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MODELS AND MEASURES FOR EFFICIENCY DOMINANCE IN DEA
Part II: Free Disposal Hull (FDH) and Russell Measure (RM) Approaches*

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Abstract Models and measures of efficiency dominance as treated by Free Disposal Hull and Russell Measure approaches to efficiency evaluation are examined as they relate to additive models and MED (Measures of Efficiency Dominance) in DEA.

1. Introduction

Mixed integer versions of the additive models of DEA with associated measures were introduced in [4] as a way to deal with "efficiency dominance". Other models and measures are examined in this paper. The focus is on (a) the "Free Disposal Hull" (FDH) and (b) the "Russell Measure" (RM) approaches, each of which has formed the basis for major research efforts—the first by H. M. Tulkens and his associates at the University of Louvain and the second by C. A. K. Lovell and his associates at the University of Georgia. See, e.g., [11] and [22]–[25]. See also [13]–[16].

2. FDH = Free Disposal Hull

We start with FDH after defining X_j, and Y_j as input and output vectors, respectively, with components x_{ij}, $i = 1, \cdots, m$ and $y_{rj}, r = 1, \cdots, s$, for each of a collection of DMU_j (Decision Making Units) where $j = 1, \cdots, n$. Then, we use DMU_o to designate the DMU_j to be evaluated and say that the efficiency of its observed performance is dominated by DMU_k if $X_o \geq X_k$ and $Y_o \leq Y_k$ with strict inequality holding in at least one input or output component.

As noted in Bardhan *et al.* [4], the assumption of "free disposal" gives rise to a danger of identifying a dominated DMU as efficient when non-zero slack is present. This danger is avoided in the FDH approach, however, by a first-stage algorithm which uses paired vector comparisons to identify nondominated DMUs.[1] The possible presence of alternate optima can nonetheless lead to problems like those we now describe.

For simplicity we follow Lovell [16, pp. 38–40] and restrict attention to outputs by assuming that all DMUs use only a single input in the same amount. Then, using y_{rj} to

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[1] See Tulkens and Vanden Eeckaut [24, pp. 3-5] who further distinguish between input and output dominance—and introduce additional categories such as "weak dominance", etc., which we do not pursue here because these refinements do not lead to important differences in our results.

represent output r for DMU_j, the efficiency measure used in FDH is obtained from

$$\phi^* = \max_{\lambda} \min_{r=1,\cdots,s} \left\{ \frac{\sum_{j\in D} y_{rj}\lambda_j}{y_{ro}} \;\middle|\; \sum_{j\in D} \lambda_j = 1, \lambda_j \in \{0,1\}, \forall j \in D \right\} = \frac{y_{rk}}{y_{ro}}. \tag{1}$$

Here $j \in D$ means that the indexes are selected from the set D of non-dominated DMUs, as determined in the first stage algorithm, so that $y_{rk}/y_{ro} \geq 1$, $\forall r$, with equality achieved when the DMU_o to be evaluated is efficient—as indicated by the choice of $\lambda_j = 1$ from the vector λ.

The following example can help to show what is involved,

$$\begin{aligned} &\max \phi \\ &\text{subject to} \quad 12\phi \leq 14\lambda_1 + 24\lambda_2, \quad 12\phi \leq 15\lambda_1 + 14\lambda_2, \\ &\qquad\qquad\quad 12\phi \leq 14\lambda_1 + 14\lambda_2, \qquad 1 = \lambda_1 + \lambda_2, \end{aligned} \tag{2}$$

where λ_1 and λ_2 must be either zero or one.[2] We then choose between DMU_1 and DMU_2 which both dominate DMU_o, the DMU to be evaluated, with 12 units recorded for each of its three outputs. To put this in the form of (1), we divide each constraint in (2) by the corresponding output of DMU_o and obtain,

$$\phi \leq \frac{14}{12}\lambda_1 + \frac{24}{12}\lambda_2, \qquad \phi \leq \frac{15}{12}\lambda_1 + \frac{14}{12}\lambda_2, \qquad \phi \leq \frac{14}{12}\lambda_1 + \frac{14}{12}\lambda_2, \qquad 1 = \lambda_1 + \lambda_2. \tag{3}$$

The measure defined by (1) is associated with alternate optima for (3) since the choices $\lambda_1^* = 1$ and $\lambda_2^* = 1$ both give

$$\phi_1^* = \phi_2^* = \frac{14}{12}, \tag{4}$$

as the measure of efficiency. This value of ϕ^* also gives the proportion in which all of DMU_o's outputs are to be augmented. With the choice of either DMU_1 or DMU_2 as the evaluator we thus find that DMU_o is inefficient—i.e., $\phi^* > 1$—and this same value of ϕ^* indicates that DMU_o should have produced 14 units in each of its three outputs.

Although each of these alternate optima yield the same value of ϕ^*, the two solutions are associated with quite different amounts of slack. This is dealt with by Tulkens *et al.* by simply listing the slacks but this is inadequate when alternate optima are present (as in the above example) if ϕ^* is to be used for ranking. In addition, simple listing of non-zero slacks is also questionable when alternate optima may be present. See the example in Bardhan *et al.* [4]. Nor is this the end of possible troubles for, as can be seen, we need to only replace the output values of 14 units by 12 units in (2) to obtain a value of $\phi^* = 1$ which accords DMU. The same measure of efficiency as is accorded to the two DMUs which dominate it.

Figure 1 below will help to illuminate what is involved. The solid line represents the graph of a step function referred to as the Free Disposal Hull. This FDH is derived from the observed values of y_1 and y_2 for each DMU via the following considerations. All points on or below the solid line to the left of R are dominated by the observations recorded at R as $y_1 = 8$, $y_2 = 2\frac{1}{2}$. This solid line is not extrapolated to the vertical axis, however, because Q with $y_1 = 4$ and $y_2 = 3$ is not dominated by R, or *vice versa* and so on.

Points Q and R, which form the vertices from which this FDH is generated, are obtained from paired comparisons in the first stage. These vertices correspond to the set D of non-dominated vectors used in (1). Relative to this set, the point P will have a value $\phi^* = \frac{3}{2}$

[2]The non-dominated DMUs assigned to D in the first pass of the algorithm are all given an efficiency measure of unity.

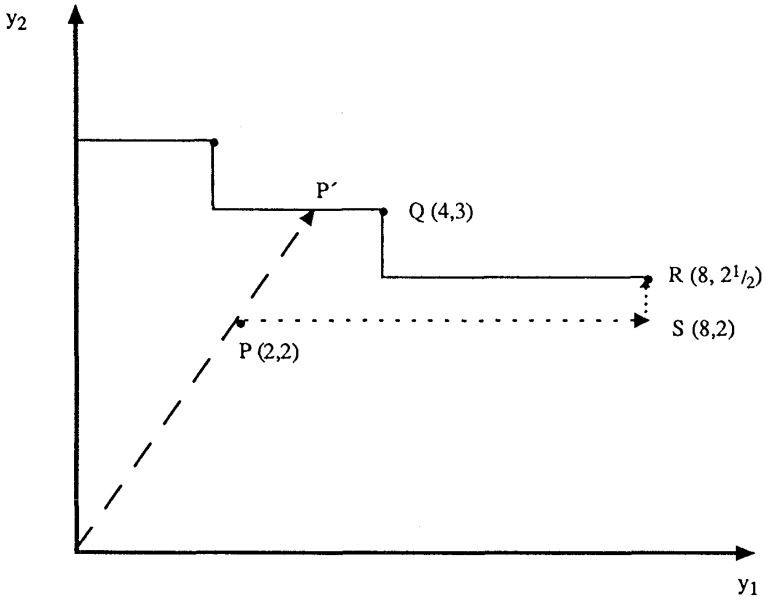

Figure 1: FDH Portrayal

and therefore be inefficient. This same value of $\phi^* = \frac{3}{2}$ is used to obtain $\frac{3}{2}(2,2) = (3,3)$, the coordinates of P′. This projection to P′ therefore results in a point which is on the Free Disposal Hull. However, the slack value of $s_1^* = 4 - 3 = 1$ is unaccounted for in this measure of efficiency. Under the assumption of "free disposal" this slack might be ignored but Tulkens and his colleagues generally proceed to at least list these values. However, they have been unable to include them in a more comprehensive measure—as Tulkens notes [23, p. 15]: "I doubt that seeking for a single numerical index of efficiency which includes the slacks would be a useful extension. Additional columns exhibiting the values of all slacks would be more informative". Such a simple listing creates difficulties with respect to the rankings that Tulkens and his associates use, however, and issues can also be raised with respect to alternate optima with differing amounts of slack while, finally, problems also arise because a value of $\theta^* = 1$ may be assigned to a dominated DMU.

3. MID and MED Measures

We approach these problems via the following model taken from Bardhan *et al.* [4],

$$\begin{aligned} \max \sum_{i=1}^{m} \frac{s_i^+}{x_{io}} &+ \sum_{r=1}^{s} \frac{s_r^-}{y_{ro}} \\ \text{subject to:} \quad x_{io} &= \sum_{j=1}^{n} x_{ij}\lambda_j + s_i^+, \qquad i = 1, \cdots, m, \\ y_{ro} &= \sum_{j=1}^{n} y_{rj}\lambda_j - s_r^-, \qquad r = 1, \cdots, s, \\ 1 &= \sum_{j=1}^{n} \lambda_j. \end{aligned} \tag{5}$$

We gain some perspective on this model by focusing on the case where DMU_k is used as the evaluator of DMU_o—as in the following expression for the i^{th} input and r^{th} output of

DMU_o,

$$\frac{s_i^{+*}}{x_{io}} + \frac{s_r^{-*}}{y_{ro}} = \frac{x_{io} - x_{ik}}{x_{io}} + \frac{y_{rk} - y_{ro}}{y_{ro}}. \tag{6}$$

The first term represents the input excess relative to the amount of input used and the second represent the output shortfall relative to the output produced by DMU_o compared to DMU_k. Both terms are stated in l_1 measure, as is true for all the other DMU_o to be evaluated in (5). It is therefore meaningful to sum these measures and interpret the result as a measure of relative distance from what is needed to arrive at the full DEA efficiency.

This same measure of relative distance is used for each DMU_j in (5). Full efficiency for any one of these DMU_j can be achieved only by traversing this distance. A ranking based on this measure of relative distance can therefore be used. There are, of course, other criteria that can be used for ranking such as total cost, total profitability, etc., but the above measure, however, is invariant to the measures used in the x_{io} and y_{ro}. See Bardhan *et al.* [4].

The rankings obtained from (5) in this fashion may differ from those obtained from (1). The latter yields a radial measure with a value of $1 \leq \phi^*$. A normed measure can also be derived for the inefficiency of DMU_o relative to the output of DMU_k by focusing only on the output terms in (6) and writing

$$1 + \frac{s_r^{-*}}{y_{ro}} = \frac{y_{rk}}{y_{ro}}.$$

To avoid possible values exceeding unity, we use the reciprocal of this expression to obtain

$$\frac{y_{ro}}{y_{ro} + s_r^{-*}} = \frac{y_{ro}}{y_{rk}}.$$

We then have a measure of relative efficiency for output r for DMU_o which cannot exceed unity and which may also be interpreted as the reciprocal of ϕ^* in (1). Moreover, we can average over all outputs to obtain

$$0 \leq \frac{\sum_{r=1}^{s} y_{ro}/y_{rk}}{s} \leq 1 \tag{7}$$

with unity attained if and only if DMU_o is fully efficient.

We illustrate with P in Figure 1 where we observe that (5) designates R rather than Q for its evaluation, as indicated by the dotted line going from P to S and then from S to R—to represent the l_1 distance measure that is used.[3] This gives $\frac{2}{8} = \frac{1}{4}$ as the relative measure of efficiency in the l_1 component for y_1 and $\frac{2}{2\frac{1}{2}} = \frac{4}{5}$ as the efficiency measure for y_2, also in l_1 measure, and the average of these two ratios is

$$\frac{\frac{1}{4} + \frac{4}{5}}{2} = \frac{21}{40}.$$

This provides the measure which was labeled as MED (=Measure of Efficiency Dominance) in Bardhan *et al.* [4] with a complementary measure of inefficiency given by

$$1 - \frac{21}{40} = \frac{19}{40},$$

[3] Also called the "city-block" metric in the operations research literature. See Appendix A in [6] for a discussion of this and other metrics.

which was referred to as MID (Measure of Inefficiency Dominance).

These measures are obtained from the DMU which is most dominant in this l_1 metric, as exhibited by the choice of R rather than Q in Figure 1. However, these measures do not result in a scalar which will project P into R. Except in the special case of fully efficient performance, such identifications will require recourse to the components in the MED or MID measures.

4. Russell Measures

C. A. K. Lovell [16, p. 29] has expressed the following concerns with some of the available measures,

> "I do not like the idea of aggregating slacks, and I like even less the idea of combining slacks with a radial efficiency score."

Possibly for these reasons,[4] he turned (with Fried and others) to a use of "Russell measures "[5] in a series of papers evaluating the performance of U.S. Credit Unions. We therefore investigate this approach and relate it to our preceding discussions via the following model

$$\begin{aligned} \max \sum_{r=1}^{s} \phi_r - \sum_{i=1}^{m} \theta_i & \\ \text{subject to} \quad \phi_r y_{ro} &\leq \sum_{j=1}^{n} y_{rj}\lambda_j, \qquad r = 1, \cdots, s, \\ \theta_i x_{io} &\geq \sum_{j=1}^{n} x_{ij}\lambda_j, \qquad i = 1, \cdots, m, \\ 1 &= \sum_{j=1}^{n} \lambda_j \end{aligned} \tag{8}$$

where we also require $\lambda_j \in \{0, 1\}$, $\forall j$, and add the $m + s$ additional conditions $0 \leq \theta_i \leq 1$, $1 \leq \phi_r$ to ensure that the requirements for dominance are satisfied. Unlike the situation for (5), however, it is now necessary to assume that all data are positive since the possibility of infinite solutions must otherwise be admitted because recourse to the Ali-Seiford translation discussed in Bardhan *et al.* [4] is not available for these ϕ_r and θ_i values.

The objective in the above formulation differs from the one in Färe, Grosskopf and Lovell [12] as well as the one in Färe and Lovell [13].[6] However, it suits our immediate purposes to treat these other formulations of the above objective later in this paper. We can then begin our comparisons here with the above efficiency measures and models by presenting an optimal solution to (8) in the following form

$$\phi_r^* y_{ro} = y_{rl}, \quad r = 1, \cdots, s \qquad \text{and} \qquad \theta_i^* x_{io} = x_{il}, \quad i = 1, \cdots, m, \tag{9}$$

when $\lambda_l^* = 1$ is an optimal choice. Similarly we write

$$y_{ro} = y_{rk} - s_r^{-*}, \quad r = 1, \cdots, s \qquad \text{and} \qquad x_{io} = x_{ik} + s_i^{+*}, \quad i = 1, \cdots, m, \tag{10}$$

when $\lambda_k^* = 1$ is an optimal choice for (5).

[4] In a subsequent paper, Lovell and Pastor [17] develop a measure which is analogous to the ones described in the concluding sections of Bardhan *et al.* [4].

[5] See Färe and Lovell [13]. See also Russell [21].

[6] See also Russell [21].

We now also use (9) to define new non-negative variables

$$\begin{aligned} s_r^- &= \phi_r^* y_{ro} - y_{ro} = y_{rl} - y_{ro}, \quad & r = 1, \cdots, s, \\ s_i^+ &= x_{io} - \theta_i^* x_{io} = x_{io} - x_{il}, \quad & i = 1, \cdots, m, \end{aligned} \tag{11}$$

where the terms on the right are obtained from (9). We also use (10) to introduce the new variables

$$\begin{aligned} \phi_r y_{ro} &= y_{ro} + s_r^{-*} = y_{rk}, \quad & r = 1, \cdots, s, \\ \theta_i x_{io} &= x_{io} - s_i^{+*} = x_{ik}, \quad & i = 1, \cdots, m, \end{aligned} \tag{12}$$

where $0 \leq \theta_i \leq 1$, $1 \leq \phi_r$ so (8) is satisfied and, similarly, the expressions (11) satisfy (5).

From (11) we see that these s_r^- and s_i^+ are associated with $\lambda_l = 1$ as a solution to (5), for which $\lambda_k^* = 1$ is optimal, while from (12) we see that these ϕ_r and θ_i are associated with $\lambda_k = 1$ for which $\lambda_l^* = 1$ is optimal. Hence suppose we could have for (8),

$$\sum_{r=1}^{s} \phi_r^* - \sum_{i=1}^{m} \theta_i^* > \sum_{r=1}^{s} \phi_r - \sum_{i=1}^{m} \theta_i \tag{13}$$

with the choices $\lambda_l^* = 1$ and $\lambda_k = 1$. Using (11) and (12), however, this would give,

$$\sum_{r=1}^{s} \frac{s_r^-}{y_{ro}} + \sum_{i=1}^{m} \frac{s_i^+}{x_{io}} > \sum_{r=1}^{s} \frac{s_r^{-*}}{y_{ro}} + \sum_{i=1}^{m} \frac{s_i^{+*}}{x_{io}} \tag{14}$$

which contradicts the optimality assumed for $\lambda_k^* = 1$ in (5). Turning in the opposite direction, assume that the choices $\lambda_k^* = 1$ and $\lambda_l = 1$ give

$$\sum_{r=1}^{s} \frac{s_r^{-*}}{y_{ro}} + \sum_{i=1}^{m} \frac{s_i^{+*}}{x_{io}} > \sum_{r=1}^{s} \frac{s_r^-}{y_{ro}} + \sum_{i=1}^{m} \frac{s_i^+}{x_{io}} \tag{15}$$

for (5). However, again, using (11) and (12) we obtain

$$\sum_{r=1}^{s} \phi_r - \sum_{i=1}^{m} \theta_i > \sum_{r=1}^{s} \phi_r^* - \sum_{i=1}^{m} \theta_i^* \tag{16}$$

which contradicts the optimality assumed for λ_l^* in (8). We therefore have the following

Theorem 1. *A solution to* (8) *is optimal if and only if it is optimal for* (5).

By virtue of this theorem we can write

$$\phi_r^* y_{ro} - y_{ro} = s_r^{-*}, \quad r = 1, \cdots, s \qquad \text{and} \qquad x_{io} - \theta_i^* x_{io} = s_i^{+*}, \quad i = 1, \cdots, m, \tag{17}$$

where the slacks and the ϕ_r^*, θ_i^* are applicable to the same optimal choice—either λ_l^* or λ_k^*, as used above—when alternate optima are present. As (17) now makes clear, the two objectives in (8) and (5) are complementary. One minimizes efficiency accomplishments and the other maximizes inefficiencies that may be present in evaluating the performance of DMU_o.

With (17) in hand, we can do more than has heretofore been done with Russell Measures. Consider, for instance, how one might treat "nondiscretionary" variables. Various approaches are described in Banker *et al.* [2, pp. 152–153]. Here we focus only on the Banker-Morey [3] approach (or, rather, its extension into the present case) for added insight into the assumption of "free disposal". To adapt this approach to a use of Russell measures, one simply assigns values of unity to the pertinent ϕ_r or θ_i and carries these values into whatever aggregate efficiency measure is used. As described in Cooper *et al.* [10], the non-zero slacks associated with these ϕ_r or θ_i are then interpretable as "free goods" which do not enter into the overall efficiency scores except insofar as they can improve the values of other ϕ_r^* or θ_i^* in the efficiency scores. Turning to (5), on the other hand, we recall that the slack values in the objective (but not in the constraints) enter into our efficiency scores. The zeros associated with (17) under a choice of θ_i^* or $\phi_r^* = 1$ are then interpreted to mean that this is the value to be used in MED or the other ratings and evaluations discussed in [4]. This is achieved by assigning a zero coefficient to these slacks in the objective. Hence, in addition to being a free good in the constraints, non-zero slacks are also to be regarded as not involving any cost in their disposal in a manner analogous to the case when free disposal is assumed.

Putting this all together, this study can see that we have generalized the Banker-Morey treatment of non-discretionary slacks to cases in which some (but not all) of the inputs and outputs are non-discretionary. This applies to the BCC and CCR models used by Banker and Morey and extends to other models as well. Indeed, as described in Banker *et al.* [2], even more general treatments are available to accommodate situations involving "floors" and "ceilings" with ranges for which some of the variables are discretionary and other ranges when they are nondiscretionary.

We now make our promised return to the objective used in Färe *et al.* [12], which has the form

$$\min \frac{\sum_{i=1}^{m} \theta_i + \sum_{r=1}^{s} \frac{1}{\phi_r}}{m+s} = \frac{\frac{m\sum_{i=1}^{m}\theta_i^*}{m} + \frac{s\sum_{r=1}^{s} 1/\phi_r^*}{s}}{m+s}. \tag{18}$$

This objective provides a measure that is (a) bounded by unity and (b) attains this bound if and only if DMU_o is efficient. As seen on the right of (18), the FGL objective is to minimize a weighted average of arithmetic and harmonic means. It is difficult to interpret and awkward to use so we replaced it by the simpler objective in (8), and we now justify this choice by the following,

Theorem 2. *An optimal solution to* (8) *is also optimal when the objective is replaced by* (18).

To prove this theorem, let θ_i^*, ϕ_r^* be optimal for (8) and let $\hat{\theta}_i, \hat{\phi}_r$ be optimal when the objective in (8) is replaced with (18). Now assume that we could have

$$\sum_{i=1}^{m} \hat{\theta}_i + \sum_{r=1}^{s} \frac{1}{\hat{\phi}_r} < \sum_{i=1}^{m} \theta_i^* + \sum_{r=1}^{s} \frac{1}{\phi_r^*} \tag{19}$$

and therefore

$$\sum_{i=1}^{m} \hat{\theta}_i - \sum_{r=1}^{s} \phi_r^* + \sum_{r=1}^{s} \frac{1}{\hat{\phi}_r} < \sum_{i=1}^{m} \theta_i^* - \sum_{r=1}^{s} \phi_r^* + \sum_{r=1}^{s} \frac{1}{\phi_r^*}, \tag{20}$$

which gives

$$\sum_{i=1}^{m} \hat{\theta}_i - \sum_{r=1}^{s} \left(\phi_r^* + \frac{1}{\phi_r^*} - \frac{1}{\hat{\phi}_r}\right) < \sum_{i=1}^{m} \theta_i^* - \sum_{r=1}^{s} \phi_r^*. \tag{21}$$

But $1/\hat{\phi}_r > 0$, $\forall r$, so, *a fortiori*,

$$\sum_{i=1}^{m} \hat{\theta}_i - \sum_{r=1}^{s} \tilde{\phi}_r < \sum_{i=1}^{m} \theta_i^* - \sum_{r=1}^{s} \phi_r^* \tag{22}$$

where $\tilde{\phi}_r := \phi_r^* + 1/\phi_r^* \geq 1$, $\forall r$. All constraints being satisfied, we have a contradiction to the assumtion that these θ_i^* and ϕ_r^* were optimal for (8). To avoid this contradiction, we must replace (19) with

$$\sum_{i=1}^{m} \theta_r^* + \sum_{r=1}^{s} \frac{1}{\phi_r^*} \leq \sum_{i=1}^{m} \hat{\theta}_i + \sum_{r=1}^{s} \frac{1}{\hat{\phi}_r}. \tag{23}$$

However, $\hat{\theta}_i, \hat{\phi}_r$ were assumed to be optimal under the objective (18). Hence, equality must hold and so, as the theorem asserts, we have

$$\sum_{i=1}^{m} \theta_i^* + \sum_{r=1}^{s} \frac{1}{\phi_r^*} = \sum_{i=1}^{m} \hat{\theta}_i + \sum_{r=1}^{s} \frac{1}{\hat{\phi}_r}. \tag{24}$$

By virtue of the above two theorems, we also have the following,

Corollary 1. *If $\lambda_k^* = 1$ is optimal for (5), then it is also optimal for (8) with (18) as its objective.*

We are now in position to reap several advantages from the preceding developments. As already noted, we can use (17) to move back and forth, or we may proceed directly by setting $\theta_i^* = x_{ik}/x_{io}$ and $\phi_r^* = y_{rk}/y_{ro}$, etc. In any case, (8), is easier to compute and simpler to interpret.[7] It is also more general. For instance, it eliminates the need for the $m+s$ additional constraints required for each DMU_o to be evaluated in (6) to insure that dominance is not violated, and there is no need to assume that all inputs and outputs are positive for every DMU, as is the case for (8). Finally, the use of (5) puts us in contact with the even more general "additive model" of DEA with properties which are related to still other DEA models, as described in Ahn, Charnes and Cooper [1], and this makes it possible to exploit features and extension of these models—which is what we did when we showed how to handle non-discretionary inputs and outputs in our discussion of (17).

To take full advantage of what has just been said, we complete the present development by removing the restriction to integer solutions so that we can extend our results to the general additive model as first given in Charnes *et al.* [9]. With this restriction removed from both (5) and (8), we then have

Corollary 2. *Let λ_k represent a vector with non-negative components formed from a solution to (5). If λ_k^* is optimal for (5) with the integrality restriction removed, then it is also optimal for (8) using the objective (18) with the integrality restriction removed.*

A simple proof can be obtained by writing

$$\begin{aligned} x_{io} &= \sum_{j=1}^{n} x_{ij}\lambda_j^* + s_i^{+*} \geq \sum_{j=1}^{n} x_{ij}\lambda_j^* = x_{ik}^*, \qquad i = 1, \cdots, m, \\ y_{ro} &= \sum_{j=1}^{n} y_{rj}\lambda_j^* - s_r^{-*} \leq \sum_{j=1}^{n} y_{rj}\lambda_j^* = y_{rk}^*, \qquad r = 1, \cdots, s. \end{aligned} \tag{25}$$

We then have

$$x_{io} - x_{ik}^* = s_i^{+*}, \quad i = 1, \cdots, m \qquad \text{and} \qquad y_{rk}^* - y_{ro} = s_r^{-*}, \qquad r = 1, \cdots, s. \tag{26}$$

[7] Färe, Grosskopf and Lovell [13] also simplify matters for some applications by using an input oriented (only) objective which minimizes the θ_i and an output (only) oriented objective which maximizes the ϕ_r. Reference to (17), however, shows that this is equivalent to maximizing only the input slacks or the output slacks, which means that these approaches are suited only to very special situations.

We also have

$$\theta_i^* x_{io} = x_{ik}^*, \quad i = 1, \cdots, m \qquad \text{and} \qquad \phi_r^* y_{ro}^* = y_{rk}^*, \quad r = 1, \cdots, s. \tag{27}$$

The remainder of the proof follows the same route as before. That is, we show that these θ_i^* and ϕ_r^* are optimal for (8) and that they remain optimal when the objective is replaced by (18). We can then conclude this extension of the preceding developments by noting that the same route can be followed to either of the two classes of additive models by omitting or imposing a convexity constraint that the solutions must satisfy.

The preceding developments are not intended to mean that further research on uses of Russell measures should be abandoned. Indeed, such research can build on what has already been accomplished. The choice of suitable measures provides an example since the measure prescribed in (18) is not the only possibility. Others might be explored as follows.

The component sums in the objective for (8) satisfy

$$\frac{\sum_{r=1}^{m} \phi_r}{s} \geq 1, \qquad \frac{\sum_{i=1}^{m} \theta_i}{m} \leq 1, \tag{28}$$

so, combining the two, we have

$$\frac{s}{m} \frac{\sum_{r=1}^{m} \theta_i}{\sum_{r=1}^{s} \phi_r} \leq 1, \tag{29}$$

and this ratio is equal to unity in these Russell measures if and only if DMU_o is efficient.

The above measure, which compares average input to average output efficiencies, may be used in the same manner as MED. That is, its value may be obtained from optimal solutions to problems with suitably formulated objectives like those incorporated in the preceding article. Alternatively, the formulation in (29) may be used in place of the objective in (8). The Charnes-Cooper [7] transformation of fractional programming might then be used (as in other DEA models) to obtain a computationally tractable model.

There are, of course, other prospects for research in Russell Measures but such research in either FDH or Russell measures is likely to be most productive if it builds on what has already accomplished by Tulkens and Lovell and their associates. See references.

5. Conclusion and Further Extensions

The present paper has utilized the special version of the additive model of DEA introduced in Bardhan *et al.* [4] in order to study, in a unified manner, other major approaches which have attempted to deal with efficiency dominance such as the FDH approach of Tulkens *et al.* and the Russell measures used by Lovell *et al.* In addition to highlighting some of the shortcomings of these approaches relative to our additive models and measures, we have also suggested opportunities for additional research and extensions that could relate the developments here to other topics such as goal programming and multiple objective optimization.

Some parts of the literature on goal programming as well as DEA have pointed to any use of "unweighted" objectives as a "huge disadvantage". In such discussions, however, one needs to distinguish between the slacks in the constraints and in the objective and this has not always been noticed in these criticisms. The slacks in the objective are to be regarded as "equally weighted", of course, if they are to be added and since the FDH and Russell Measure approach use this same assumption, it is of interest to put the matter in better perspective for possible uses in DEA.

A variety of possible weights have been presented in Bardhan *et al.* [4] and examples given on how they might be employed. However, one needs to bear in mind uses of DEA as a "data-based approach" for discovering "what is happening", "where it is happening" and in "what amounts". The situation is akin to that encountered in statistical regressions which are similarly "data oriented". One may differentially weight the observations or reorient the objectives in estimating statistical regressions,[8] but, of course, this is not done very often because of accompanying risks of concealing what is really in the data. Indeed, the very common use of such "unweighted" regressions even carries over into stochastic-frontier regressions which (like DEA) are oriented toward performance evaluations.

Available methods of regression estimation often take a form in which the data are stated in terms of deviations from their means divided by their standard deviations. In this way, the effects of differences in units of measure are eliminated from influencing the parameter estimates. Use of the x_{io} and y_{ro} in the objective of (5) has a similar purpose but, of course, this is not the only way this can be done—and, indeed, one can use the standard deviation of each input or output as a denominator for the corresponding slack, just as is done in statistical regressions.[9]

"Assurance regions" and "cone-ratio envelopment" approaches also offer possibilities which recognize that it may be preferable to establish upper or lower bounds on admissible values in place of choosing exact weights *ab initio*. See Cooper *et al.* [10]. Some of the onerous conditions and arbitrary choices encountered in choosing weights might thereby be replaced or ameliorated by allowing DEA to choose exact weights within the limits of such bounds on variables in the dual problems when they are available. These duality relations are, of course, not available when integrality conditions are imposed on the variable choices, but we cannot pursue that topic here. It should nevertheless be possible to develop suitable bounding techniques and this would also be in keeping with DEA as a new approach to choosing wights—which are allowed to vary from one DMU to another in accordance with the principle that the choice of weights should be best, or most pertinent, for *each* DMU_o to be evaluated.

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[8] Cf. Manski [19]

[9] See Romero [20] where a variety of such approaches are discussed in the context of goal programming. See also Lovell and Pastor [17]. The very recent paper by Lovell and Pastor [18] utilizes standard deviations in place of the denominator in (5) to obtain an additive model which is both "units invariant" and "translations invariant".

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