cdots, m$. Therefore, the proof of Theorem 1 in this case becomes complete, by substituting the equation

$$R(s, n; p_{i1}, p_{i2}, \cdots, p_{in}) = \Pr\{\mathbf{Y}_i = \mathbf{0}\},$$

the equations (3.1) and (3.5) for the equation (3.4).

(B) The case of $h > i \geq 0$

When $h > i = 0$, that is $i = 0$ and $h \geq 1$, from the definition of $R(i; h_1, h_2, \cdots, h_n)$, $R(i; h_1, h_2, \cdots, h_n) = 1$ and when $h > i \geq 1$, as $\bigcap_{j=1}^{n} \mathbf{V}_j(i,h_j) = \Omega$, $R(i; h_1, h_2, \cdots, h_n) = 1$.

Therefore, the equation (3.2) holds in this case.

(C) The case of $h = 0$

when $h = 0$, from the definition of $R(i; h_1, h_2, \cdots, h_n)$, $R(i; h_1, h_2, \cdots, h_n) = 0$.

Q.E.D.

From Theorem 1, if Φ, $R(s, n; p_{i1}, p_{i2}, \cdots, p_{in})$ and $F_i(s, n; \delta)$, for $i = 1, 2, \cdots, m$ and $\delta \in \Phi$, are given, $R(i; h_1, h_2, \cdots, h_n)$'s can be obtained by the equation (3.2). The reliability of the circular connected-(r, s)-out-of-(m, n):F lattice system is given as $R(m; r, r, \cdots, r)$ from the equation (3.3).

As $R(s, n; p_{i1}, p_{i2}, \cdots, p_{in})$ means the reliability of the circular consecutive-s-out-of-n:F system, there are many well known methods for it, for example, *Hwang* [6].

The recursive algorithm for Φ and the closed formula for $F_i(s, n; \delta)$ are given in the Appendix.

Next, the results of Theorem 2 are important, as the computing time for the system reliability can be reduced when all the components on circle i have an equal reliability p_i for i = 1, 2,···, m.

Theorem 2.

If $p_{ij} = p_i$, for j = 1, 2,···, n and i = 1, 2,···, m,

$$R(i; h_1, h_2, \cdots, h_n) = R(i; h_j, h_{j+1}, \cdots, h_n, h_1, \cdots, h_{j-1}), \quad (3.6)$$

for j = 1, 2,···, n, i = 1, 2,···, m, $h_\beta = 0,1,2,\cdots,r$ and $\beta = 1, 2,\cdots, n$.

Proof: Theorem 2 is proven by mathematical induction for i.

Firstly, from the assumptions of theorem 2,

$$F_i(s,n;\delta_1,\delta_2,\cdots,\delta_n) = F_i(s,n;\delta_j,\cdots,\delta_n,\delta_1,\cdots,\delta_{j-1}), \quad (3.7)$$

$$R(s, n; p_i) = R(s, n; p_{ij}, p_{i,j+1}, \cdots, p_{i,j-1}),$$

for j = 1, 2, ···, n and i = 1, 2, ···, m.

(A) the case of i = 0 or i = 1

When i = 0 or (i = 1 and h $\neq$1), it is obvious that the equation (3.6) holds, from the definition of $R(i; h_1, h_2, \cdots, h_n)$.

When i = 1 and h = 1, the equation (3.6) is proven in the following. As $r \geq 1$,

$$R(0; r, r, \cdots, r) = 1,$$

from the equation (3.2). Therefore,

$$R(1; h_1, h_2, \cdots, h_n)$$
$$= R(s,n;p_{11},p_{12},\cdots,p_{1n}) + \sum_{\delta\in\Phi} R(0;\varepsilon)\cdot F_1(s,n;\delta) \quad (3.8)$$

holds, where $\varepsilon_j = h_j - 1$, if $\delta_j = 1$ and

$\varepsilon_j = r$, if $\delta_j = 0$, for j = 1, 2,···, n,

from the equation (3.2). $R(0;\varepsilon)$ is 1 or 0 for $\delta \in \Phi$. Let Θ be a set

$$\{\delta \mid R(0;\varepsilon) = 1 \text{ and } \delta \in \Phi\}.$$

Therefore,

$$R(1; h_1, h_2, \cdots, h_n)$$
$$= R(s,n;p_{1,j},p_{1,j+1},\cdots,p_{1,j-1}) + \sum_{\delta\in\Theta} F_1(s,n;\delta_j,\cdots,\delta_n,\delta_1,\cdots,\delta_{j-1})$$
$$= R(1;h_j,\cdots,h_n,h_1,\cdots,h_{j-1}),$$

for j = 1, 2,···, n, from the equations(3.7) and (3.8).

(B) the case of $i \geq 2$

Suppose that the equation (3.6) holds for i = 0, 1, 2,···, t, where $1 \leq t \leq m-1$. By using the equation (3.2), the equation (3.6) holds for i = t+1. ***Q.E.D.***

3.2 RECURSIVE ALGORITHM AND EXAMPLE

It follows from the above that our algorithm has the following steps.

STEP 1. Obtain Φ by using equation (A2.1).

STEP 2. Calculate $F_i(s, n; \delta)$ for any $\delta \in \Phi$ and i = 1, ···, m by using equations (A3.1) and (A3.2). The reliability of the linear consecutive-k-out-of-n:F system in their equations can be obtained, for example, by *Hwang* [6].

STEP 3. Calculate R(m; r, r, ···, r) by using the equation (3.2) recursively , where, also, $R(s, n; p_{i1}, p_{i2}, \cdots, p_{in})$'s can be given by *Hwang* [6].

R(m; r, r, ···, r) is the reliability of the circular connected-(r, s)-out-of-(m, n):F lattice system from the equation (3.3). If all the components on circle i have an equal reliability , for i = 1, 2,···, m, from the results of Theorem 2, we can reduce the number of the $R(i; h_1, h_2, \cdots,$

h_n)'s, which must be evaluated. Especially, in the statistically independent and identically distributed case, in addition to the above, the computing time for $F_i(s, n; \delta)$' s can be reduced, because $F_i(s, n; \delta)$' s are same for any fixed $\delta \in \Phi$.

In order to demonstrate our algorithm, an example will be shown as follows.

Example. The case of the circular connected-(2,2)-out-of-(m, 4):F lattice system

As an example, we will calculate the reliability of the circular connected-(2, 2)-out-of-(m, 4):F lattice system in statistically independent and identically distributed case, i.e., $p_{11} = p_{12} = \cdots = p_{m3} = p_{m4} = p$.

STEP 1. Obtain Φ (see Figure 3.3).

$$\Phi = \{ (1,0,0,0), (0,1,0,0), (0,0,1,0), (0,0,0,1), (1,1,0,0), (0,1,1,0), (0,0,1,1), (1,0,0,1), (1,1,1,1)\}$$

STEP 2. Calculate $F_i(s, n; \delta)$ for any $\delta \in \Phi$ and $i = 1, \cdots, m$

$F_i(2,4;1,0,0,0) = F_i(2,4;0,1,0,0) = F_i(2,4;0,0,1,0) = F_i(2,4;0,0,0,1) = p^2q^2$,
$F_i(2,4;1,1,0,0) = F_i(2,4;0,1,1,0) = F_i(2,4;0,0,1,1) = F_i(2,4;1,0,0,1) = pq^3$
and $F_i(2,4;1,1,1,1) = q^4$,
for $i = 1, 2, \cdots, m$, are obtained respectively.

STEP 3. Calculate $R(m; r, r, \cdots, r)$.

As $R(2,4;p)=p^2+2p^2q-p^2q^2$, for $i = 1, 2, \cdots, m$, the following recursive formulas are given, based on the equation (3.2).

$$R(i; h_1, h_2, h_3, h_4) = \begin{cases} (p^2+2p^2q-p^2q^2)\bullet R(i-1;2,2,2,2) + \sum_{\delta\in\Phi} R(i-1;\varepsilon_1,\varepsilon_2,\varepsilon_3,\varepsilon_4)\bullet F_i(2,4;\delta), & (m \ge i \ge h \ge 1) \\ 1, & (2 \ge h > i \ge 0) \\ 0, & (h = 0) \end{cases}$$

where $\varepsilon_j = h_j - 1$ if $\delta_j = 1$
$\varepsilon_j = r$, if $\delta_j = 0$, for $j = 1, 2, 3, 4$.

By using the results of Theorem 2, we get

$R(i;1,0,0,0) = R(i;0,1,0,0) = R(i;0,0,1,0) = R(i;0,0,0,1)$
and $R(i;1,1,0,0) = R(i;0,1,1,0) = R(i;0,0,1,1) = R(i;1,0,0,1)$,
for $i = 0, 1, 2, \cdots, m$. Therefore, the reliability of the circular connected-(2, 2)-out-of-(m, 4):F lattice system can be given as $R(m; 2, 2, 2, 2)$ by the equation (3.3).

Table 1 includes the reliabilities of the circular connected-(2, 2)-out-of-(m, 4):F lattice system in the statistically independent and identically distributed case.

3.3 EVALUATION

In this section, we evaluate our algorithm as compared with the other methods.

TABLE 1 Reliabilities of the circular connected -(2, 2)-out-of-(m, 4):F lattice system

p	m=4	m=5	m=10	m=20	m=30	m=50
0.60	0.7834	0.7249	0.4916	0.2262	0.1040	0.0220
0.65	0.8611	0.8209	0.6459	0.3999	0.2476	0.0949
0.70	0.9193	0.8946	0.7807	0.5946	0.4529	0.2627
0.75	0.9586	0.9455	0.8826	0.7690	0.6700	0.5087
0.80	0.9822	0.9765	0.9481	0.8939	0.8428	0.7491
0.85	0.9942	0.9923	0.9827	0.9639	0.9455	0.9097
0.90	0.9988	0.9984	0.9965	0.9926	0.9887	0.9810
0.95	0.9999	0.9999	0.9998	0.9995	0.9993	0.9988

Firstly, we will obtain the order of the computing time for our algorithm. The order is given in the following. The cardinality of Φ does not exceed 2^n . As the reliability of the linear consecutive-s-out-of-n:F system is computed in $O(ns)$ time [6], the computing time for each $F_i(s,n;\delta)$ is in $O(n^2s)$. The reliability of the circular consecutive-s-out-of-n:F system is computed in $O(n^2s)$ time [6]. The number of $R(i; h_1, h_2, \cdots, h_n)$'s is $(r+1)^n(m+1)$. Therefore, the system reliability can be computed in $O(r^n n^2 sm)$ time by our algorithm.

We have compared our algorithm with the following two methods. One is the enumeration method, in which 2^{mn} states must be checked. Another is the ALW method that is a kind of the sum of disjoint products (SDP) methods. This method is available for the reliability of the general network system and was proposed by *Abraham* [1], *Locks* [7] and *Wilson* [10]. If a minimal-path set or a minimal-cut set is given, the ALW method gives the efficient algorithm for the system reliability by expressing it as the sum of probabilities of the exclusive events. It was *Zuo*[11] who proposed the ALW method was able to be used to compute the reliability of the linear (circular) connected-(r, s)-out-of-(m, n):F lattice system. We investigated the computing time for the circular connected-(r, s)-out-of-(m, n):F lattice system by the C language on a personal computer, NEC PC9801RA, which is a i80386 machine. For the circular connected-(2, 2)-out-of-(4, 4):F lattice system, our algorithm took less than 1 second as compared with about 12 seconds for the ALW method and about 41 seconds for the enumeration method. For the circular connected-(2, 2)-out-of-(5, 4):F lattice system, the enumeration method took about 13 minutes and the ALW method could not obtain system reliability for lack of main memories. Our algorithm took less than 1 second. For the circular connected-(2, 2)-out-of-(50, 4):F lattice system, it is impossible for the enumeration method and the ALW method to obtain the system reliability on our machine. On the other hand, our algorithm took about 2 seconds.

4. UPPER & LOWER BOUNDS

As described in section 3, the order of our algorithm is polynomial for m but exponential for n. As n becomes larger, our algorithm needs much computing time. Therefore, in this section, we will propose the upper and lower bounds, which are easily obtained, for the reliability of the circular connected-(r, s)-out-of-(m, n):F lattice system. Before upper and lower bounds are given in Theorem 3, we define some notations.

w_{ij} ; $1-\prod_{k=i}^{i+r-1} q_{kj}$, for $i = 1, 2, \cdots, m-r+1$ and $j = 1, 2, \cdots, n$

$R((r, s), (k, n); (i, n); [p_{ij}])$

; $\Pr\left\{\bigcap_{a=i}^{i+k-r}\bigcap_{b=1}^{n}\left\{Z_{\alpha\beta}=0, a\le\alpha\le a+r-1, b\le\beta\le b+s-1\right\}^c\right\}$, that is, the reliability of the circular connected-(r, s)-out-of-(k, n):F lattice system, which consists of k circles from circle i to circle i+k - 1 on the system considered, for $k = r, r+1, \cdots, m-i+1$ and $i = 1, 2, \cdots, m-r+1$

Theorem 3.

$$\min\left\{UB1(k_1), UB2\right\} \ge R((r,s),(m,n);[p_{ij}]) \ge LB(k_2), \tag{4.1}$$

for $k_1 = r, r+1, \cdots, m$ and $k_2 = r, r+1, \cdots, m$,

where

$$UB1(k) = \prod_{i=1}^{[m/k]} R((r,s),(k,n);((k-1)i+1,n);[p_{ij}]) \bullet$$
$$R((r,s),(m-[m/k],n);([m/k]k+1,n);[p_{ij}]),$$

for $k = r, r+1, \cdots, m$,

$$\mathrm{UB2} = R(s,n;w_{11},w_{12},\cdots,w_{1n})\bullet\prod_{i=r}^{m-1}\Big[1-\Big\{1-R\big(s,n;w_{(i-r+2),1},w_{(i-r+2),2},\cdots,w_{(i-r+2),n}\big)\Big\}\bullet R\big(s,n;p_{(i-r+1),1},p_{(i-r+1),2},\cdots,p_{(i-r+1),n}\big)\Big]$$

$$\mathrm{LB}(k) = \prod_{i=1}^{m-k+1} R((r,s),(k,n);(i,n);[p_{ij}]),$$

for $k = r, r+1, \cdots, m$.

For the proof of Theorem 3, we define the following events.

B(i) $:\left\{\bigcap_{b=1}^{n}\{\mathbf{Z}_{i\beta}=0, b\le\beta\le b+s-1\}^{c}\right\}$, that is, event that the circular consecutive-s-out-of-n:F system functions, which is composed of circle i on the system considered, for i = 1, 2,$\cdots$, m

C(i, k):$\left\{\bigcap_{a=i}^{i+k-r}\bigcap_{b=1}^{n}\{\mathbf{Z}_{\alpha\beta}=0, a\le\alpha\le a+r-1, b\le\beta\le b+s-1\}^{c}\right\}$, that is, event that the circular connected-(r, s)-out-of-(k, n):F lattice system functions, which is composed of k circles from circle i to circle i+k - 1 on the system considered, for k = r, r+1,$\cdots$, m - i+1 and i =1, 2, $\cdots$, m - r+1

By using the above notations, it follows from the discussions in *Boehme, Kossow and Preuss* [2] that

$\Pr\{\mathbf{A}(r)\} = R(s, n; w_{11}, w_{12}, \cdots, w_{1n})$,

$\Pr\{\mathbf{C}(i-r+2, r)\} = R(s, n; w_{(i-r+2),1}, w_{(i-r+2),2}, \cdots, w_{(i-r+2),n})$

and

$\Pr\{\mathbf{B}(i-r+1)\} = R(s, n; p_{(i-r+1),1}, p_{(i-r+1),2}, \cdots, p_{(i-r+1),n})$,

for i = r+1, r+ 2, $\cdots$, m-1.

The relations among the events, **A**(i), **B**(i) and **C**(i, r) are given in Lemma 4.1.

Lemma 4.1.

$\Pr\{\mathbf{A}(i+1)^{c} \mid \mathbf{A}(i)\} \ge \Pr\{\mathbf{C}(i-r+2, r)^{c}\}\Pr\{\mathbf{B}(i-r+1)\}$,

for i = r, r+1,$\cdots$, m - 1.

Proof of Lemma 4.1: The proof of Lemma 4.1 is given in Appendix.

Proof of Theorem 3:

(A) UB2

From the same token for Fu's equation [5], it is shown that

$$R((r,s),(m,n);[p_{ij}]) = \Pr\{\mathbf{A}(r)\}\prod_{i=r}^{m-1}\left[1-\Pr\{\mathbf{A}(i+1)^{c}|\mathbf{A}(i)\}\right]$$

By Lemma 4.1,

$$R((r,s),(m,n);[p_{ij}]) \le \Pr\{\mathbf{A}(r)\}\prod_{i=r}^{m-1}\left[1-\Pr\{\mathbf{C}(i-r+2,r)^{c}\}\Pr\{\mathbf{B}(i-r+1)\}\right]$$

holds.

(B) UB1(k)

$R((r,s),(m,n);[p_{ij}])$

$$\le \Pr\left\{\bigcap_{i=1}^{[m/k]}\mathbf{C}((i-1)k+1, k)\bigcap\mathbf{C}([m/k]k+1, m-[m/k]k)\right\}$$

$$= \prod_{i=1}^{[m/k]} R((r, s), (k, n); ((i-1)k+1, n); [p_{ij}])\bullet$$

$$R\big((r,\ s),\ (m\ -\ [m/k]\,k,\ n);\ ([m/k]\,k+1,\ n)\ ;[p_{ij}]\big),$$

for $k = r, r+1, \cdots, m$.

(C) **LB(k)** Note that

$$\Pr\left\{C\,(i,\ k)\,\middle|\,\bigcap_{a=i+1}^{m-k+1} C\,(a,\ k)\right\} \geq \Pr\{C(i,\ k)\},$$

for $i = 1, 2, \cdots, m - k+1$ and $k = r, r+1, \cdots, m$. Therefore, it is shown that

$$R((r,s),(m,n);[p_{ij}]) = \Pr\left\{\bigcap_{i=1}^{m-k+1} C(i,\ k)\right\} \geq \prod_{i=1}^{m-k+1}\Pr\{C(\,i,\ k)\} = \prod_{i=1}^{m-k+1} R\big((r,\ s),\ (k,\ n);\ (i,\ n)\ ;\ [p_{ij}]\ \big),$$

for $k = r, r+1, \cdots, m$. ***Q.E.D.***

If taking the small number as k, R((r, s), (k, n); (i, n) ; $[p_{ij}]$), for $i = 1, 2, \cdots, m - r+1$, can be given without much computing time by Theorem 1. The reliability of the circular consecutive-s-out-of-n:F system can be obtained in the polynomial computing time for n [6]. Therefore, $UB1(k_1)$, UB2 and $LB(k_2)$ are obtained easily. If taking $k_1 = r$ and $k_2 = r$, $UB1(k_1)$ and $LB(k_2)$ can be expressed by the reliabilities of some circular consecutive-s-out-of-n:F systems. Especially, in the statistically independent and identically distributed case, we can obtain Corollary 1.

Corollary 1.

In the statistically independent and identically distributed case,

$$\min\{UB1,\ UB2\} \geq R(\ (r,\ s),\ (m,\ n);\ 1\ -\ q) \geq LB,$$

where

$$UB1 = R\,(s,\ n;\ 1\ -\ q^r)^{[m/r]}$$

$$UB2 = R\,(s,\ n;\ 1\ -\ q^r)\,[\,1\ -[1\ -\ R\,(s,\ n;\ 1\ -\ q^r)]\,R\,(s,\ n;\ 1\ -\ q)\,]^{m-r+1}$$

$$LB = R\,(s,\ n;\ 1\ -\ q^r)^{m-r+1}$$

Proof: From *Boehme, Kossow and Preuss* [2] and Theorem 3, it is obvious that Corollary 1 holds. ***Q.E.D.***

Numerical results are shown in Table 2, where EXACT denotes the exact reliability obtained by our recursive algorithm in section 3. LB will give good approximations in all cases. It is obtained that the values of UB1 are not uniformly smaller than the values of UB2. UB2 tends to take the smaller value than UB1 when the component reliability is high.

TABLE 2 The upper and lower bounds for reliability of the circular connected-(2, 2)-out-of-(m, n):F lattice system

Table 2.1 m = 10, n = 10

p	LB	EXACT	UB1	UB2
0.60	0.1289	0.1682	0.3204	0.5142
0.65	0.2934	0.3343	0.5060	0.6013
0.70	0.5087	0.5380	0.6869	0.7047
0.75	0.7170	0.7316	0.8313	0.8117
0.80	0.8704	0.8753	0.9258	0.9028
0.85	0.9564	0.9574	0.9755	0.9633
0.90	0.9911	0.9912	0.9951	0.9918
0.95	0.9943	0.9994	0.9997	0.9995

Table 2.2 m = 50, n = 10

p	LB	EXACT	UB1	UB2
0.60	0.0000	0.0001	0.0034	0.0037
0.65	0.0013	0.0027	0.0332	0.0934
0.70	0.0252	0.0352	0.1529	0.1783
0.75	0.1635	0.1843	0.3968	0.3440
0.80	0.4698	0.4856	0.6801	0.5850
0.85	0.7844	0.7892	0.8835	0.8192
0.90	0.9526	0.9531	0.9755	0.9566
0.95	0.9969	0.9970	0.9984	0.9970

5. LIMIT THEOREM

In this section, we study the reliability of the large circular connected-(r, s)-out-of-(m, n):F lattice system in the statistically independent and identically distributed case.

In this paper the limit theorem implies the following. For example, considering a series system consisted of n components, the system reliability R_n can be expressed as $(1 - q_n)^n$, where letting q_n be the failure probability of component as a function of n. Then when $q_n = \lambda n^{-\eta}$, R_n approaches to 1 if $\eta > 1$, $\exp[-\lambda]$ if $\eta = 1$ and 0 if $\eta < 1$ as n becomes larger. By the result of the limit theorem, we can obtain an approximate value of the system reliability when n is large. The limit theorem for the linear consecutive-k-out-of-n:F system was given by *Fu* [5].

The limit theorem for the linear connected-(r, s)-out-of-(m, n):F lattice system was given by *Yamamoto and Miyakawa* [8]. But, the limit theorems for the circular case have not been given. The limit theorems are given in Lemma 5.1 for the circular consecutive-k-out-of-n:F system and in Theorem 4 for the circular connected-(r, s)-out-of-(m, n):F lattice system.

Lemma 5.1.

Let $q_n = \lambda n^{-1/k}$, where λ is a positive constant. Then,

$$\lim_{n\to\infty} R(k, n; 1 - q_n) = \lim_{n\to\infty} (1 - q_n{}^k)^n = \exp[-\lambda^k].$$

Proof : The proof of Lemma 5.1 is given in Appendix.

Theorem 4.

Let μ, λ and η be constant respectively.

If $\quad m = \mu n^{\eta-1}$ (5.1)

and $\quad q_n = \lambda n^{-\eta/rs}$, (5.2)

then $\quad \lim_{n\to\infty} R((r, s), (m, n); 1 - q_n) = \exp[-\mu\lambda^{rs}]$ (5.3)

where $\mu, \lambda > 0$, $\eta > r$.

Proof : It follows from the result of Lemma 5.1, $\eta > r$ and the equation (5.2), that

$$\lim_{n\to\infty} R(s, n; 1 - q_n) = \lim_{n\to\infty} R(s, n; 1 - q_n{}^r) = 1.$$

As $R(s, n; 1 - q_n{}^r)$ approaches to1 faster than $R(s, n; 1 - q_n)$, it is shown that

$$\lim_{n\to\infty} UB2 = \lim_{n\to\infty} R(s, n; 1 - q_n{}^r)^{\mu n^{\eta-1}}. \quad \text{(by the equation (5.1))} \qquad (5.4)$$

Also, it follows that

$$\lim_{n\to\infty} LB = \lim_{n\to\infty} R(s, n; 1 - q_n{}^r)^{\mu n^{\eta-1}}. \quad \text{(by the equation (5.1))} \qquad (5.5)$$

It is shown, by Lemma 5.1 and the equation (5.2), that

$$\lim_{n\to\infty} R(s, n; 1 - q_n{}^r)^{\mu n^{\eta-1}} = \lim_{n\to\infty} (1 - q_n{}^{sr})^{\mu n^{\eta}} = \exp[-\mu\lambda^{rs}] \qquad (5.6)$$

Therefore, from Corollary 1, the equations (5.4), (5.5) and (5.6), in the case of $\eta > r$,

$$\lim_{n\to\infty} R((r, s), (m, n); 1 - q_n) = \exp[-\mu\lambda^{rs}]$$

holds. ***Q.E.D.***

By the result of theorem 4, we can obtain $\exp[-q^{rs}mn]$ as an approximate value of the reliability of the large circular connected-(r, s)-out-of-(m, n):F lattice system, because the assumptions of theorem 4 imply that $\mu\lambda^{rs} = q^{rs}mn$.

APPENDIX

A1 Proof of Lemmas

A1.1 Proof of Lemma 4.1

It can be shown that

$$\Pr\{\mathbf{A}(i+1)^c|\mathbf{A}(i)\}$$
$$\geq \Pr\{\mathbf{A}(i+1)^c \cap \mathbf{B}(i-r+1)\,|\mathbf{A}(i)\}$$
$$= \Pr\{\mathbf{A}(i+1)^c\,|\mathbf{A}(i) \cap \mathbf{B}(i-r+1)\}\Pr\{\mathbf{B}(i-r+1)\,|\mathbf{A}(i)\},$$

for i = r, r+1, , m - 1. Furthermore, we can get

$$\Pr\{\mathbf{A}(i+1)^c|\mathbf{A}(i) \cap \mathbf{B}(i-r+1)\} = \Pr\{\mathbf{C}(i-r+2, r)^c\},$$

for i = r, r+1, , m - 1, and

$$\Pr\{\mathbf{B}(i-r+1)\,|\,\mathbf{A}(i)\} \geq \Pr\{\mathbf{B}(i-r+1)\},$$

for i = r, r+1, , m - 1. Therefore,Lemma 4.1 has been proven. *Q.E.D.*

A1.2 Proof of Lemma 5.1

Let RL(k, n; 1 - q) be the reliability of the linear consecutive-k-out-of-n:F system with an equal component failure probability q. Then, from *Hwang* [6],

$$R(k, n; 1-q) = \sum_{0 \leq s+t<k} (1-q)^2 q^{s+t} RL(k, n-2-(s+t); 1-q).$$

Therefore, as $\lim_{n\to\infty} q_n = 0$,

$$\lim_{n\to\infty} R(k,n;1-q_n) = \lim_{n\to\infty}(1-q_n)^2 RL(k,n-2;1-q_n) = \exp[-\lambda^k],$$

by using the result of Fu's Theorem[5]. *Q.E.D.*

A2. How to obtain Φ

It is easy to see that, for $\delta \in \Phi$, δ satisfies the following condition.

If δ_j and $\delta_{j+a+1}=1$, then $\delta_{j+1}=\cdots=\delta_{j+a}=0$, for j= 1, 2,···, n and a=1, 2,···, s - 1. Considering the condition, we can get the recursive formula for Φ.

To give Φ, we define **L**(s, k), for k = 0, 1,···, n - 2, as a set of the k-dimensional binary vectors $\eta \equiv (\eta_1, \eta_2, \cdots, \eta_k)$ as follows.

$$\mathbf{L}(s,k) \equiv \begin{cases} \{\eta \,|\, \text{if } \eta_j = 1 \text{ and } \eta_{j+a+1} = 1, \text{then } \eta_{j+1} = \cdots = \eta_{j+a} = 0, \\ \qquad \text{for } j = 1, 2, \cdots, k-a-1 \text{ and } a = 1, 2, \cdots, s-1\}, & \text{if } k = 1, 2, \cdots, n-2, \\ \{\phi\} & \text{if } k = 0, \end{cases}$$

where $\{\phi\}$ denotes the empty set.

L(s, k), for k=0, 1,···, n - 2, can be obtained by the recursive formulas of k [8].

Let us assume that $\delta_1, \delta_2, \cdots, \delta_n$ are located on a circle in a clockwise rotation as shown in Figure A2.1. We define two sets, $\mathbf{H}^1(s\,;t,u)$ for $n-s \geq t+u$, $t \geq 1$ and $u \geq 0$ and $\mathbf{H}^0(s\,;t,u)$ for $n \geq u+\max(s,t)$, $t \geq 1$ and $u \geq 1$ in the following.

$\mathbf{H}^1(s\,;t,u)$

$$\equiv \begin{cases} \{\delta \,|\, \delta_1 = \cdots = \delta_t = \delta_{n-u+1} = \cdots = \delta_n = 1, \\ \quad \delta_{t+1} = \cdots = \delta_{t+s} = \delta_{n-u+1-s} = \cdots = \delta_{n-u} = 0, \\ \quad (\delta_{t+s+1}, \cdots, \delta_{n-u-s}) \in \mathbf{L}(s, n-(t+u)-2s)\}, & \text{for } n-2s \geq t+u, \\ \{\delta \,|\, \delta_1 = \cdots = \delta_t = \delta_{n-u+1} = \cdots = \delta_n = 1, \\ \quad \delta_{t+1} = \cdots = \delta_{n-u} = 0\}, & \text{for } n-s \geq t+u > n-2s, \end{cases}$$

where $\delta_{n+1} \equiv \delta_1$, for $n-s \geq t+u$, $t \geq 1$ and $u \geq 0$.

$\mathbf{H}^0(s\,;t,u)$

$$\equiv \begin{cases} \{\delta \mid \delta_1 = \cdots = \delta_t = \delta_{n-\max(s-t,0)+1} = \cdots = \delta_n = 0, \\ \quad \delta_{t+1} = \cdots = \delta_{t+u} = 1, \delta_{t+u+1} = \cdots = \delta_{t+u+s} = 0, \\ \quad\quad (\delta_{t+u+s+1}, \cdots, \delta_{n-\max(s-t,0)}) \in \mathbf{L}(s, n-s-u-\max(s,t))\}, \\ \qquad\qquad\qquad\qquad \text{for } n-s \geq u+\max(s,t), \\ \{\delta \mid \delta_1 = \cdots = \delta_t = \delta_{n-\max(s-t,0)+1} = \cdots = \delta_n = 0, \\ \quad \delta_{t+1} = \cdots = \delta_{t+u} = 1, \delta_{t+u+1} = \cdots = \delta_{n-\max(s-t,0)} = 0\}, \\ \qquad\qquad\qquad\qquad \text{for } n \geq u+\max(s,t) > n-s, \end{cases}$$

where $\delta_{n+1} \equiv \delta_1$, for $n \geq u+\max(s,t)$, $t \geq 1$ and $u \geq 1$.

As shown in Figure A2.2, $\mathbf{H}^1(s\,;t,u)$ implies a part set of $\{\delta \mid \delta_1=1, \delta \in \Phi$ except $(0, 0,\cdots, 0)\}$ and a set of the n-dimensional binary vectors, on whose circle, t and u respectively is the number of 1's until the first 0 clockwise and counterclockwise from δ_1 and δ_n, respectively. As shown in Figure A2.3, $\mathbf{H}^0(s\,;t,u)$ implies a part set of $\{\delta \mid \delta_1=0, \delta \in \Phi$ except $(0, 0, \cdots, 0)\}$ and a set of the n-dimensional binary vectors, on whose circle, t is the number of 0's until the first 1 clockwise from δ_1 and u is the number of 1's until the first 0 clockwise from δ_{t+1}.

As $\mathbf{H}^0(s\,;t,u)$ and $\mathbf{H}^1(s\,;t,u)$ are disjoint, Φ can be expressed as

$$\Phi = \left(\bigcup_{t=1}^{n-s}\bigcup_{u=0}^{n-s-t}\mathbf{H}^1(s;t,u)\right) \cup \left(\bigcup_{t=1}^{n-1}\bigcup_{n=1}^{n-\max(s,t)}\mathbf{H}^0(s;t,u)\right) \cup \{(1,1,\cdots,1)\} \tag{A2.1}$$

A3. How to obtain $F_i(s, n; \delta)$

It is easy to see that if $\delta=(1, 1,\cdots, 1)$, $F_i(s, n; \delta)=\prod_{j=1}^{n} q_{ij}$. If $\delta \neq (1, 1,\cdots, 1)$, the closed formula for $F_i(s, n; \delta)$ is given in the following.

Without loss of generality, we assume that component (i, n) functions and the component (i, 1), (i, 2),$\cdots$, (i, s) fail. $k(\delta)$ represents the cardinality of set $\{j \mid \delta_j = 1$, for $j = 1, 2,\cdots, n\}$ and $j(u; \delta)$ is defined to be *u-th* element of the sequence, which is obtained by arranging elements of the set $\{j \mid \delta_j = 1\}$ in ascending order in a line, for $u = 1, 2,\cdots, k(\delta)$. That is, $j(1; \delta) = 1$, from the above assumption.

We let $i(u; \delta) \equiv n+1$, if $u = k(\delta)+1$, for convenience'sake. Also, let $P_i(u, \delta)$ be the probability that components $(i, j(u; \delta))$, $(i, j(u; \delta)+1)$, $\cdots$, $(i, j(u+1; \delta)-1)$ are in the states, which are necessary for the event $\{\mathbf{Y}_i=\delta\}$ to occur, for $u = 1, 2, \cdots, k(\delta)$.

It is easy to see that $F_i(s, n; \delta)$ is obtained by equation (A3.1).

$$F_i(s, n; \delta) = \prod_{u=1}^{k(\delta)} P_i(u,\delta). \tag{A3.1}$$

And $P_i(u, \delta)$ is obtained by the following equation (A3.2).

For $u = 1, 2,\cdots, k(\delta)$,

$P_i(u, \delta)$

$$
= \begin{cases}
q_{i,j(u,\delta)} & \left(j(u+1;\delta) - j(u;\delta) = 1\right) \\
\left(\prod_{j=j(u;\delta)}^{j(u+1;\delta)-2} q_{ij}\right) \bullet p_{i,j(u+1,\delta)-1}, & \left(j(u+1;\delta) - j(u;\delta) = s+1\right) \\
\left(\prod_{j=j(u;\delta)}^{j(u+1;\delta)+s-1} q_{ij}\right) \bullet p_{i,j(u+1,\delta)+s} & \\
\quad \bullet RL_i\left(s; j(u,\delta)+s+1, j(u+1,\delta)-2; [p_{ij}]\right) \bullet p_{i,j(u+1,\delta)-1}, & \left(j(u+1;\delta) - j(u;\delta) > s+1\right)
\end{cases}
\tag{A3.2}
$$

where, $RL_i(s; u, v; [p_{ij}])$ is defined to be the reliability of a linear consecutive-s-out-of-(v - u+1):F system composed of components (i, u), (i, u+1), ⋯, (i, v) on the system considered, for $1 \le u, v \le n$, where $RL_i(s; u, v; [p_{ij}]) \equiv 1$ if $u > v$.

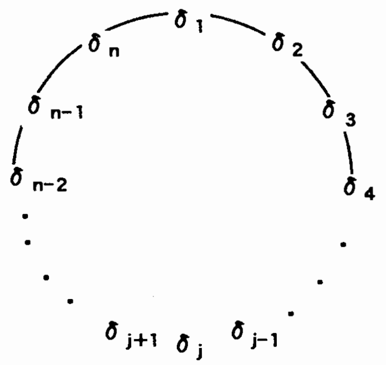
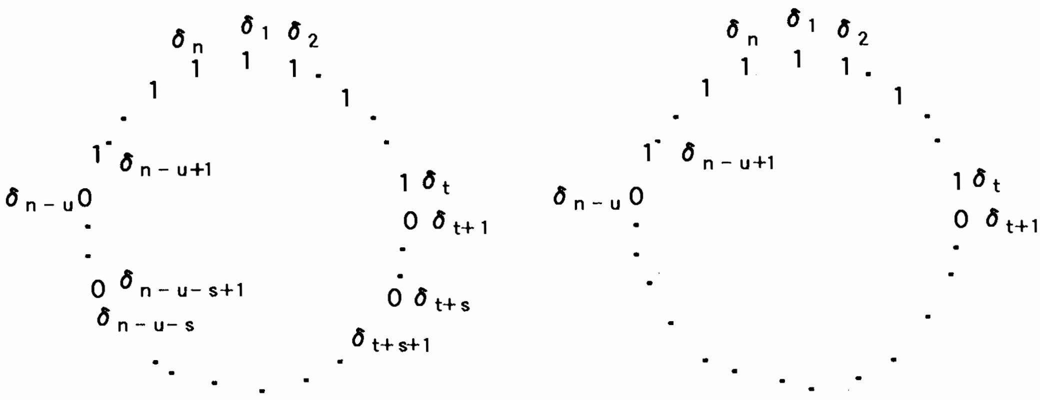


Figure A2.1 The Location of δ_j's



(a) $n - 2s \geqq t+u$ (b) $n - s \geqq t+u > n - 2s$

Figure A2.2 $H^1(s; t, u)$

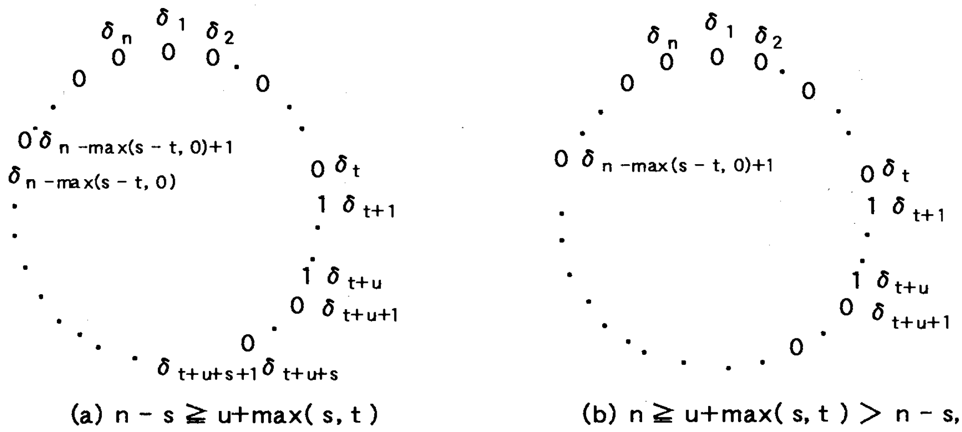


Figure A2.3 $H^0(s\,;\,t,\,u)$

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