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BOND PORTFOLIO OPTIMIZATION PROBLEMS AND THEIR APPLICATIONS TO INDEX TRACKING: A PARTIAL OPTIMIZATION APPROACH

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Abstract We will discuss exact and efficient parametric simplex algorithms for solving a class of nonconvex minimization problems associated with bond portfolio optimization models which one of authors proposed in the late 1980's. We will show that globally optimal solutions of both total and partial optimization problems can now be calculated on a real time basis. Also we will present some computational results of a partial optimization model applied to a tracking of an index portfolio.

1 Introduction

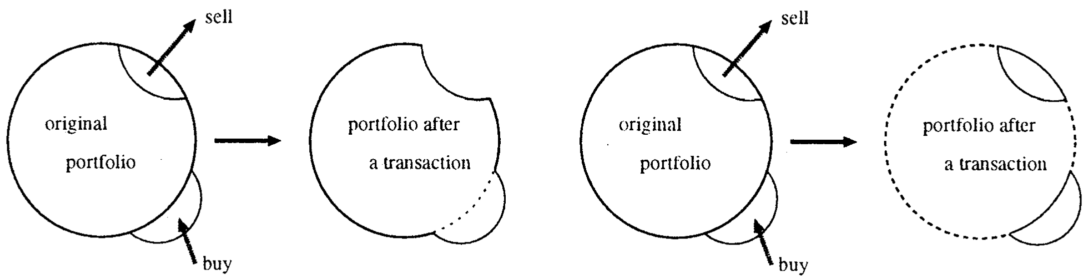
In the late 1980's, one of the authors proposed a pair of bond portfolio optimization models taking into account a variety of objectives and constraints associated with the dealing activities of bond managers of institutional investors.[7].

Of the two models, one is the "total" optimization model which intends to optimize (either maximize or minimize) a certain index associated with a total portfolio after a transaction (Fig 1(a)). This model was formulated as a bilinear fractional programming problem, whose good locally optimal solution (which may or may not be a globally optimal solution) can be calculated reasonably fast by a heuristic algorithm based upon the simplex method for linear programming problems. Hence, this model has been used in practice by several institutional investors both in Japan and in U.S..

The "partial" optimization model, on the other hand tries to optimize the difference of a given index associated with the bundle of assets purchased from the market and sold to the market (Fig 1(b)). Though odd looking from a viewpoint of standard approach in portfolio optimization, partial optimization is often considered more attractive from the viewpoint of practitioners as evidenced by a series of interviews with a number of bond managers of institutional investors. The primary reason is that only a small fraction, typically less than 10% of the total assets owned by an investor is sold and/or purchased in a typical transaction.

Therefore the resulting improvement of the objective function before and after a transaction in terms of total optimization model is very small since at least 90% of the assets remains the same, while the improvement in terms of partial optimization model can be very significant. In fact, the competence of a bond manager is appreciated more in terms of partial optimization than total optimization.

Unfortunately, however this partial optimization model results in a difficult nonconvex minimization problem, for which the author could not devise an exact and efficient algorithm in [7] and thus this model was not used in practice until recently.



(a) Total Optimization Model (b) Partial Optimization Model

Figure 1: Bond Portofolio Optimization Models

The purpose of this paper is to demonstrate that this nonconvex minimization problem resulting from the partial optimization model can be solved to global optimality very fast as a result of recent developments in global optimization methods [6]. In fact they can be solved in approximately the same amount of computation time as that needed for solving a linear programming problem of the same size. This means that we can solve both total and partial optimization models on a real time basis.

The readers will find that the model developed in this paper is different from the classical and standard bond portfolio optimization models in which the portfolio is constructed so as to meet the (fixed) future cash flows with minimal cost under the assumption that the investor holds the portfolio until maturity. (The readers are referred to the standard text such as Elton–Gruber [4] and Fong–Fabozzi [5], for conventional models.

To the contrary, the model developed in this paper is concerned with the dealing of bonds where the portfolio manager tries to hold a portfolio with maximal performance index in an effort to maximize his or her profit. We believe that this kind of model would play more and more important roles in bond dealing business and bond management associated with asset allocation.

In the next section, we will explain several basic notions associated with bond portfolio management. In section 3, we will introduce total and partial optimization models. Section 4 will be devoted to exact and fast algorithms for solving these models which is based upon a variant of parametric simplex algorithm [9] for solving a class of global optimization problems. Finally in Section 5, we show some preliminary results of our numerical experiments using the historical data of government bonds available in the market. Here we will concentrate on the application of partial optimization model to index tracking.

2 Bond Portfolio Optimization Models

Let us assume that an investor holds u_j units of bonds $B_j (j = 1, \ldots, n)$. Associated with B_j are four basic attributes[4,5,7].

c_j: coupon to be paid at a fixed rate (yen/bond/year)
f_j: principal value to be refunded at maturity (yen/bond)
p_j: unit transaction price in the market (yen/bond)
t_j: maturity (number of years until the principal value is refunded)

It is well known that if the interest rate i remains constant throughout the life of a bond, then the theoretical price p_j of B_j is given by the following formula:

$$p_j = \sum_{t=1}^{t_j} \frac{c_j}{(1+i)^t} + \frac{f_j}{(1+i)^{t_j}} \tag{1}$$

To emphasize the dependence of p_j on the interest rate i. we employ an alternative notation $p_j(i)$ in the sequel. The duration and convexity of B_j are then defined as follows:

$$d_j(i) = -p_j'(i)/p(i) \tag{2}$$

$$C_j(i) = p_j''(i)/p(i) \tag{3}$$

In reality however, the interest rate varies from period to period. Under such circumstances, the theoretical price of bond (1) has to be replaced by taking its fluctuation into account. Let $T = \max_{1\leq j\leq n} t_j$ and let c_{jt} be the cash flow from B_j during period t. Then the theoretical bond price is given by

$$p_j(i_1, \ldots, i_{t_j}) = \frac{c_{j1}}{(1+i_1)} + \frac{c_{j2}}{(1+i_1)(1+i_2)} + \cdots + \frac{c_{jT}}{(1+i_1)(1+i_2)\cdots(1+i_T)} \tag{1'}$$

where i_t is the interest rate to be applied during period t $(t = 1, \ldots, T)$. Also the duration d_j and C_j has to be replaced by

$$d_j(i_1, \ldots, i_T) = -p_j'(i_1, \ldots, i_T)/p_j(i_1, \ldots, i_T) \tag{2'}$$

$$C_j(i_1, \ldots, i_T) = p_j''(i_1, \ldots, i_T)/p_j(i_1, \ldots, i_T) \tag{3'}$$

where

$$p_j'(i_1, \ldots, i_T) = \lim_{\Delta i - 0} \frac{p_j(i_1+\Delta i, \ldots, i_T+\Delta i) - p_j(i_1, \ldots, i_T)}{\Delta i}$$

$$p_j''(i_1, \ldots, i_T) = \lim_{\Delta i - 0} \frac{p_j'(i_1+\Delta i, \ldots, i_T+\Delta i) - p_j'(i_1, \ldots, i_T)}{\Delta i}$$

Let us note that the term structure $(i_1, \ldots, i_T)$ can be calculated very fast by using several efficient methods including[8].

Bond managers of institutional investors, when evaluating a portfolio $u = (u_1, \ldots, u_n)$ take into account such performance indices as:

(a) average unit price $\pi(u)$

$$\pi(u) = \sum_{j=1}^{n} p_j u_j / \sum_{j=1}^{n} u_j \tag{4}$$

(b) average direct yield (average coupon rate) $c(u)$

$$c(u) = \sum_{j=1}^{n} c_j u_j / \sum_{j=1}^{n} u_j \tag{5}$$

(c) average maturity $t(u)$

$$t(u) = \sum_{j=1}^{n} t_j u_j / \sum_{j=1}^{n} u_j \tag{6}$$

(d) average duration $d(u)$

$$d(u) = \sum_{j=1}^{n} d_j p_j u_j / \sum_{j=1}^{n} p_j u_j \tag{7}$$

(e) average convexity $C(u)$

$$C(u) = \sum_{j=1}^{n} C_j p_j u_j / \sum_{j=1}^{n} p_j u_j \tag{8}$$

Most fund managers prefer to have a portfolio with larger average unit price and average direct yield. Also many, if not all managers, prefer to have a shorter average maturity, because a larger risk is associated with a portfolio with a longer maturity. Further they prefer to have average duration and average convexity to remain within a certain interval to avoid risk associated with the fluctuation of interest rates.

In addition to five indices listed above, most fund managers want to have larger average yield to maturity, to be defined below. The traditional definition of yield to maturity ν_j of a bond B_j is the smallest nonnegative solution of the following nonlinear equation:

$$p_j(1+\xi)^{t_j} = \sum_{t=1}^{t_j} c_j(1+\xi)^t + f_j$$

where p_j is the price of B_j in the market. Instead, we will employ an alternative (and more meaningful) definition:

$$p_j(1+\nu_j)^{t_j} = \sum_{t=1}^{t_j} c_j(1+i_1)\cdots(1+i_t) + f_j \tag{9}$$

The left hand side is the total amount of cash obtained by saving p_j at the compound annual interest rate ν_j until maturity while the right hand side is the total amount of cash obtained by saving all coupon payments under the interest rate structure i_t $(t = 1, \ldots, t_j)$ plus the principal value. The average effective yield $\nu(u)$ of a portfolio u is defined as follows:

(f) Average effective yield $\nu(u)$

$$\nu(u) = \sum_{j=1}^{n} \nu_j p_j u_j / \sum_{j=1}^{n} p_j u_j \tag{10}$$

It is easy to see that the trader will get the effective yield $\nu(u)$ per period by holding a portfolio $(u_1, \ldots, u_n)$. Thus he or she prefers to have larger average effective yield.

3 Mathematical Description of Bond Portfolio Optimization Models.

Let us assume again that an investor holds u_j units of $B_j, (j = 1, \ldots, n)$ out of which n_1 are chosen as candidates for sale in the market. In a typical situation, n is a few hundred and n_1 is less than one hundred. Also let us assume that U_k units of bond $B'_k(k = 1, \ldots, n_2)$ are available in the market . The bond trader sells $B_j(j = 1, \ldots, n_1)$ to the market and purchases B'_k from the market to improve a portfolio. In a typical situation, he chooses a particular index out of (a)~(f) explained in Section 2 and tries to optimize (either maximize or minimize) it while putting others into constraints by specifying the least desirable level for each of them.

Let x_j = amount of B_j to be sold, $j = 1, \ldots, n_1$ (11)

X_k = amount of B'_k to be purchased, $k = 1, \ldots, n_2$ (12)

These variables have to satisfy the conditions:

$$0 \le x_j \le u_j, \; j = 1, \ldots, n_1 \tag{13}$$

$$0 \le X_k \le U_k, \; k = 1, \ldots, n_2 \tag{14}$$

In addition to these, several constraints are associated with transactions such as the restrictions on the total amount of bonds to be sold and/or purchased, total amount of profit and/or loss and the total amount of liquidation, all of which are represented as a linear function of x_j 's and X_k 's.

Let us note that all of indices (a)~(f) as well as the constraints above belong to a class of linear or bilinear fractional functions of u if the price p_j 's are constants. Unfortunately, however some of p_j 's are variables rather than constants in a typical transaction environment. When a bond trader simultaneously sells and buys bonds through a same agent, he is entitled to choose the price of each bond within a certain interval provided the agent agrees to this transaction. The reason why a trader agrees to sell a bond B_j for the price lower than the market price p_j is that he wants to reduce the amount of profit (difference between selling price p_j and the book price p_{j_0}) out of this transaction, thereby reduce the amount of tax. The agent may instead agree to sell another bond, B'_k for the price lower than the market price P_k to compensate a loss of the trader. The transaction price is, however not permitted to deviate more than a few percent from the reference price due to a regulation imposed by the government.

Let y_i and Y_k be the unit transaction price of B_j and B'_k, respectively. They have to satisfy

$$(1-\lambda_j)p_j \le y_j \le (1+\lambda_j)p_j,\ j=1,\ldots,n_1 \tag{15}$$

$$(1-\lambda'_k)P_k \le Y_k \le (1+\lambda'_k)P_k,\ k=1,\ldots,n_2 \tag{16}$$

where p_j and P_k are the price of B_j and B'_k , respectively in the market and λ_j and λ'_k are constants called "price adjustment coefficients". Thus a generic total optimization model [7] can be formulated as follows:

$$(P)\quad \begin{array}{ll} \text{maximize} & \dfrac{\sigma_0 - \sum_{j=1}^{n_1}(q_j+q'_j y_j)x_j + \sum_{k=1}^{n_2}(Q_k+Q'_k Y_k)X_k}{\pi_0 - \sum_{j=1}^{n_1}(r_j+r'_j y_j)x_j + \sum_{k=1}^{n_2}(R_k+R'_k Y_k)X_k} \\ \text{subject to} & \beta_i \le \dfrac{\phi_0 - \sum_{j=1}^{n_1}(f_{ij}+f'_{ij}y_j)x_j + \sum_{k=1}^{n_2}(F_{ik}+F'_{ik}Y_k)X_k}{\psi_0 - \sum_{j=1}^{n_1}(d_{ij}+d'_{ij}y_j)x_j + \sum_{k=1}^{n_2}(D_{ik}+D'_{ik}Y_k)X_k} \le \alpha_i, \\ & \qquad i=1,\ldots,m_1 \\ & \sum_{j=1}^{n_1} a_{lj}x_j + \sum_{k=1}^{n_2} a'_{lk}X_k \le a_{l0},\ l=1,\ldots,m_2 \\ & 0\le x_j\le u_j,\ y_j^0\le y_j\le y_j^1,\ j=1,\ldots,n_1 \\ & 0\le X_k\le U_k,\ Y_k^0\le Y_k\le Y_k^1,\ k=1,\ldots,n_2 \end{array} \tag{17}$$

This model would serve as a reference model when a significant portion, say more than one third of the assets owned by an investor is subject to sale.

Let us note that we can assume without loss of generality that the divisors of the fractional terms of the objective function and constraints are positive for all solutions satisfying other constraints. Therefore the problem (P) is equivalent to the following bilinear fractional programming problem:

$$\begin{array}{ll} \text{maximize} & \dfrac{\sigma_0 - \sum_{j=1}^{n_1}(q_j+q'_j y_j)x_j + \sum_{k=1}^{n_2}(Q_k+Q'_k Y_k)X_k}{\pi_0 - \sum_{j=1}^{n_1}(r_j+r'_j y_j)x_j + \sum_{k=1}^{n_2}(R_k+R'_k Y_k)X_k} \\ \text{subject to} & \tilde{\beta}_i \le \sum_{j=1}^{n_1}(g_{ij}+g'_{ij}y_j)x_j + \sum_{k=1}^{n_2}(G_{ik}+G'_{ik}Y_k)X_k \le \tilde{\alpha}_i, i=1,\ldots,m_1 \\ & \sum_{j=1}^{n_1} a_{lj}x_j + \sum_{k=1}^{n_2} a'_{lk}X_k \le a_{l0},\ l=1,\ldots,m_2 \\ & 0\le x_j\le u_j,\ y_j^0\le y_j\le y_j^1,\ j=1,\ldots,n_1 \\ & 0\le X_k\le U_k,\ Y_k^0\le Y_k\le Y_k^1,\ k=1,\ldots,n_2 \end{array} \tag{18}$$

Let us now turn to the partial optimization model, the main topic of this paper in which the difference of a given objective, say O_1 associated with the bundle of assets $(X_1, \dots, X_{n_2})$ purchased from the market and $(x_1, \dots, x_{n_1})$ sold to the market subject to the same constraints as (18) (See Fig 1(b)). This means that we evaluate the objective and constraints relative to the total portfolio owned by an investor after the transaction except the objective O_1 .

The problem can now be formulated as follows:

$$\begin{array}{ll} \text{maximize} & \dfrac{\sum_{k=1}^{n_2}(Q_k + Q'_k Y_k)X_k}{\sum_{k=1}^{n_2}(R_k + R'_k Y_k)X_k} - \dfrac{\sum_{j=1}^{n_1}(q_j + q'_j y_j)x_j}{\sum_{j=1}^{n_1}(r_j + r'_j y_j)x_j} \\ \text{subject to} & \tilde{\beta}_i \le \sum_{k=1}^{n_2}(G_{ik} + G'_{ik}Y_k)X_k + \sum_{j=1}^{n_1}(g_{ij} + g'_{ij}y_j)x_j \le \tilde{\alpha}_i,\ i = 1, \dots, m_1 \\ & \sum_{j=1}^{n_1} a_{lj}x_j + \sum_{k=1}^{n_2} a'_{lk}X_k \le a_{l0},\ l = 1, \dots, m_2 \\ & 0 \le x_j \le u_j,\ y_j^0 \le y_j \le y_j^1,\ j = 1, \dots, n_1 \\ & 0 \le X_k \le U_k,\ Y_k^0 \le Y_k \le Y_k^1,\ k = 1, \dots, n_2 \end{array} \tag{19}$$

where as before, we will assume that the divisors of the objective functions are positive for all feasible solutions. Also, we assume that the divisors in the objective function are of similar magnitude, so that it is meaningful to compare the difference. A typical example is the one explained is Section 5 in which the variables $(x_1, \dots, x_{n_1})$ and $(X_1, \dots, X_{n_2})$ are constrained so that

$$\sum_{k=1}^{n_2} Y_k X_k = \sum_{j=1}^{n_1} y_j x_j + C \tag{20}$$

This means that the total amount of cash spent for purchasing the portfolio $(X_1, \dots, X_{n_2})$ is the total amount of cash from the sale of the portfolio $(x_1, \dots, x_{n_1})$ plus the income C from the coupon payment during the given period. It then follows that divisors of performance index (a)~(f) associated with portfolios x and X are of the similar magnitude.

4 Algorithms for Solving Total and Partial optimization Models

Let us now discuss the algorithms for calculating globally optimal solutions of the problems (18) and (19). The first step to introduce auxiliary variables:

$$z_j = y_j x_j,\ j = 1, \dots, n_1 \tag{21}$$

$$Z_k = Y_k X_k,\ k = 1, \dots, n_2 \tag{22}$$

Then the constraints $y_j^0 \le y_j \le y_j^1$ and $Y_k^0 \le Y_k \le Y_k^1$ are equivalent to the following conditions:

$$y_j^0 x_j \le z_j \le y_j^1 x_j,\ j = 1, \dots, n_1 \tag{23}$$

$$Y_k^0 X_k \le Z_k \le Y_k^1 X_k,\ k = 1, \dots, n_2 \tag{24}$$

Thus the problems (18) is reformulated as a linear fractional programming problem:

$$\left|\begin{array}{ll} \text{maximize} & \dfrac{\sigma_0 - \sum_{j=1}^{n_1}(q_j x_j + q'_j z_j) + \sum_{k=1}^{n_2}(Q_k X_k + Q'_k Z_k)}{\pi_0 - \sum_{j=1}^{n_1}(r_j x_j + r'_j z_j) + \sum_{k=1}^{n_2}(R_k X_k + R'_k Z_k)} \\ \text{subject to} & \tilde{\beta}_i \le \sum_{j=1}^{n_1}(g_{ij} x_j + g'_{ij} z_j) + \sum_{k=1}^{n_2}(G_{ik} X_k + G'_{ik} Z_k) \le \tilde{\alpha}_i,\ i = 1,\ldots,m_1 \\ & \sum_{j=1}^{n_1} a_{lj} x_j + \sum_{k=1}^{n_2} a'_{lk} X_k \le a_{l0},\ l = 1,\ldots,m_2 \\ & 0 \le x_j \le u_j,\ y_j^0 x_j \le z_j \le y_j^1 x_j,\ j = 1,\ldots,n_1 \\ & 0 \le X_k \le U_k,\ Y_k^0 X_k \le Z_k \le Y_k^1 X_k,\ k = 1,\ldots,n_2 \end{array}\right. \tag{25}$$

which can be solved by standard methods [2]. Also, the problem (19) is reformulated as follows:

$$\left|\begin{array}{ll} \text{maximize} & \dfrac{\sum_{k=1}^{n_2}(Q_k X_k + Q'_k Z_k)}{\sum_{k=1}^{n_2}(R_k X_k + R'_k Z_k)} - \dfrac{\sum_{j=1}^{n_1}(q_j x_k + q'_j z_j)}{\sum_{j=1}^{n_1}(r_j x_j + r'_j z_j)} \\ \text{subject to} & \tilde{\beta}_i \le \sum_{j=1}^{n_1}(g_{ij} x_j + g'_{ij} z_j) + \sum_{k=1}^{n_2}(G_{ik} X_k + G'_{ik} Z_k) \le \tilde{\alpha}_i,\ i = 1,\ldots,m_1 \\ & \sum_{j=1}^{n_1} a_{lj} x_j + \sum_{k=1}^{n_2} a'_{lk} X_k \le a_{l0},\ l = 1,\ldots,m_2 \\ & 0 \le x_j \le u_j,\ y_j^0 x_j \le z_j \le y_j^1 x_j,\ j = 1,\ldots,n_1 \\ & 0 \le X_k \le U_k,\ Y_k^0 X_k \le Z_k \le Y_k^1 X_k,\ k = 1,\ldots,n_2 \end{array}\right. \tag{26}$$

This looks much simpler than (19) but it cannot be solved by standard nonlinear programming algorithms [1] since the objective function is not (quasi-)convex. Fortunately however, we will show in the sequel that a global optimum of this problem can be obtained by a variant of parametric simplex algorithm developed by the authors [9].

To explain the algorithm, let us first introduce a new set of variables:

$$v_j = \begin{cases} x_k, & j = 1,\ldots,n_1 \\ z_{j-n_1}, & j = n_1+1,\ldots,2n_1 \\ X_{j-2n_1}, & j = 2n_1+1,\ldots,2n_1+n_2 \\ Z_{j-2n_1-n_2}, & j = 2n_1+n_2+1,\ldots,2n_1+2n_2 \end{cases} \tag{27}$$

and denote (26) as follows:

$$\left|\begin{array}{ll} \text{maximize} & \dfrac{\sum_{j=1}^{n} \rho'_j v_j}{\sum_{j=1}^{n} \rho_j v_j} - \dfrac{\sum_{j=1}^{n} \sigma'_j v_j}{\sum_{j=1}^{n} \sigma_j v_j} \\ \text{subject to} & \sum_{j=1}^{n} \alpha_{ij} v_j \le \alpha_{i0},\ i = 1,\ldots,m \\ & 0 \le v_j \le v_j^0,\ j = 1,\ldots,n \end{array}\right. \tag{28}$$

where $n = 2n_1 + 2n_2$. Let

$$w = \frac{1}{\sum_{j=1}^{n} \sigma_j v_j} \tag{29}$$

By assumption, we have $w > 0$ for all v_j's satisfying the constraints of (26). Therefore the problem is equivalent to

$$\left| \begin{array}{ll} \text{maximize} & \dfrac{\sum_{j=1}^{n} \rho'_j v_j \cdot w}{\sum_{j=1}^{n} \rho_j v_j \cdot w} - \sum_{j=1}^{n} \sigma'_j v_j \cdot w \\ \text{subject to} & \sum_{j=1}^{n} \alpha_{ij} v_j w \leq \alpha_{i0} w, \ i = 1, \ldots, m \\ & 0 \leq v_j w \leq v_j^0 w, \ j = 1, \ldots, n \\ & \sum_{j=1}^{n} \sigma_j v_j w = 1 \end{array} \right. \tag{30}$$

Let us define

$$V_j = v_j \cdot w, \ j = 1, \ldots, n \tag{31}$$

then (30) is equivalent to

$$\left| \begin{array}{ll} \text{maximize} & \dfrac{\sum_{j=1}^{n} \rho'_j V_j}{\sum_{j=1}^{n} \rho_j V_j} - \sum_{j=1}^{n} \sigma'_j V_j \\ \text{subject to} & \sum_{j=1}^{n} \alpha_{ij} V_j - \alpha_{i0} w \leq 0, \ i = 1, \ldots, m \\ & \sum_{j=1}^{n} \sigma_j V_j = 1 \\ & 0 \leq V_j \leq v_j^0 w, \ j = 1, \ldots, n \end{array} \right. \tag{32}$$

Finally, let

$$S = \left\{ (V_1, \ldots, V_n, w) \ \middle| \ \begin{array}{ll} \sum_{j=1}^{n} \alpha_{ij} V_j - \alpha_{i0} w \leq 0, & i = 1, \ldots, m; \\ \sum_{j=1}^{n} \sigma_j V_j = 1, \ 0 \leq V_j \leq v_j^0 w, & j = 1, \ldots, n \end{array} \right\} \tag{33}$$

and let

$$\xi = \sum_{j=1}^{n} \rho_j V_j \tag{34}$$

also let

$$\xi_{\max} = \max \left\{ \textstyle\sum_{j=1}^{n} \rho_j V_j \ \middle| \ (V_1, \ldots, V_n, w) \in S \right\} \tag{35}$$

$$\xi_{\min} = \min \left\{ \textstyle\sum_{j=1}^{n} \rho_j V_j \ \middle| \ (V_1, \ldots, V_n, w) \in S \right\} \tag{36}$$

Then (32) can be put in the following form:

$$
\begin{array}{ll}
\text{maximize} & \dfrac{1}{\xi}\sum_{j=1}^{n}\rho'_j V_j - \sum_{j=1}^{n}\sigma'_j V_j \\
\text{subject to} & \sum_{j=1}^{n}\rho_j V_j = \xi \\
& \sum_{j=1}^{n}\alpha_{ij}V_j - \alpha_{i0}w \le 0,\ i=1,\dots,m \\
& \sum_{j=1}^{n}\sigma_j V_j = 1 \\
& 0 \le V_j \le v_j^0 w,\ j=1,\dots,n \\
& \xi_{\min} \le \xi \le \xi_{\max}
\end{array}
\tag{37}
$$

Let us note that the problem (37) reduces to a linear programming problem if we fix the value of ξ. Let $\xi_0 \in [\xi_{\min}, \xi_{\max}]$ and consider a linear programming problem.

$$
\begin{array}{ll}
\text{maximize} & \sum_{j=1}^{n}\left(\dfrac{\rho'_j}{\xi_0} - \sigma'_j\right) V_j \\
\text{subject to} & \sum_{j=1}^{n}\rho_j V_j = \xi_0 \\
& \sum_{j=1}^{n}\alpha_{ij}V_j - \alpha_{i0}w \le 0,\ i=1,\dots,m \\
& \sum_{j=1}^{n}\sigma_j V_j = 1 \\
& 0 \le V_j \le v_j^0 w,\ j=1,\dots,n
\end{array}
\tag{38}
$$

Let B be an optimal basic solution associated with (38). This basis remains optimal for all values of ξ in the interval $[\underline{\xi}, \overline{\xi}]$ such that the primal feasibility and dual feasibility are maintained. Thus we can apply a parametric simplex algorithm for solving (37) analytically for all ξ in the interval $[\xi_{\min}, \xi_{\max}]$. Let us note that the objective function has a simple form $a\xi + b\dfrac{1}{\xi} + c$ in each subinterval, so that we can calculate a global optimum of (37) in finitely many steps. Readers not familiar with parametric simplex algorithms are referred to either [9] or to a standard text of linear programming[3].

5 Results of Numerical Experiments

We will present some preliminary results of numerical simulation on the partial optimization model using the available data of about 90 government bonds circulated in the market. We adopted the average effective yield as an objective to be optimized and solved the following problem:

$$
\begin{array}{ll}
\text{maximize} & \nu(X) - \nu(x) \\
\text{subject to} & d(W) = d_0 \\
& C(W) \ge C_0 \\
& \sum_{k=1}^{n_2} P_k X_k = \sum_{j=1}^{n_1} p_j x_j + C \\
& 0 \le x_j \le u_j,\ j=1,\dots,n_1 \\
& 0 \le X_k \le U_k,\ k=1,\dots,n_2
\end{array}
\tag{39}
$$

where $\nu(X)$ and $\nu(x)$ are average effective yield of the bundle of assets X and x, respectively. Also $d(W)$ and $C(W)$ are average duration and average convexity of the portfolio Y owned by an investor after the transaction, i.e.,

$$W_j = \begin{cases} u_j - x_j, & j = 1, \ldots, n_1 \\ X_{j-n_1}, & j = n_1 + 1, \ldots, n_1 + n_2 \end{cases} \tag{40}$$

while d_0 and C_0 are the duration and convexity of the index portfolio, respectively. The third set of constraints means that we purchase the portfolio X by the available fund from the sale of portfolio x plus the amount of coupon payment C during the given period. Also, we assumed that the price adjustment coefficients are all zeros, i.e., that all the prices are held constant.

The upper bound U_k on X_k is chosen to be 5% of the total amount of the bond available in the market. Also, we assume that all bonds in the portfolio can be sold to the market. We started from a unit value portfolio consisting of a single bond and repeated solving the problem (39) for thirty nine periods using W as the starting portfolio in the next period.

We repeated this process eighty two times by choosing all available government bonds as starting "single bond" portfolios.

Figure 2 shows the values of the best and the worst one among these eighty two portfolios in terms of the values at the end of the thirty nine periods horizon. The best one labeled No. 211 outperforms the Yamaichi index while the worst one labeled No. 111 is a little behind the index. We see from this figure that we can keep track of the index portfolio very well by using our model. Also Figure 3 shows the monthly rates of return of the calculated portfolio and the index portfolio. Let us note that the portfolio consists of at most seven bonds, which is very desirable from the viewpoint of bond managers. Also, the computation time for solving the problem (39) is about 5 seconds on SONY NWS–3800 Workstation.

6 Conclusions and the Future Direction of Research

We showed in this paper that both partial optimization models and total optimization models proposed in [7] for bond portfolio management of institutional investors can be solved to optimality in an efficient way by using a global optimization algorithm developed by one of the authors[9]. In the last several years, a simplified version of the total optimization model has been used by several institutional investors. Instead, a partial optimization model was not used because of computational difficulty. However, it is now ready for use for the first time in a real transaction environment. As stated in the Introduction, there has been a significant needs of bond managers of institutional investors for a partial optimization model since only a fraction, typically less than one tenth of the total assets is subject to transaction. The partial optimization model meets the request of such bond managers to evaluate their transaction.

Due to the limited availability of the real market data, we could not conduct an extensive test to fully demonstrate the potential power of this model. Therefore we applied a partial optimization model to the index tracking of government bonds, by using the available market data. The result shows that this model generates a remarkably good result both in terms of computational efficiency and quality of the resulting portfolio. To demonstrate the usefulness of our approach, a more extensive simulation has to be conducted, whose results as well as further improvements of the model will be discussed elsewhere.

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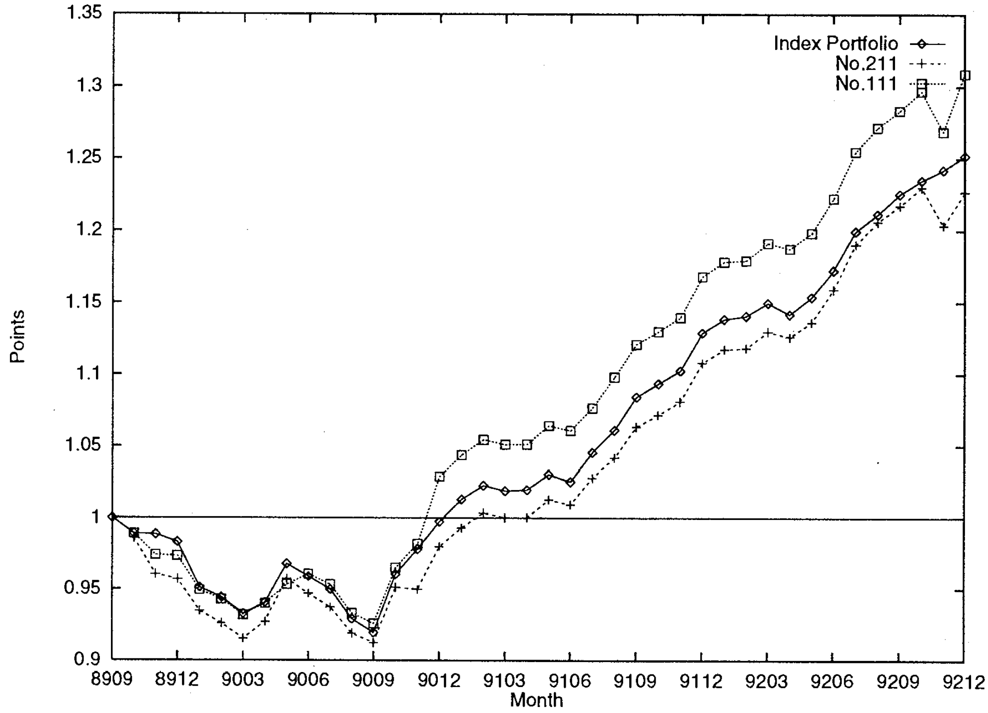

Figure 2: Values of The Portfolios

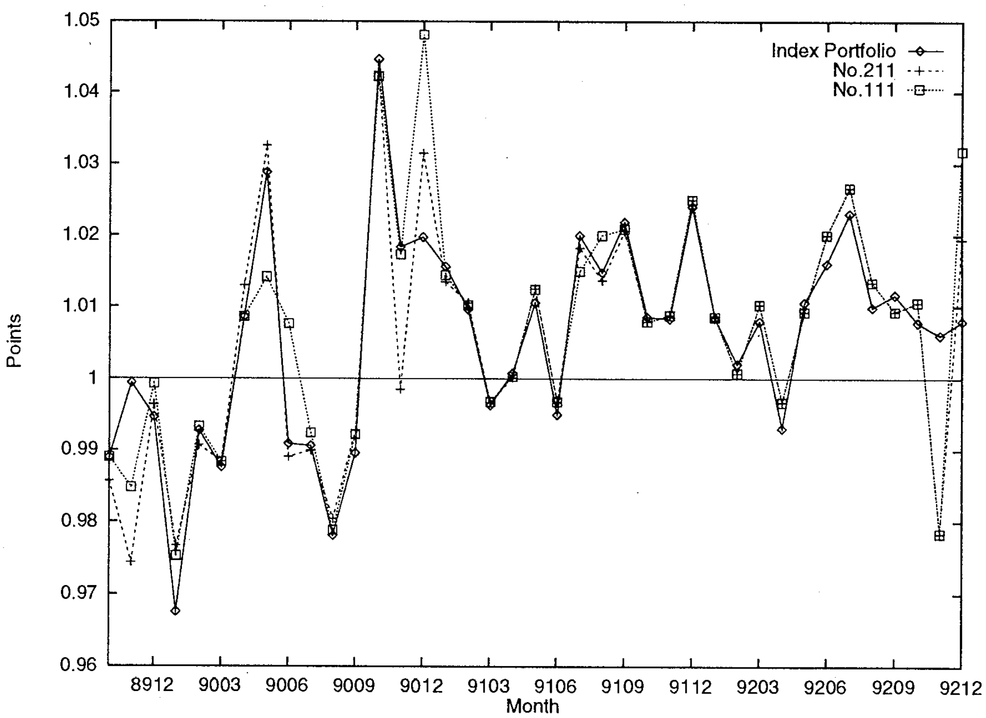

Figure 3: Monthly Rate of Returns

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