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GLOBAL OPTIMIZATION PROBLEM WITH MULTIPLE REVERSE CONVEX CONSTRAINTS AND ITS APPLICATION TO OUT-OF-ROUNDNESS PROBLEM

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Abstract We consider a global minimization problem: $\min\{c^\top x+d^\top y \mid x\in X, y\in Y\setminus\bigcup_{h=1}^{m_2}G_h,\ (x,y)\in F\}$, where X and Y are polytopes in R^{n_1} and R^{n_2}, respectively; F is a closed convex set in $R^{n_1+n_2}$; and G_h $(h=1,\cdots,m_2)$ is an open convex set in R^{n_2}. We propose an alogorithm based on a combination of polyhedral outer approximation, branch-and-bound and cutting plane techniques. We also show that the out-of-roundness problem can be solved by the algorithm.

1 Introduction

In this paper we consider the following minimization problem:

$$(1.1)\qquad \left|\begin{array}{ll} \min & c^\top x+d^\top y \\ \text{s.t.} & x\in X, \\ & y\in Y\setminus \bigcup_{h=1}^{m_2} G_h, \\ & (x,y)\in F, \end{array}\right.$$

where X and Y are polytopes in R^{n_1} and R^{n_2}, respectively; F is a closed convex set in $R^{n_1+n_2}$; and G_h $(h=1,\ldots,m_2)$ is an open convex set in R^{n_2}; the vectors c and d are in R^{n_1} and R^{n_2}, respectively. In many applications the constraints in (1.1) are usually given as a system of inequalities, then we assume in this paper that

$$(1.2)\qquad \begin{array}{rcl} X &=& \{x\mid A_0x\le a_0\}, \\ Y &=& \{y\mid B_0y\le b_0\}, \\ F &=& \{(x,y)\mid f_i(x,y)\le 0,\ i=1,\ldots,m_1\}, \\ G_h &=& \{\ y\mid g_h(y)<0\ \},\ h=1,\ldots,m_2, \\ G &=& \bigcup_{h=1}^{m_2} G_h, \\ W &=& \{\ (x,y)\mid (x,y)\in (X\times Y)\cap F, y\notin G\ \}. \end{array}$$

where A_0, B_0 and a_0, b_0 are matrices and vectors of appropriate sizes; f_i $(i=1,\ldots,m_1)$ and g_h $(h=1,\ldots,m_2)$ are convex functions. Obviously, the constraint $y\notin\cup_{h=1}^{m_2}G_h$ can be rewritten as $g_h(y)\ge 0$ for $h=1,...,m_2$. The constraint $g_h(y)\ge 0$ is often called *reverse convex constraint* (see, e.g., Horst and Tuy [9]). By setting

$$f(x,y)=\max_{i=1,\ldots,m_1} f_i(x,y),$$

Problem (1.1) is equivalent to the following *noncanonical d.c. problem*:

$$(1.3)\qquad \left|\begin{array}{ll} \min & c^\top x + d^\top y \\ \text{s.t.} & x \in X, y \in Y, \\ & f(x,y) \le 0, \\ & g_h(y) \ge 0,\ h = 1,\dots,m_2. \end{array}\right.$$

Note that $f(\cdot)$ is a convex function.

Problem (1.3) includes several important classes of global optimization problems, such as a special d.c. programming problem, a class of problems with multiplicative terms [7]. Moreover it certainly contains the *canonical d.c. programming* (see, e.g. Horst and Pardalos [5]).

Recently several algorithms [4, 7, 10] are proposed for solving a special case of (1.3) in which only one additional reverse convex constraint is considered. Since a single reverse convex constraint is not available to represent the set defined by multiple reverse convex constraints, their algorithm is not directly applicable to Problem (1.3). The general branch-and-bound algorithm is a sole method for solving Problem (1.3) (section X.2 in [9]), which however does not make use of the structure of the problem. Since Problem (1.3) possesses a special structure that the reverse convex constraints are defined only on the y-space, we devise an algorithm which takes advantage of this specific structure.

Our new algorithm is based mainly on a combination of polyhedral outer approximation method and conical branch-and-bound in which the partition is made only in the y-space. Since only linear programming problems are solved in each step, it should not be costly to determine a solution for subproblem even if adding a new cut changes the feasible region. Therefore we can incorporate the cutting plane method whenever a cut is available. The algorithm can be regarded as a generalization of the first algorithm proposed in [7].

To use polyhedral outer approximation and conical subdivision we assume that

(A1) $\operatorname{int}((X \times Y) \cap F) \neq \emptyset$.

(A2) $\bigcap_{h=1}^{m_2} G_h \neq \emptyset$ and a point $y^0 \in \bigcap_{h=1}^{m_2} G_h$ is available.

Furthermore we require that the reverse convex constraints are essential, i.e.,

(A3) there exists a point $(x^\star, y^\star)$ such that $(x^\star, y^\star) \in (X \times Y) \cap F$, $y^\star \in G$ and $c^\top x^\star + d^\top y^\star < c^\top x + d^\top y$ for any $(x, y) \in W$.

The remainder of this paper is organized as follows. Section 2 describes a partition of $R^{n_1+n_2}$ based on a conical partition of R^{n_2}. We also show how to find the lower and upper bounds in Section 2. Section 3 gives the algorithm and proves its convergence. In Section 4 we show that the out-of-roundness problem [1, 11] can be formulated as Problem (1.3) and describe the details of the method for obtaining the lower and upper bounds for this specific problem.

2 Branching and bounding operations

We establish a subdivision underlying the branch-and-bound algorithm in this paper. Owing to the special structure of the problem we first subdivide the subspace R^{n_2} and then build the subdivision on the whole space $R^{n_1+n_2}$. We use a conical partition as the subdivision of the subspace R^{n_2}. The bounding operations are carried out by solving a series of linear programming problems.

2.1 Conical partition

Let y^0, z^i $(i = 1, \dots, n_2)$ be $n_2 + 1$ affinely independent points of R^{n_2}. We call the convex polyhedral cone $\{ y \in R^{n_2} \mid y = \sum_{i=1}^{n_2} \lambda^i (z^i - y^0) + y^0, \lambda^i \ge 0 \}$ the cone generated by points $y^0, z^1, \dots, z^{n_2}$. The cone has exactly n_2 edges emanating from point y^0. Without loss of generality, we assume that all z^i are on the surface of the unit ball $B(y^0) = \{ z \in R^{n_2} \mid \| z - y^0 \| \leq 1 \}$ with center at y^0.

For a subset $Y' \subseteq R^{n_2}$ a collection $\mathcal{C}$ of finitely many cones $\{ C_1, ..., C_t \}$ defined above is called a *conical partition* (see figure 1) of Y' if $\cup_{j=1}^t C_j = Y'$ and $\operatorname{int} C_i \cap \operatorname{int} C_j = \emptyset$ for $i \neq j$. We also call a collection $\mathcal{D} = \{ D_1, \ldots, D_t \}$, where $D_j = R^{n_1} \times C_j$, a *conical partition* of $R^{n_1} \times Y'$.

Let $\mathcal{C}$ and $\mathcal{C}'$ be conical partitions of R^{n_2}. Conical partition $\mathcal{C}'$ is said to be a *refinement* of $\mathcal{C}$ if for any $C' \in \mathcal{C}'$ there exists a cone $C \in \mathcal{C}$ such that $C' \subseteq C$.

In our algorithm we repeatedly refine the conical partition of R^{n_2} to yield a sequence $\{ \mathcal{C}_k \}_{k=1,2,...}$ of conical partitions. The refinement process is called *exhaustive* if for every strictly nested sequence $\{ C_k \}_{k=1,2,...}$ satisfying $C_k \in \mathcal{C}_k$ and $C_{k+1} \subset C_k$ for every k, there exists a vector $\bar{z}$ on $B(y^0)$ such that

$$\lim_{k \to \infty} z_k^i = \bar{z} \quad \text{for all} \quad i = 1, ..., n_2.$$

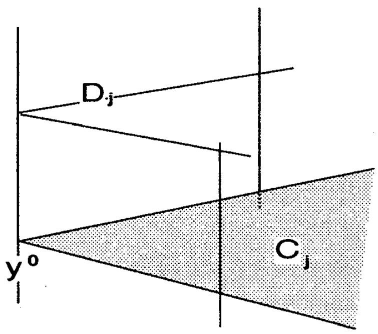

Figure 1: Conical Partition

2.2 Lower and upper bounds

Let P be a polytope containing all optimal solutions of (1.3), e.g., one can take $X \times Y$ as P. Then we can assume without loss of generality that P is contained in $X \times Y$. Let $\mathcal{D} = \{ D_1, \ldots, D_t \}$ be a conical partition of $R^{n_1+n_2}$. We consider how to compute a lower bound L_j of $c^\top x + d^\top y$ over $(P \cap W) \cap D_j = (P \cap W) \cap (R^{n_1} \times C_j)$ for $j = 1, \ldots, t$. For the sake of brevity, we omit the subscript j and let $D = R^{n_1} \times C$ denote a cone of the conical partition $\mathcal{D} = \{ D_1, \ldots, D_t \}$ throughout this and next subsections.

Assume that the polytope P is defined by the following system of inequalities:

$$Ax + By \leq b,$$

where A, B, and b are matrices and a vector of appropriate sizes. Note that P does not necessarily contain the whole set $(X \times Y) \cap F$. We propose a procedure for calculating the lower bound L of $c^\top x + d^\top y$ over the set $P \cap W \cap D$.

For the cone C determined by $y^0, z^1, ..., z^{n_2}$, let $\Theta = 2 \max\{\theta \mid y^0 + \theta(z - y^0) \in Y, z \in B(y^0) \}$ and let

$$\theta^{ih} = \min\{ \Theta, \ \sup\{\theta \mid y^0 + \theta(z^i - y^0) \in G_h \}\} \tag{2.1}$$

for $i = 1, ..., n_2$. And define for every G_h $(h = 1, \ldots, m_2)$ a set of n_2 points

$$v^{ih} = y^0 + \theta^{ih}(z^i - y^0). \tag{2.2}$$

We denote by θ^h the n_2-dimensional vector $(\theta^{1h}, ..., \theta^{n_2 h})^\top$, U^h the $n_2 \times n_2$-matrix $(v^{1h} - y^0, ..., v^{n_2 h} - y^0)$. Define a half space H^h in R^{n_2} as

$$H^h \;=\; \{\, y \in R^{n_2} \,|\, y = y^0 + U^h \lambda^h, e^\top \lambda^h \geq 1 \,\},$$

where $e = (1, ..., 1)^\top$. From the choices of y^0 and $v^{1h}, ..., v^{n_2 h}$, U^h is a nonsingular matrix, then H^h can be written as

$$H^h \;=\; \{\, y \in R^{n_2} \,|\, e^\top (U^h)^{-1}(y - y^0) \geq 1 \,\}.$$

Then the intersection of H^h and the cone C is written as

$$H^h \cap C = \{\, y \in R^{n_2} \,|\, y = y^0 + U^h \lambda^h, e^\top \lambda^h \geq 1, \lambda^h \geq 0 \,\}, \tag{2.3}$$

and for every point $(x, y) \in P \cap (R^{n_1} \times \cap_{h=1}^{m_2} H^h) \cap D$, there exist nonnegative vectors $\lambda^h = (\lambda^{1h}, ..., \lambda^{n_2 h})^\top$ for $h = 1, ..., m_2$, such that $e^\top \lambda^h \geq 1$ and

$$(x, y) = (0^{n_1}, y^0) + (x, U^h \lambda^h). \tag{2.4}$$

Lemma 2.1 *For every $(x, y) \in P \cap (R^{n_1} \times \cap_{h=1}^{m_2} H^h) \cap D$, λ^h in (2.4) is bounded for $h = 1, ..., m_2$.*

Proof. Let h be an arbitrary index of $\{1, ..., m_2\}$. From (2.4) point y with $(x, y) \in P \cap (R^{n_1} \times \cap_{h=1}^{m_2} H^h) \cap D$ is written as

$$y = y^0 + U^h \lambda^h,$$

that is

$$\lambda^h = (U^h)^{-1}(y - y^0).$$

Then we obtain

$$\| \lambda^h \| \leq \| (U^h)^{-1} \| \| y - y^0 \|,$$

which is bounded since y is in the polytope Y. □

Let

$$\begin{aligned} L &= \min\{\, c^\top x + d^\top y \,|\, (x, y) \in P \cap (R^{n_1} \times \cap_{h=1}^{m_2} H^h) \cap D \,\} \\ &= \min\{\, c^\top x + d^\top y \,| Ax + By \leq b; y = y^0 + U^h \lambda^h, \\ &\qquad e^\top \lambda^h \geq 1,\ \lambda^h \geq 0,\ h = 1, ..., m_2 \,\}. \end{aligned} \tag{2.5}$$

The following lemma shows that L is a lower bound of $c^\top x + d^\top y$ over the set $P \cap W \cap D$.

Lemma 2.2

(i) If $P \cap (R^{n_1} \times \cap_{h=1}^{m_2} H^h) \cap D$ is empty, then the optimal solution of (1.3) is not in $P \cap W \cap D$.

(ii) If $P \cap (R^{n_1} \times \cap_{h=1}^{m_2} H^h) \cap D$ is not empty, then

$$L \leq \min\{\, c^\top x + d^\top y \,|\, (x, y) \in P \cap W \cap D \,\}.$$

Proof. From the definitions of H^h, C and G, we see that $\{\, y \,|\, y \notin G \,\} \cap C \subseteq (\cap_{h=1}^{m_2} H^h) \cap C$, then

$$P \cap W \cap D = P \cap W \cap (R^{n_1} \times C) \subseteq P \cap (R^{n_1} \times (\cap_{h=1}^{m_2} H^h \cap C)) = P \cap (R^{n_1} \times \cap_{h=1}^{m_2} H^h) \cap D.$$

Therefore we obtain (ii). Moreover if $P \cap (R^{n_1} \times \cap_{h=1}^{m_2} H^h) \cap D = \emptyset$, then $P \cap W \cap D = \emptyset$. It implies (i). $\square$

From the above lemma, it seems that we have to solve a linear program with a lot of linear constraints when m_2 is large. However from Lemma 2.3 below, it is likely that we can remove many of such constraints of $H^h \cap C$ in computing (2.5).

Lemma 2.3 *For $h_1, h_2 \in \{1, \dots, m_2\}$ if $\theta^{h_1} \geq \theta^{h_2}$, then $H^{h_1} \cap C \subseteq H^{h_2} \cap C$.*

Proof. Let y be a point of $H^{h_1} \cap C$, then there exists a vector $\lambda^{h_1} \geq 0$ such that $e^\top \lambda^{h_1} \geq 1$ and

$$\begin{aligned} y &= y^0 + U^{h_1}\lambda^{h_1} \\ &= y^0 + (v^{1h_1} - y^0, \dots, v^{n_2 h_1} - y^0)(\lambda^{1h_1}, \dots, \lambda^{n_2 h_1})^\top \\ &= y^0 + [(z^1 - y^0)\theta^{1h_1}, \dots, (z^{n_2} - y^0)\theta^{n_2 h_1}](\lambda^{1h_1}, \dots, \lambda^{n_2 h_1})^\top. \end{aligned}$$

Let $t^i = \theta^{ih_1}/\theta^{ih_2}$, which is well defined by $\theta^{ih_2} > 0$, then $t^i \geq 1$ and we obtain

$$\begin{aligned} y &= y^0 + [(z^1 - y^0)\theta^{1h_2}t^1, \dots, (z^{n_2} - y^0)\theta^{n_2 h_2}t^{n_2}]\,(\lambda^{1h_1}, \dots, \lambda^{n_2 h_1})^\top \\ &= y^0 + [(z^1 - y^0)\theta^{1h_2}, \dots, (z^{n_2} - y^0)\theta^{n_2 h_2}](t^1\lambda^{1h_1}, \dots, t^{n_2}\lambda^{n_2 h_1})^\top. \end{aligned}$$

Note that $t^i\lambda^{ih_1} \geq 0$ for all i and $\sum t^i \lambda^{ih_1} \geq 1$. This means $y \in H^{h_2} \cap C$. $\square$

The following lemma is derived from Assumption (A3).

Lemma 2.4 *Let (x^*, y^*) be a global optimal solution of (1.3), then y^* is on the boundary of G.*

Proof. Suppose that the optimal solution (x^*, y^*) of (1.3) is not on the boundary of G. Then $g_h(y^*) > 0$ for any $h \in \{1, \dots, m_2\}$. Let $(x(\lambda), y(\lambda)) = (\lambda(x^\star, y^\star) + (1-\lambda)(x^*, y^*))$ for $(x^\star, y^\star)$ of Assumption (A3). Then for any $\lambda \in (0, 1]$

$$c^\top x(\lambda) + d^\top y(\lambda) < c^\top x^* + d^\top y^*.$$

Therefore $c^\top x(\bar\lambda) + d^\top y(\bar\lambda) < c^\top x^* + d^\top y^*$ for some $\bar\lambda \in (0, 1]$ and $g_h(y(\bar\lambda)) > 0$ for all h. By the convexity of X, Y and F, we also see that $x(\bar\lambda) \in X, y(\bar\lambda) \in Y$ and $(x(\bar\lambda), y(\bar\lambda)) \in F$. It implies that $(x(\bar\lambda), y(\bar\lambda))$ is a feasible solution of (1.3). This is a contradiction. $\square$

After solving the linear programming problem of (2.5) we obtain an optimal solution $(\bar x, \bar y)$ and the corresponding objective function value L. If the point $(\bar x, \bar y)$ lies in $P \cap W$, it is an optimal solution of $\min\{\, c^\top x + d^\top y \mid (x, y) \in P \cap W \cap D \,\}$. Then the currently considered D need not be subdivided further. Moreover, L serves as an upper bound of the optimal value of Problem (1.3).

If $(\bar x, \bar y) \notin P \cap W$, then we possibly find a feasible point of (1.3) by moving from $(\bar x, \bar y)$ along some specific direction. A possible choice of the direction is (c, d). Let

(2.6) $$\hat\tau = \min\{\, 2\max\{\, \tau \mid \bar y + \tau d \in Y \}, \sup\{\, \tau \mid \bar y + \tau d \in G \} \,\},$$

and define a point $(\hat x, \hat y)$ by

(2.7) $$(\hat x, \hat y) = (\bar x, \bar y) + \hat\tau(c, d).$$

If $(\hat x, \hat y) \in W$, then the value $c^\top \hat x + d^\top \hat y$ is an upper bound of the optimal value of (1.3). If $(\hat x, \hat y) \notin W$, then it is difficult in general to find a feasible point of (1.3) by moving $(\hat x, \hat y)$. The following technique, however, works well, for example, for the case of the *out-of-roundness* problem in Section 4. Namely we fix $\hat{y}$ and search a test point. Let

$$\ddot{x} = \arg\max\{\, c^\top x \mid Ax \leq b - B\hat{y} \,\} \tag{2.8}$$

and let $\hat{\lambda} = \sup\{\lambda \mid [(\hat{x}, \hat{y}), (\hat{x} + \lambda(\ddot{x} - \hat{x}), \hat{y})] \cap (X \times Y) \cap F = \emptyset\}$, where $[\cdot, \cdot]$ stands for a closed line segment. If $\hat{\lambda} < +\infty$, then we let the test point $(\tilde{x}, \hat{y})$ be determined by

$$\tilde{x} = \hat{x} + \hat{\lambda}(\ddot{x} - \hat{x}). \tag{2.9}$$

Lemma 2.5 *If $\hat{\lambda} < +\infty$, then $(\tilde{x}, \hat{y}) \in (X \times Y) \cap F$.*
Proof. Suppose $(\tilde{x}, \hat{y}) \notin (X \times Y) \cap F$. Then we have

$$[(\hat{x}, \hat{y}), (\tilde{x}, \hat{y})] \cap ((X \times Y) \cap F) = \emptyset.$$

By virtue of compactness of $(X \times Y) \cap F$, there exists $\varepsilon > 0$ such that

$$[(\hat{x}, \hat{y}), (\hat{x} + (\hat{\lambda} + \varepsilon)(\ddot{x} - \hat{x}), \hat{y})] \cap ((X \times Y) \cap F) = \emptyset.$$

It contradicts the definition of $\hat{\lambda}$. □

2.3 Polyhedral outer approximation and cutting plane

At the beginning of the algorithm, we take the polytope $X \times Y$ as an initial polytope P_1 containing $(X \times Y) \cap F$. The algorithm generates a sequence of polytopes $\{\, P_k \mid k = 1, 2, ...\}$ such that $P_1 \supseteq P_2 \cdots$ and each P_k contains an optimal solution of (1.3).

At the kth iteration, we construct a conical partition over some cone D chosen in the (k-1)st iteration. By solving linear programming problem (2.5) for all cones in the partition, we obtain several lower bounds. We also obtain several, possibly no, feasible points of (1.3), which are generated by solving (2.5) or by (2.7)-(2.9). After bounding operations (see Section 3 for the details) we choose a point with minimal lower bound to obtain a point $(\bar{x}_k, \bar{y}_k)$, which is an optimal solution of (2.5) for some cone of $\mathcal{D}$. If we find some feasible points, then choose one of them, say $(\dot{x}, \dot{y})$ having the smallest objective function value. We can take the inequality

$$c^\top x + d^\top y \leq c^\top \dot{x} + d^\top \dot{y} \tag{2.10}$$

as a cutting plane if the value $c^\top \dot{x} + d^\top \dot{y}$ is less than the current upper bound. Adding (2.10) to the constraints of P_k will not cut off the optimal solution of (1.3). Moreover, if $(\bar{x}_k, \bar{y}_k) \notin F$, compute a subgradient $S(\bar{x}_k, \bar{y}_k)$ of f at $(\bar{x}_k, \bar{y}_k)$ and let

$$l_k(x, y) = [\,(x, y) - (\bar{x}_k, \bar{y}_k)\,]S(\bar{x}_k, \bar{y}_k) + f(\bar{x}_k, \bar{y}_k). \tag{2.11}$$

Then the inequality

$$l_k(x, y) \leq 0 \tag{2.12}$$

will cut off the point $(\bar{x}_k, \bar{y}_k)$ but no feasible points of (1.3) in P_k. Therefore we can define the polytope P_{k+1} for the next iteration by

$$P_{k+1} = P_k \cap \{\, (x, y) \mid l_k(x, y) \leq 0,\ c^\top x + d^\top y \leq c^\top \dot{x} + d^\top \dot{y} \,\},$$

However, on the situation that only one of the cutting planes (2.10) and (2.12) or no cutting planes can be constructed, P_{k+1} is defined by adding the corresponding cutting plane to P_k or by $P_{k+1} = P_k$, respectively. The latter occurs if $(\bar{x}_k, \bar{y}_k) \in F \cap G$.

3 Algorithm

Based on the above discussion we propose an algorithm for solving Problem (1.3) as follows.

Algorithm

begin
  Construct a polytope $P_1 : P_1 \supseteq W$ and a conical partition $\mathcal{C}$ of R^{n_2};
  $\mathcal{M}_1 := \mathcal{C}$; $\gamma := +\infty$; $k := 1$;
  while $\mathcal{M}_k \neq \emptyset$ **do**
    begin
      for each $C \in \mathcal{M}_k$ **do**
        begin
          Solve linear program (2.5);
          $(\bar{x}(C), \bar{y}(C))$:= the optimal solution; $L(C) := c^\top \bar{x}(C) + d^\top \bar{y}(C)$;
          if $(\bar{x}(C), \bar{y}(C)) \in W$ **and** $\gamma > L(C)$ **then**
            begin
              $\gamma := L(C)$; $(\dot{x}, \dot{y}) := (\bar{x}(C), \bar{y}(C))$
            end
          else
            begin
              Compute $(\hat{x}, \hat{y})$ by (2.7);
              if $(\hat{x}, \hat{y}) \in W$ **and** $\gamma > c^\top \hat{x} + d^\top \hat{y}$ **then**
                begin
                  $\gamma := c^\top \hat{x} + d^\top \hat{y}$; $(\dot{x}, \dot{y}) := (\hat{x}, \hat{y})$
                end
              else
                begin
                  Compute $(\tilde{x}, \hat{y})$ by (2.8) with (2.9);
                  if $(\tilde{x}, \hat{y}) \in W$ **and** $\gamma > c^\top \tilde{x} + d^\top \hat{y}$ **then**
                  begin
                    $\gamma := c^\top \tilde{x} + d^\top \hat{y}$; $(\dot{x}, \dot{y}) := (\tilde{x}, \hat{y})$
                  end
                end
            end
        end;
      if $\{ C \in \mathcal{M}_k \mid L(C) < \gamma \} \neq \emptyset$ **then** $\mathcal{M}_{k+1} := \{ C \in \mathcal{M}_k \mid L(C) < \gamma \}$
      else goto *Out_of_while* ;
      Choose a set $C \in \mathcal{M}_{k+1}$ satisfying $c^\top \bar{x}(C) + d^\top \bar{y}(C) = \min\{ L(C) \mid C \in \mathcal{M}_{k+1} \}$;
      $C_k := C$; $(\bar{x}_k, \bar{y}_k) := (\bar{x}(C_k), \bar{y}(C_k))$;
      if γ is updated **then** $P_{k+1} := P_k \cap \{ (x, y) \mid c^\top x + d^\top y \leq \gamma \}$
      else $P_{k+1} := P_k$;
      if $(\bar{x}_k, \bar{y}_k) \notin F$ **then**
        begin
          $l_k := [(x, y) - (\bar{x}_k, \bar{y}_k)] S(\bar{x}_k, \bar{y}_k) + f(\bar{x}_k, \bar{y}_k) \leq 0$;
          $P_{k+1} := P_{k+1} \cap \{ (x, y) \mid l_k(x, y) \leq 0 \}$
        end
      Construct a conical partition $\mathcal{C}_k$ of C_k;
      $\mathcal{M}_{k+1} := \mathcal{M}_{k+1} \setminus \{ C_k \} \cup \mathcal{C}_k$; $k := k + 1$
    end

Out_of_while ;
  if $\gamma := +\infty$ **then** **writeln**(' The problem is infeasible ')
  else **writeln**(' The solution is ', $(\dot{x},\ \dot{y})$)
end.

3.1 Proof of the validity of the algorithm

If the algorithm terminates within a finite number of iterations and $\gamma < +\infty$, then clearly we obtain an optimal solution of Problem (1.3). If it terminates with $\gamma = +\infty$, we see that the problem has no feasible solution.

Theorem 3.1 *If $\gamma = +\infty$ when the algorithm terminates, the problem (1.3) has no feasible solution.*

Proof. Note that the algorithm terminates only if $\mathcal{M}_{k+1}$ becomes vacant and that it has two steps where $\mathcal{M}_k$ is updated:

(3.1) $$\mathcal{M}_{k+1} := \{\, C \in \mathcal{M}_k \mid L(C) < \gamma \,\}$$

and

(3.2) $$\mathcal{M}_{k+1} := \mathcal{M}_{k+1} \setminus \{C_k\} \cup \mathcal{C}.$$

Since $\mathcal{M}_{k+1}$ does not become vacant at Step (3.2), we conclude that

(3.3) $$L(C) \geq \gamma = +\infty,$$

meaning (2.5) is infeasible for any $C \in \mathcal{M}_k$. Suppose (1.3) has a feasible solution, say (x, y). Then it is feasible for (2.5), and hence $L(C)$ is finite if and only if the corresponding cone C contains y. Since $\gamma = +\infty$ throughout the execution of the algorithm, $\mathcal{M}_k$ keeps a cone, say C' containing (x, y). This implies that $L(C')$ is finite, which contradicts (3.3). □

Suppose that an infinite sequence $\{\,(\bar{x}_k, \bar{y}_k)\,\}_{k=1,2,\dots}$ is generated by the algorithm. Since the sequence is in compact set $X \times Y$, and hence bounded, it has a cluster point $(x^*, y^*) \in X \times Y$. We see that the conical partition $\mathcal{C}_k$ consists of finitely many cones, therefore there exists at least one cone of $\mathcal{C}_k$ containing infinitely many points of $(\bar{x}_k, \bar{y}_k)$. Consequently, we can choose a subsequence $\{\,(\bar{x}_{k_q}, \bar{y}_{k_q})\,\}_{q=1,2,\dots}$ of the above sequence and a sequence $\{C_{k_q}\}_{q=1,2,\dots}$ of nested cones such that $(\bar{x}_{k_q}, \bar{y}_{k_q}) \in C_{k_q}$.

The following two cases can happen.

Case(1) there exists a $\bar{q}$ such that for all $q > \bar{q}$, $(\bar{x}_{k_q}, \bar{y}_{k_q}) \in F$;

Case(2) for any $\bar{q}$ there exists $q > \bar{q}$ such that $(\bar{x}_{k_q}, \bar{y}_{k_q}) \notin F$.

In order to prove the convergence of the algorithm we first prove $(x^*, y^*) \in F$. If case (1) happens, then clearly $(x^*, y^*) \in F$. Therefore we only consider case (2). For simplicity we assume that the points $(\bar{x}_{k_q}, \bar{y}_{k_q})$ does not belong to F for every q by taking a suitable subsequence of $\{\,(\bar{x}_{k_q}, \bar{y}_{k_q})\,\}$ if necessary. For a positive ε let us introduce a closed ε-neighborhood $P(\varepsilon)$ of P_1, i.e.,

$$P(\varepsilon) = \{\,(x, y) \mid \text{ there exists } (x', y') \in P_1,\ \|\,(x, y) - (x', y')\,\| \leq \varepsilon\,\}.$$

Then $P_1 \subseteq \mathrm{int}P(\varepsilon)$.

Lemma 3.2 *The cutting plane functions $\{\, l_{k_q} \,\}_{q=1,2,\dots}$ are uniformly equicontinuous on P_1.*

Proof. From the compactness of $P(\varepsilon)$ we see that the convex function $f(x,y)$ is bounded on $P(\varepsilon)$, i.e., $f(x,y) \le M$ for some M. By the definition of subgradient

$$f(x,y) \ge [\,(x,y) - (\bar{x}_{k_q}, \bar{y}_{k_q})\,]S(\bar{x}_{k_q}, \bar{y}_{k_q}) + f(\bar{x}_{k_q}, \bar{y}_{k_q}) \text{ for all } (x,y) \in P(\varepsilon),$$

and consequently

$$[\,(x,y) - (\bar{x}_{k_q}, \bar{y}_{k_q})\,]S(\bar{x}_{k_q}, \bar{y}_{k_q}) \le M \tag{3.4}$$

holds for all $(x,y) \in P(\varepsilon)$ and for all k. Suppose that $\{\, S(\bar{x}_{k_q}, \bar{y}_{k_q}) \mid q = 1,2,\ldots \}$ is unbounded, then there exists at least one unbounded component of $S(\bar{x}_{k_q}, \bar{y}_{k_q})$. We assume without loss of generality that the first component of $S(\bar{x}_{k_q}, \bar{y}_{k_q})$ is unbounded. By $(\bar{x}_{k_q}, \bar{y}_{k_q}) \in P_1$, we can take a point $(x,y) \in P(\varepsilon)$ such that $(x,y) - (\bar{x}_{k_q}, \bar{y}_{k_q}) = (\pm\varepsilon, 0, \ldots, 0)$. By choosing an appropriate sign of ε, we have $[(x,y) - (\bar{x}_{k_q}, \bar{y}_{k_q})]S(\bar{x}_{k_q}, \bar{y}_{k_q}) > M$ for a sufficiently large q, a contradiction to (3.4). Therefore $\{\, S(\bar{x}_{k_q}, \bar{y}_{k_q}) \mid q = 1,2,\ldots \}$ is bounded.

Let M_1, M_2 and M_3 be sufficiently large numbers such that $\|\,(x,y) - (\bar{x}_{k_q}, \bar{y}_{k_q})\,\| \le M_1$, $\|\, S(\bar{x}_{k_q}, \bar{y}_{k_q})\,\| \le M_2$ and $|f(x,y)| \le M_3$ hold for all $(x,y) \in P_1$ and for all q. Then $|l_{k_q}(x,y)|$ is bounded by $M_1M_2 + M_3$ for all $(x,y) \in P_1$ and for all q. Therefore, both $\sup\{\, l_{k_q}(x,y) \mid (x,y) \in P_1,\ q = 1,2,\ldots \}$ and $\inf\{\, l_{k_q}(x,y) \mid (x,y) \in P_1,\ q = 1,2,\ldots \}$ are finite. The desired result follows from Theorem 10.6 of [13]. □

Since $\{\, l_{k_q}(x,y) \mid (x,y) \in P_1,\ q = 1,2,\ldots \}$ is bounded, by Theorem 10.8 of [13], $l_{k_1}, l_{k_2}, \ldots$ converge uniformly to a continuous function l, i.e.,

$$\lim_{q\to\infty} \sup_{(x,y)\in P_1} |l_{k_q}(x,y) - l(x,y)\,| = 0.$$

We have

Lemma 3.3 $\lim_{q\to\infty} l_{k_q}(\bar{x}_{k_q}, \bar{y}_{k_q}) = l(x^*, y^*)$.

Proof. Since l_{k_q} converges uniformly to l, we have that for every $\varepsilon > 0$ there exists q_1 such that

$$\sup_{(x,y)\in P_1} |\, l_{k_q}(x,y) - l(x,y)| \le \varepsilon/2 \quad \text{for all } q \ge q_1.$$

From $\lim_{q\to\infty} (\bar{x}_{k_q}, \bar{y}_{k_q}) = (x^*, y^*)$ and the continuity of l, we have $\lim_{q\to\infty} l(\bar{x}_{k_q}, \bar{y}_{k_q}) = l(x^*, y^*)$, i.e., for every $\varepsilon > 0$ there exists q_2 such that

$$|\, l(\bar{x}_{k_q}, \bar{y}_{k_q}) - l(x^*, y^*)\,| \le \varepsilon/2 \quad \text{for all } q \ge q_2.$$

Then we have for all $q \ge \max\{\, q_1, q_2 \,\}$,

$$\begin{aligned}
& |\, l_{k_q}(\bar{x}_{k_q}, \bar{y}_{k_q}) - l(x^*, y^*)\,| \\
\le\ & |\, l_{k_q}(\bar{x}_{k_q}, \bar{y}_{k_q}) - l(\bar{x}_{k_q}, \bar{y}_{k_q})\,| + |\, l(\bar{x}_{k_q}, \bar{y}_{k_q}) - l(x^*, y^*)\,| \\
\le\ & \sup_{(x,y)\in P_1} |l_{k_q}(x,y) - l(x,y)\,| + |\, l(\bar{x}_{k_q}, \bar{y}_{k_q}) - l(x^*, y^*)\,| \\
\le\ & \varepsilon/2 + \varepsilon/2 \\
=\ & \varepsilon.
\end{aligned}$$

□

Lemma 3.4 *If $l(x^*, y^*) \le 0$ then $(x^*, y^*) \in F$.*

Proof. Note that $l_{k_q}(x,y) = [(x,y) - (\bar{x}_{k_q}, \bar{y}_{k_q})]S(\bar{x}_{k_q}, \bar{y}_{k_q}) + f(\bar{x}_{k_q}, \bar{y}_{k_q})$,

$$\lim_{q\to\infty} l_{k_q}(x,y) = l(x,y) \quad \text{and} \quad \lim_{q\to\infty} f(\bar{x}_{k_q}, \bar{y}_{k_q}) = f(x^*, y^*).$$

By the boundedness of $S(\bar{x}_{k_q}, \bar{y}_{k_q})$, we can find a subsequence of $S(\bar{x}_{k_q}, \bar{y}_{k_q})$ converging to a vector $\bar{S}$. Therefore

$$l(x,y) = [(x,y) - (x^*, y^*)]\bar{S} + f(x^*, y^*),$$

implying $f(x^*, y^*) = l(x^*, y^*)$. By the definition of F, we have the lemma. □

Lemma 3.5 $\lim_{q\to\infty} l_{k_q}(\bar{x}_{k_{q+1}}, \bar{y}_{k_{q+1}}) = l(x^*, y^*)$.

Proof. By $\lim_{q\to\infty} l_{k_q}(\bar{x}_{k_q}, \bar{y}_{k_q}) = l(x^*, y^*)$ and $\lim_{q\to\infty}(\bar{x}_{k_q}, \bar{y}_{k_q}) = (x^*, y^*)$, we see that for every $\varepsilon > 0$ there exists q_1 such that

$$|\, l(x^*, y^*) - l_{k_q}(\bar{x}_{k_q}, \bar{y}_{k_q})\,| < \varepsilon/3 \quad \text{for all } q \geq q_1$$

and for every $\delta > 0$ there exists q_2 such that

$$\|\, (x^*, y^*) - (\bar{x}_{k_q}, \bar{y}_{k_q})\, \| < \delta \quad \text{for all } q \geq q_2.$$

From Lemma 3.2, we see that $\{\, l_{k_q}(x,y)\,\}$ is equicontinuous at (x^*, y^*), i.e., for every $\varepsilon > 0$ there exists $\delta > 0$ such that

$$\text{if } \; \|\, (x,y) - (x^*, y^*)\, \| \leq \delta \text{ implies}$$

$$|\, l_{k_q}(x^*, y^*) - l_{k_q}(x,y)\,| < \varepsilon/3 \quad \text{for all } q.$$

Therefore for every $\varepsilon > 0$ we have that for $q \geq \max\{\, q_1, q_2\,\}$

$$\begin{aligned} & |\, l(x^*, y^*) - l_{k_q}(\bar{x}_{k_{q+1}}, \bar{y}_{k_{q+1}})\,| \\ \leq\ & |\, l(x^*, y^*) - l_{k_q}(\bar{x}_{k_q}, \bar{y}_{k_q})\,| + |\, l_{k_q}(x^*, y^*) - l_{k_q}(\bar{x}_{k_q}, \bar{y}_{k_q})\,| \\ & + |\, l_{k_q}(x^*, y^*) - l_{k_q}(\bar{x}_{k_{q+1}}, \bar{y}_{k_{q+1}})\,| \\ \leq\ & \varepsilon/3 + \varepsilon/3 + \varepsilon/3 \\ =\ & \varepsilon. \end{aligned}$$

□

Theorem 3.6 $(x^*, y^*) \in F$.

Proof. Note that $k_{q+1} \geq k_q + 1$. Then by the definition of $(\bar{x}_{k_{q+1}}, \bar{y}_{k_{q+1}})$, we see $(\bar{x}_{k_{q+1}}, \bar{y}_{k_{q+1}}) \in P_{k_{q+1}} \subseteq P_{k_q+1}$, meaning $l_{k_q}(\bar{x}_{k_{q+1}}, \bar{y}_{k_{q+1}}) \leq 0$. Therefore $l(x^*, y^*) \leq 0$ by Lemma 3.5. Then Lemma 3.4 proves the assertion. □

We have proved that every cluster point of the sequence $\{\, \bar{x}_k, \bar{y}_k\,\}$ generated by the algorithm belongs to F. Moreover, from $P_k \subseteq X \times Y$ for $k = 1, 2...$ we have $(x^*, y^*) \in (X \times Y) \cap F$.

Theorem 3.7 *Suppose the conical partitions generated by the algorithm are exhaustive. If $\gamma < +\infty$ then every cluster point of the sequence $\{\, (\bar{x}_k, \bar{y}_k)\,\}$ is an optimal solution of Problem (1.3).*

Proof. Let (x^*, y^*) be a cluster point of $\{\, (\bar{x}_k, \bar{y}_k)\,\}$. Assume that $\{\, (\bar{x}_{k_q}, \bar{y}_{k_q})\,\}$ is a subsequence of $\{\, (\bar{x}_k, \bar{y}_k)\,\}$ converging to (x^*, y^*) such that $(\bar{x}_{k_q}, \bar{y}_{k_q}) \in D_{k_q} = R^{n_1} \times C_{k_q}$ for a nested sequence $\{\, C_{k_q}\,\}$ of cones, i.e., $C_{k_{q+1}} \subset C_{k_q}$ for all q.

First we show $y^* \notin G$. From the assumption that $\{C_{k_q}\}$ is exhaustive, there exists a vector $\bar{z} \in R^{n_2}$ such that y^* is on the ray $\{y \,|\, y = y^0 + \theta(\bar{z} - y^0), \theta \geq 0\}$. On the other hand, by the definitions of $(\bar{x}_{k_q}, \bar{y}_{k_q})$ and C_{k_q},

$$\bar{y}_{k_q} = y^0 + U^h_{k_q}\lambda^h_{k_q} = y^0 + (v^{1h}_{k_q} - y^0, ..., v^{n_2h}_{k_q} - y^0)(\lambda^{1h}_{k_q}, ..., \lambda^{n_2h}_{k_q})^\top, \tag{3.5}$$

where $e^\top\lambda^h_{k_q} \geq 1$ and $\lambda^h_{k_q} \geq 0$. From the definition of $v^{ih}_{k_q}$ in (2.2) (where index k_q is omitted) we see that

$$\bar{y}_{k_q} = y^0 + [\theta^{1h}_{k_q}(z^1_{k_q} - y^0), ..., \theta^{n_2h}_{k_q}(z^{n_2}_{k_q} - y^0)](\lambda^{1h}_{k_q}, ..., \lambda^{n_2h}_{k_q})^\top.$$

By Lemma 2.1, we have that $\{\lambda^h_{k_q} \,|\, h = 1, ..., m_2, q = 1, 2, ...\}$ is also bounded. Taking a subsequence if necessary, we obtain that

$$\lim_{q\to\infty} \theta^{ih}_{k_q} = \bar{\theta}^{ih}, \quad \lim_{q\to\infty} \lambda^{ih}_{k_q} = \bar{\lambda}^{ih} \ \text{ for } \ i = 1, ..., n_2 \ \text{ and } \ \sum \bar{\lambda}^{ih} \geq 1, \bar{\lambda}^{ih} \geq 0.$$

Hence

$$\begin{aligned} y^* = \lim_{q\to\infty} \bar{y}_{k_q} &= y^0 + [\,\bar{\theta}^{1h}(\bar{z} - y^0), ..., \bar{\theta}^{n_2h}(\bar{z} - y^0)\,](\bar{\lambda}^{1h}, ..., \bar{\lambda}^{n_2h})^\top \\ &= y^0 + [\,(\bar{z} - y^0), ..., (\bar{z} - y^0)\,](\bar{\theta}^{1h}\bar{\lambda}^{1h}, ..., \bar{\theta}^{n_2h}\bar{\lambda}^{n_2h})^\top \\ &= y^0 + (\bar{z} - y^0)\sum_{i=1}^{n_2} \bar{\theta}^{ih}\bar{\lambda}^{ih} \qquad \text{for all } h. \end{aligned} \tag{3.6}$$

Suppose $y^* \in G$, then there exists at least one h_0 such that $y^* \in G_{h_0}$. We will show that

$$y^0 + \bar{\theta}^{ih_0}(\bar{z} - y^0) \in \partial G_{h_0} \text{ for some } i. \tag{3.7}$$

Suppose that $y^0 + \bar{\theta}^{ih_0}(\bar{z} - y^0) \notin \partial G_{h_0}$ for all i. Note that $\theta^{ih_0}_{k_q}$ is taken either as Θ in (2.1) or such that $y^0 + \theta^{ih_0}_{k_q}(z^i_{k_q} - y^0) \in \partial G_{h_0}$. Then for sufficiently large q

$$\theta^{ih_0}_{k_q} = \Theta,$$

which implies

$$\bar{\theta}^{ih_0} = \Theta \ \text{ for all } \ i.$$

Therefore from (3.6) we obtain

$$y^* = y^0 + (\bar{z} - y^0)\sum_{i=1}^{n_2} \bar{\theta}^{ih_0}\bar{\lambda}^{ih_0} = y^0 + (\bar{z} - y^0)\Theta\sum_{i=1}^{n_2} \bar{\lambda}^{ih_0}. \tag{3.8}$$

From the definition of Θ, (3.8) contradicts the fact that $y^* \in Y$. Therefore (3.7) holds true. Moreover, it follows that from the compactness of ∂G_{h_0}

$$y^0 + \bar{\theta}^{ih_0}(\bar{z} - y^0) \in \partial G_{h_0} \text{ while } \ \bar{\theta}^{ih_0} \neq \Theta \text{ for all } \ i. \tag{3.9}$$

Taking $\hat{\theta}^{h_0} = \min_i \bar{\theta}^{ih_0}$, by virtue of (3.7) and (3.9), we see that $y^0 + \hat{\theta}^{h_0}(\bar{z} - y^0) \in \partial G_{h_0}$. From $y^* \in G_{h_0}$ therefore $\sum \bar{\theta}^{ih_0}\bar{\lambda}^{ih_0} < \hat{\theta}^{h_0} = \min_i \bar{\theta}^{ih_0}$. On the other hand, $\sum \theta^{ih_0}\bar{\lambda}^{ih_0} \geq \sum \hat{\theta}^{h_0}\bar{\lambda}^{ih_0} \geq \hat{\theta}^{h_0}$, a contradiction. It implies that $y^* \notin G$.

Combining the above result with Theorem 3.6 we see that (x^*, y^*) is a feasible solution of Problem (1.3), i.e., $(x^*, y^*) \in W$.

Let V^* be the optimal value of (1.3). Note that $c^\top\bar{x}_{k_q} + d^\top\bar{y}_{k_q}$ is a lower bound of V^*, therefore we see that

$$c^\top x^* + d^\top y^* = \lim_{q\to\infty} c^\top\bar{x}_{k_q} + d^\top\bar{y}_{k_q} \leq V^*.$$

It implies that (x^*, y^*) is an optimal solution of Problem (1.3). □

4 Out-of-roundness problem

Let $\bar{P}$ be a set of finitely many points $\bar{p}^1, \ldots, \bar{p}^m$ in R^n. The *out-of-roundness* problem is formulated as follows.

$$
(4.1) \qquad \left| \begin{array}{ll} \min & R - r \\ \text{s.t.} & \| p - \bar{p}^h \| \leq R, \ h = 1, \ldots, m, \\ & \| p - \bar{p}^h \| \geq r, \ h = 1, \ldots, m, \\ & p \in C(\bar{P}), \end{array} \right.
$$

where $C(\bar{P})$ is the convex hull of the set $\bar{P}$.

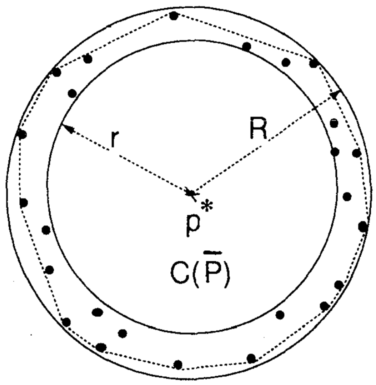

Figure 2: Out-of-Roundness Problem

The problem is to find a pair of concentric balls one of which contains all the points $\bar{p}^1, \ldots, \bar{p}^m$ and the other contains none of them such that the difference of two radii is minimized. If the objective function $R - r$ is small enough, we can conclude that the given points $\bar{p}^1, \ldots, \bar{p}^m$ lie on the surface of a ball.

There are several algorithms [1, 11] dealing with the problem. The proposed algorithms [1, 11] can solve 2-dimensional out-of-roundness problems in $O(m^2)$ time. To the authors' knowledge there are no algorithms developed for solving problems with dimensions higher than two. The algorithms in [1, 11] are based on constructing the nearest and furthest neighbor Voronoi diagrams. However, constructing the nearest neighbor Voronoi diagrams alone needs $O(m^{\lceil \frac{n+1}{2} \rceil})$ time, which increases exponentially with the dimension n (see [2]). Moreover, whether the approaches in [1, 11] can be generalized to solve problems with $n > 2$ is not clear. In the remaining part of this section we show that the out-of-roundness problem can be formulated as Problem (1.3), where the structures of X, Y, F and G of constraint sets are rather simple. By taking advantages of the structures, we show that the algorithm proposed in the previous section can be applied efficiently to the problem.

We consider the problem (4.1) with the last constraint $p \in C(\bar{P})$ dropped, i.e.,

$$
(4.2) \qquad \left| \begin{array}{ll} \min & R - r \\ \text{s.t.} & \| p - \bar{p}^h \| \leq R, \ h = 1, \ldots, m, \\ & \| p - \bar{p}^h \| \geq r, \ h = 1, \ldots, m. \end{array} \right.
$$

In practice the given points $\bar{p}^1, \ldots, \bar{p}^m$ represent the location of sample points on the surface of an almost round object. Therefore it is very likely that the solution of (4.2) lies somewhere in the convex hull of $\bar{P}$. Let us set an assumption as follows:

Assumption 4.1 *The optimal solution of (4.2) is in the convex hull $C(\bar{P})$.*
The out-of-roundness problem is equivalent to Problem (4.2) under Assumption (4.1). Let

$$\begin{aligned} p^0 &= \frac{1}{m}\sum_{h=1}^{m} \bar{p}^h, \\ \alpha_j^0 &= \min\{\,(e^j)^\top \bar{p}^h \mid h = 1,\dots,m\,\},\ j = 1,\dots,n, \\ \beta_j^0 &= \max\{\,(e^j)^\top \bar{p}^h \mid h = 1,\dots,m\,\},\ j = 1,\dots,n, \end{aligned}$$

where e^j is a jth unit vector in R^n. Further we define

$$\begin{aligned} F &= \{\,(R,r,p) \mid \|\, p - \bar{p}^h \,\| \le R,\ h = 1,\dots,m\,\}, \\ G_h &= \{\,(r,p) \mid \|\, p - \bar{p}^h \,\| < r\,\},\ h = 1,\dots,m, \\ G &= \bigcup_{h=1}^{m} G_h, \\ \rho^0 &= \max_h \|\, p^0 - \bar{p}^h \,\|, \\ X &= \{\,R \mid 0 \le R \le 2\rho^0\,\}, \\ Y &= \{\,(r,p) \mid 0 \le r \le 2\rho^0,\ \alpha_j^0 \le p_j \le \beta_j^0,\ j = 1,\dots,n\,\} \end{aligned}$$

and consider

$$(4.3) \qquad \left|\begin{array}{ll} \min & R - r \\ \text{s.t.} & R \in X,\ (r,p) \in Y, \\ & (R,r,p) \in F,\ (r,p) \in R^{n+1} \setminus G. \end{array}\right.$$

Note that

$$G_h = \{\,(r,p) \mid \|\, p - \bar{p}^h \,\| - r < 0\,\},\ h = 1,\dots,m$$

is an open convex set. The polytope X is just an interval and the polytope Y is a hypercube of dimension $n+1$. From the nature of this problem the ratio of m/n is usually very large.

Theorem 4.2 *Under Assumption 4.1 the problems (4.2) and (4.3) are equivalent.*

Proof. Let $(R^*(2), r^*(2), p^*(2))$ and $(R^*(3), r^*(3), p^*(3))$ be optimal solutions of Problem (4.2) and Problem (4.3), respectively. Since $(R^*(3), r^*(3), p^*(3))$ is a feasible point of the problem (4.2),

$$(4.4) \qquad R^*(2) - r^*(2) \le R^*(3) - r^*(3).$$

On the other hand, we obtain $p^*(2) = \sum_{h=1}^{m} \lambda_h^* \bar{p}^h$ for some nonnegative λ_h^* such that $\sum_{h=1}^{m} \lambda_h^* = 1$ from Assumption 4.1 and $R^*(2) = \|\, p^*(2) - \bar{p}^{h_1} \,\|$ for some $h_1 \in \{1,\dots,m\}$. Therefore

$$\begin{aligned} R^*(2) &= \|\, p^*(2) - \bar{p}^{h_1} \,\| \\ &\le \|\, p^*(2) - p^0 \,\| + \|\, p^0 - \bar{p}^{h_1} \,\| \\ &= \|\, \sum_{h=1}^{m} \lambda_h^* \bar{p}^h - p^0 \,\| + \|\, p^0 - \bar{p}^{h_1} \,\| \\ &\le \sum_{h=1}^{m} \lambda_h^* \|\, \bar{p}^h - p^0 \,\| + \|\, p^0 - \bar{p}^{h_1} \,\| \\ &\le 2\rho^0. \end{aligned}$$

Furthermore, $p^*(2) \in C(\bar{P})$ implies $\alpha_j^0 \le p_j^*(2) \le \beta_j^0$ for $j = 1,\dots,n$. Therefore $(R^*(2), r^*(2), p^*(2))$ is a feasible point of Problem (4.3). Then we have

$$R^*(3) - r^*(3) \le R^*(2) - r^*(2). \tag{4.5}$$

□

From Assumption 4.1 and Theorem 4.2, the out-of-roundness problem is equivalent to Problem (4.3), which is solvable by the algorithm proposed in Section 3.

To start the algorithm we choose $X \times Y$ as an initial polytope P_1 containing $(X \times Y) \cap F$, where $(X \times Y) \cap F$ is defined as before. Take r^0 to be any value greater than $\rho^0 = \max_h \| p^0 - \bar{p}^h \|$. Then the point (r^0, p^0) belongs to $\cap_{h=1}^m G_h$, and can serve as point y^0 of the algorithm.

Suppose that we are at the kth iteration of the algorithm, let the polytope P_k be defined by

$$P_k = \{ (R, r, p) \mid a_R^k R + a_r^k r + B_p^k p \le b^k \},$$

where a_R^k, a_r^k and b^k are m^k-dimensional vectors, and B_p^k is an $m^k \times n$-matrix. To obtain a lower bound over a set $P_k \cap W \cap D$, let $(r^1, p^1), \dots, (r^{n+1}, p^{n+1})$ be points generating the cone C. Since each constraint $g_h(r,p) < 0$ defining the set G_h is very simple, a solution of the equation

$$\| p^0 + \theta^{ih}(p^i - p^0) - p^h \| - (r^0 + \theta^{ih}(r^i - r^0)) = 0,$$

if any yields the value of θ^{ih}, for which $(r^{ih}, p^{ih}) = (r^0, p^0) + \theta^{ih}(r^i - r^0, p^i - p^0)$ lies on the intersection of ∂G_h and the ith ray of the cone C. After computing the set of $n+1$ points $(r^{1h}, p^{1h}), \dots, (r^{n+1,h}, p^{n+1,h})$ for every h $(h = 1, \dots, m)$, we have to solve a linear program (2.5) to obtain a lower bound and possibly an upper bound. The linear program (2.5) can be written as

$$\left|\begin{array}{ll} \min & R - r \\ \text{s.t.} & a_R^k R + a_r^k r + B_p^k p \le b^k, \\ & (r, p) = (r^0, p^0) + U^h \lambda^h, \ \forall h, \\ & e^\top \lambda^h \ge 1, \ \forall h, \\ & \lambda^h, \ R, \ r \ge 0, \ \forall h. \end{array}\right. \tag{4.6}$$

Recall $U^h = [(r^{1h} - r^0, p^{1h} - p^0), \dots, (r^{n+1,h} - r^0, p^{n+1,h} - p^0)]$. Let

$$U^h = \begin{pmatrix} U_r^h \\ U_p^h \end{pmatrix},$$

then

$$r = r^0 + U_r^h \lambda^h, \ \forall h, \tag{4.7}$$
$$p = p^0 + U_p^h \lambda^h, \ \forall h. \tag{4.8}$$

Take an arbitrary number of $\{1, \dots, m\}$, for instance 1 and substitute (4.7) and (4.8) with $h = 1$ for r and p of (4.6), respectively. Then the problem (4.6) reduces to

$$\left|\begin{array}{llll} \min & R - U_r^1 \lambda^1 - r^0 & & \\ \text{s.t.} & \tilde{a}^k R + \tilde{B}^k \lambda^1 & \ge & \tilde{b}^k, \\ & U^h \lambda^h - U^1 \lambda^1 & = & 0, \ h = 2, \dots, m, \\ & e^\top \lambda^h & \ge & 1, \ h = 1, \dots, m, \\ & \lambda^h, \ R & \ge & 0, \ h = 1, \dots, m, \end{array}\right. \tag{4.9}$$

where

$$\begin{aligned}\tilde{a}^k &= -a^k_{R},\\ \tilde{B}^k &= -a^k_r U^1_r - a^k_p U^1_p,\\ \tilde{b}^k &= a^k_r r^0 + a^k_p p^0 - b^k.\end{aligned}$$

The above problem has $m^k + (n+1)(m-1) + m$ constraints and $(n+1)m+1$ variables, and the number m^k grows at each iteration as cutting planes are introduced. Therefore it is time consuming to solve this problem directly. We deal with this shortcoming as follows.

There are lots of redundant constraints in (4.9). Using Lemma 2.3 we can remove h from the set $\{1,..,m\}$ if there exists an h' such that $\theta^{h'} < \theta^h$. Let I be the remaining subset of $\{1,..,m\}$ after removing all those h, relabel the elements in I as $1,\dots,|I|$, and relabel also correspondingly U^h and λ^h. We consider the dual problem of (4.9). Let ζ, $\eta^1,\dots,\eta^{|I|-1}$, ξ be dual variables of the reduced and relabeled problem (4.9), where ζ is a vector of m^k-dimension, $\eta^1,\dots,\eta^{|I|-1}$ are vectors of $(n+1)$-dimension, and ξ is a vector of $|I|$-dimension. The dual problem is

$$(4.10)\qquad \left|\begin{array}{lll} \max & (\tilde{b}^k)^\top\zeta + e^\top\xi & \\ \text{s.t.} & (\tilde{a}^k)^\top\zeta & \le\ 1,\\ & (\tilde{B}^k)^\top\zeta - \sum_{h=1}^{|I|-1}(U^1)^\top\eta^h + \xi e & \le\ -(U^1_r)^\top,\\ & (U^h)^\top\eta^{h-1} + \xi e & \le\ 0,\ h=2,\dots,|I|,\\ & \zeta,\ \xi & \ge\ 0.\end{array}\right.$$

Note that the above problem has $(n+1)|I|+1$ constraints. It is obvious that solving (4.10) is less time consuming in comparison with solving (4.9) directly.

The other thing worth mentioning is that the methods of finding possible feasible points in (2.7) and (2.8)-(2.9) are extremely simple. By solving (4.10) we obtain a point $(\bar{R},\bar{r},\bar{p})$. Suppose $(\bar{R},\bar{r},\bar{p}) \notin (X\times Y)\cap F$. Then

$$\|\,\bar{p}-\bar{p}^h\,\| - \bar{r} < 0 \quad\text{for}\quad h=1,\dots,m.$$

The value of $\hat{r} = \sup\{\tau\,|\,\tau d + \bar{y} \in G\}$ in (2.7), where $d=(-1,0)$, $\bar{y}=(\bar{r},\bar{p})$, can be determined by

$$\hat{r} = \max\{\bar{r} - \|\,\bar{p}-\bar{p}^h\,\|\,|\,h=1,\dots,m\}.$$

In fact the point $(\hat{r}d+\bar{y}) = (\bar{r}-\hat{r},\bar{p})$ belongs to G, since

$$\|\,\bar{p}-\bar{p}^h\,\| - (\bar{r}-\hat{r}) \ge 0,\ h=1,\dots,m,$$

and at least one equation holds. Then the point $(\hat{x},\hat{y}) = (\hat{R},\hat{r},\hat{p})$ in (2.7) can be determined by $\hat{r}(c,d) + (\bar{x},\bar{y}) = \hat{r}(1,-1,0) + (\bar{R},\bar{r},\bar{p})$.

If the point $(\tilde{x},\hat{y}) = (\tilde{R},\hat{r},\hat{p})$ in (2.8) is necessary, we have to determine first $\ddot{x} = \ddot{R}$ satisfying (2.9), i.e., to solve the following maximization problem

$$(4.11)\qquad \max\{\,R\,|\,a^k_{R}R + a^k_r\hat{r} + B^k_p\hat{p} \le b^k\,\}.$$

Note that $(R,r,p)\in P_k$ satisfies $0 \le R \le 2\rho^0$. Therefore (4.11) has an optimal solution. The optimal solution is simply given by

$$\ddot{R} = \min\left\{\left.\frac{b_i}{a_i}\,\right|\, a_i > 0, i=1,\dots,m_k\right\},$$

where a_i and b_i are the ith components of the vectors a^k_{R} and $b^k - a^k_r\hat{r} - B^k_p\hat{p}$.

5 Conclusions

We have proposed an algorithm for solving the global optimization problem with a set of reverse convex constraints by means of cutting plane techniques and branch-and-bound method. The out-of-roundness problem has been discussed as a special case of the problem considered in this paper. The techniques proposed to find a feasible point in (2.7)-(2.9) become very simple when applied to the out-of-roundness problem. The proposed method formulating a computational geometry problem as a global optimization problem is also successful for the *largest empty sphere* problem (see Shi and Yamamoto [3]). This is a concrete example using continuous approaches to discrete optimization problems (see Pardalos [12]).

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