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# GRAPHICAL/STATISTICAL APPROACH TO REPAIR LIMIT REPLACEMENT PROBLEM

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*Abstract* The total time on test (TTT) concept has proved to be a very useful tool in many reliability applications. The graphical solution procedure based on this concept gives geometrical interpretations to the solutions of reliability problems as well as the optimal policies from empirical data directly. In fact, some kinds of reliability models such as age replacement and burn-in problems have been analyzed by applying the TTT concept.

In this paper, we discuss the optimal repair limit replacement policy, which determines the timing to stop repairing a unit after it fails. We propose two kinds of solution procedures for the repair limit replacement problem with a time constraint, where one is a straightforward/algebraic procedure and another is a geometrical one based on the TTT concept. Also, we develop a computer software utilizing the computer graphics to calculate the optimal repair limit policy numerically, and refer to its educational effect.

## 1. Introduction

The total time on test (TTT) concept has proved to be a very useful tool in many reliability applications. One of these generations is the TTT transform, defined as a certain function of the survivor function, and another is the TTT plot, defined as an empirical counterpart. The seminal work by Barlow and Campo [2] discussed the generalizations of TTT concept for the characteristics of lifetime distribution classes and the hypothesis test. On the other hand, some kinds of reliability models have been analyzed by using this concept [8]. The graphical solution procedure based on the TTT concept gives the geometrical interpretations to the solutions of reliability problems as well as the optimal policies from the empirical data directly. Every problem that can be transformed into a problem of minimizing a type of objective function, *e.g.* preventive maintenance problems with perfect/imperfect replacement [4-6], the discounted age replacement problem [7] and the optimal burn-in precedure [9], can be analyzed by the method based on the TTT concept.

However, a great number of reliability problems, which should be analyzed by using the TTT concept, have been remained. One of them is a repair limit replacement problem. In the typical age replacement problem (see [1]), the unit is maintained preventively at a prespecified time or correctively at failure, whichever occurs first, and then returned to operation after maintenance. On the other hand, in the repair limit replacement problem, the failed unit is repaired if the repair time is relatively short, and the failed unit is replaced if the repair time is too long. Hastings [10] considered the repair limit problem for army vehicles and proposed three methods of optimizing the repair limit policies by simulation, hill-climbing and dynamic programming. Nakagawa and Osaki [12, 13], Okumoto and Osaki [14] and Kaio and Osaki [11] extended the Hastings' model and discussed different repair limit replacement problems. In this paper, we discuss the optimal repair limit policy for a one-unit system, which has a different model framework from earlier contributions [11-14], by using the TTT concept.

The repair limit problem under consideration is the following: When the original unit in one-unit system fails, the repair is started immediately. If the repair is completed up to a prespecified time, then the repaired unit is installed as soon as the repair is completed. The unit upon repair is assumed to be as good as new. On the other hand, if the repair time is greater than the prespecified time, the failed unit is scrapped and a spare is ordered immediately. It is delivered and installed after a *lead time*. The problem is to obtain the optimal repair time limit minimizing the total expected cost per unit time in the steady-state. It is noticed that the concept of *estimated repair time* is never included in the problem, but the lead time for the spare is considered. This is a most different point from earlier models [11-14].

The paper is organized as follows: First, the straightforward and algebraic solution procedure to obtain the optimal policies is given under a different assumption from Nakagawa and Osaki [13]. Then, the necessary and sufficient condition for the existence of an optimal repair time limit is derived analytically. Second, we interpret the repair limit problem graphically and propose a visual solution procedure for the problem. Then, the TTT transform plays a significant role. It should be noted that the necessary and sufficient condition derived by the TTT transform has a dual relation with that by the algebraic method. Furthermore, we consider a nonparametric solution procedure using the TTT plot when the complete data of the repair time is given. By applying the TTT plot, we can obtain the optimal repair limit policies from the actual data without specifying the repair time distribution. Finally, numerical examples show that the TTT concept is useful for the repair limit problem. We also develop a visual computer software to represent the optimal repair limit policies, and report the results of numerical experiments.

## 2. Replacement Problem for Repairable Unit

Consider a single-unit system, where each spare is provided only following an order after a lead time $L$ and each failed unit is repairable. The original unit begins operating at time 0. The mean failure time for each unit is $m_f(>0)$. When the unit has failed, the repair is started immediately. If the repair is completed up to the time limit for repair $t_0 \in [0, \infty)$, then the unit is installed at that time. It is assumed that the unit once repaired, is presumed as good as new. However, if the repair time is greater than $t_0$, *i.e.* if the repair is not completed up to the time $t_0$, then the failed unit is scrapped, and the spare unit is ordered immediately and delivered after the lead time $L(>0)$. It is assumed that the time required for replacement is negligible (or included in the lead time $L$). From this point of view, the time $t_0$ is called the *repair time limit* (see [10-14]).

The repair time for each unit has an arbitrary distribution $G(t)$ with density $g(t)$ and finite mean $m_r(>0)$. The distribution $G(\cdot)$ is assumed to have an inverse function, *i.e.* $G^{-1}(\cdot)$, and to be continuous and strictly increasing. Under these model assumptions, we define the interval from the start of the operation to the following start as one cycle (see Fig. 1).

Next, we consider the cost structure. The costs considered in this paper are the following; a cost per unit time $k_r(>0)$ is incurred for repairing failed unit, a cost per unit time $k_f(>0)$ is incurred for the shortage and a cost $c(>0)$ is incurred for each order. In particular, we assume:

$$k_r L < c. \tag{1}$$

This assumption implies that the ordering cost is larger than the repair cost during the time interval $t \in [0, L]$, *i.e.* until the delivery of a new unit.

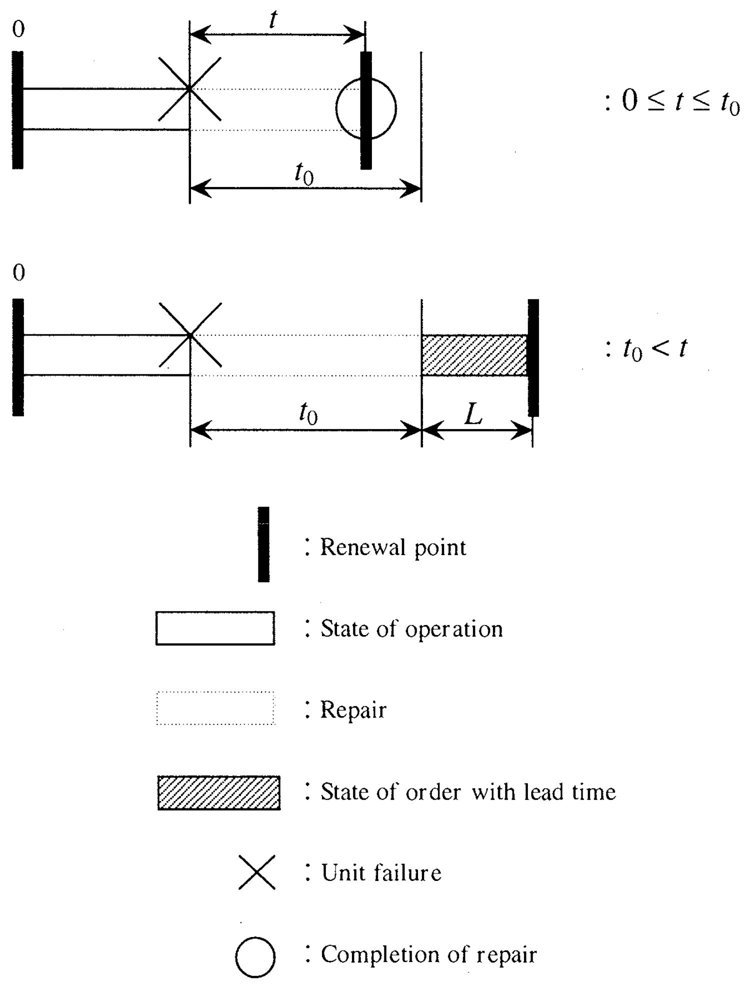



Figure 1: State Diaglam for One Cycle.

For an infinite planning horizon, it is appropriate to adopt the total expected cost per unit time in the steady-state. The following three costs are considered for one cycle:
(i) The expected repair cost is

$$k_r\Big\{\int_0^{t_0} t dG(t) + \int_{t_0}^{\infty} t_0 dG(t)\Big\} = k_r \int_0^{t_0} \overline{G}(t)dt, \tag{2}$$

where $\overline{G}(t) = 1 - G(t)$.
(ii) The expected shortage cost is

$$k_f\Big\{\int_0^{t_0} t dG(t) + \int_{t_0}^{\infty} (t_0 + L) dG(t)\Big\} = k_f\Big\{\int_0^{t_0} \overline{G}(t)dt + L\overline{G}(t_0)\Big\}. \tag{3}$$

(iii) The expected ordering cost is $c\overline{G}(t_0)$.
The mean time of one cycle is given by

$$\begin{aligned} E_T(t_0) &= \int_0^{t_0} (m_f + t)dG(t) + \int_{t_0}^{\infty} (m_f + t_0 + L)dG(t) \\ &= m_f + \int_0^{t_0} \overline{G}(t)dt + L\overline{G}(t_0). \end{aligned} \tag{4}$$

Then, the total expected cost per unit time in the steady-state is, from Ross [16, p.52],

$$C(t_0) = \lim_{t\to\infty} \frac{[\text{the total cost on } (0,t]]}{t} = \frac{E_C(t_0)}{E_T(t_0)}, \tag{5}$$

where $E_C(t_0)$ is the total expected cost for one cycle as follows:

$$E_C(t_0) = (k_r + k_f)\int_0^{t_0} \overline{G}(t)dt + (k_f L + c)\overline{G}(t_0). \tag{6}$$

Then, the problem is to obtain the optimal repair time limit $t_0{}^*$ minimizing $C(t_0)$ and is formulated as follows:

$$t_0{}^* = \text{argmin}_{0\le t_0 \le \infty} C(t_0). \tag{7}$$

Define the numerator of the derivative of $C(t_0)$ with respect to $t_0$ divided by $\overline{G}(t_0)$, as $q(t_0)$, *i.e.*

$$\begin{aligned} q(t_0) &= \Big\{k_r + k_f - (k_f L + c)r(t_0)\Big\}\Big\{m_f + \int_0^{t_0}\overline{G}(t)dt + L\overline{G}(t_0)\Big\} \\ &\quad - (1 - Lr(t_0))\Big\{(k_r + k_f)\int_0^{t_0}\overline{G}(t)dt + (k_f L + c)\overline{G}(t_0)\Big\}, \end{aligned} \tag{8}$$

where

$$r(t) \equiv g(t)/\overline{G}(t) \tag{9}$$

is called *instantaneous repair rate* and is equivalent to the hazard rate of the repair time distribution $G(t)$. We assume that $r(t)$ is differentiable without loss of generality. Two special cases, $t_0 = 0$ and $t_0 \to \infty$, are

$$C(0) = \frac{k_f L + c}{m_f + L} \tag{10}$$

and

$$C(\infty) = \frac{(k_r + k_f)m_r}{m_f + m_r}, \tag{11}$$

respectively.

The following theorem gives the existence of the optimal repair time limit.

**Theorem 2.1:** If $q(\infty) > 0$ or $q(0) < 0$, then there exists at least one optimal repair time limit $t_0^* \in [0,\infty)$ or $t_0{}^* \in (0,\infty]$ minimizing the total expected cost per unit time in the steady-state.

**Proof:** By taking the logarithm of $C(t_0)$, we have

$$\frac{d}{dt_0}\ln C(t_0) = \frac{\overline{G}(t_0)}{E_C(t_0)E_T(t_0)} q(t_0). \tag{12}$$

For a sufficiently large $t_0$, the equation above is

$$\frac{d}{dt_0}\ln C(t_0) \simeq \frac{\overline{G}(t_0)}{E_C(t_0)E_T(t_0)} q(\infty), \tag{13}$$

where

$$q(\infty) = \Big\{k_r + k_f - (k_f L + c)r(\infty)\Big\}(m_f + m_r) - (1 - Lr(\infty))\Big\{(k_r + k_f)m_r\Big\}. \quad (14)$$

On the other hand, for a sufficiently small $t_0$, we have

$$\frac{d}{dt_0}\ln C(t_0) \simeq \frac{\overline{G}(t_0)}{E_C(t_0)E_T(t_0)}q(0), \quad (15)$$

where

$$q(0) = \Big\{k_r + k_f - (k_f L + c)r(0)\Big\}(m_f + L) - (1 - Lr(0))(k_f L + c). \quad (16)$$

Thus, if $q(\infty) > 0$ or $q(0) < 0$, then there exists at least one optimal repair time limit $t_0^*$ ($0 \le t_0^* < \infty$ or $0 < t_0{}^* \le \infty$) minimizing the total expected cost per unit time in the steady-state. □

Next, we have the theorem on the uniqueness of the optimal repair time limit by supposing the monotone property of instantaneous repair rate.

**Theorem 2.2:** (1) Suppose that the repair time distribution $G(t)$ has a strictly DHR (decreasing hazard rate).

(i) If $q(\infty) > 0$ and $q(0) < 0$, then there exists a finite and unique optimal repair time limit $t_0{}^* \in (0, \infty)$ satisfying $q(t_0) = 0$ and the corresponding minimum total expected cost per unit time in the steady-state is

$$C(t_0^*) = \frac{k_r + k_f - (k_f L + c)r(t_0^*)}{1 - Lr(t_0^*)}. \quad (17)$$

(ii) If $q(0) \ge 0$, then the optimal repair time limit is $t_0{}^* = 0$, *i.e.* it is optimal to order a spare automatically, and the repair is never carried out. The corresponding minimum total expected cost per unit time is given by Eq. (10).

(iii) If $q(\infty) \le 0$, then the optimal repair time limit is $t_0{}^* \to \infty$, *i.e.* the repair is completed and no spare should be ordered. The corresponding minimum total expected cost per unit time is given by Eq. (11).

(2) Suppose that the repair time distribution has an IHR (increasing hazard rate). Then, if $(k_f L + c)(m_f + m_r) < ((k_r + k_f)m_r)(m_f + L)$, $t_0{}^* = 0$, otherwise, $t_0{}^* \to \infty$.

**Proof:** Differentiating $C(t_0)$ with respect to $t_0$ and setting it equal to zero implies the equation $q(t_0) = 0$. Further, with respect to $t_0$, we have

$$q'(t_0) = -r'(t_0)\Big\{(k_f L + c)m_f - (k_r L - c)\int_0^{t_0} \overline{G}(t)dt\Big\}, \quad (18)$$

where the symbol $'$ indicates the derivative. When the repair time distribution has a strictly DHR, *i.e.* $r'(t_0) < 0$, the function $q(t_0)$ is strictly increasing from the assumption $k_r L < c$. If $q(0) < 0$ and $q(\infty) > 0$, then there exists an optimal repair time limit $t_0{}^* \in (0, \infty)$ which minimizes the total expected cost per unit time $C(t_0)$, as a finite and unique solution to $q(t_0) = 0$, since $q(t_0)$ is strictly increasing and continuous. Substituting the relation of $q(t_0) = 0$ into $C(t_0)$ in Eq. (5) yields Eq. (17). If $q(0) \ge 0$, then $C(t_0)$ is strictly increasing, and the optimal repair time limit is $t_0{}^* = 0$. If $q(\infty) \le 0$, then $C(t_0)$ is strictly decreasing, and $t_0{}^* \to \infty$.

On the other hand, when the repair time distribution has an IHR, *i.e.* $r'(t_0) \geq 0$, then $q(t_0)$ is decreasing. If $C(0) < C(\infty)$, then $t_0{}^* = 0$, otherwise $t_0{}^* \to \infty$. The proof is completed. □

In the following section, we develop the geometrical solution procedure applying the TTT transform for the repair limit replacement problem.

## 3. Optimal Repair Limit Policy by TTT Transform

Following Barlow and Campo [2] and Bergman and Klefsjö [8], the scaled total time on test (TTT) transform of a repair time distribution is defined as

$$\phi(p) \equiv \frac{1}{m_r}\int_0^{G^{-1}(p)} \overline{G}(t)dt, \qquad 0 \leq p \leq 1, \tag{19}$$

where

$$G^{-1}(p) = \inf\{t : G(t) \geq p\}, \tag{20}$$

and where the mean repair time $m_r$ is given by

$$m_r = \int_0^{G^{-1}(1)} \overline{G}(t)dt. \tag{21}$$

The following relationship between the aging and the scaled TTT transform is well known (see Barlow [3], Osaki and Li [15] and Shaked and Shanthikumar [17]):

(i) $G(t)$ is IHR (DHR) if and only if $\phi(p)$ is concave (convex) on $[0,1]$.

(ii) If $G(t)$ is IFRA (DFRA) then $\phi(p)$ is starshaped on $[0,1]$.

(iii) If $G(t)$ is NBUE (NWUE) if and only if $\phi(p) \geq (\leq) p$ for $p \in [0,1]$.

(iv) If $G(t)$ is DMRL (IMRL) if and only if $(1-\phi(p))/(1-p)$ is nonincreasing (nondecreasing) on $[0,1]$.

From Eq. (5), we have

$$C(t_0) = (k_rL - c) \times \frac{-\dfrac{(k_r + k_f)m_f + k_rL - c}{k_rL - c} + G(t_0)}{m_f + L + \displaystyle\int_0^{t_0} \overline{G}(t)dt - LG(t_0)} + (k_r + k_f). \tag{22}$$

Consequently, we obtain the following dual optimization problem.

**Lemma 3.1:** Finding the optimal repair time limit $t_0{}^*$ minimizing the total expected cost per unit time in the steady-state $C(t_0)$ is equivalent to obtaining

$$p^* = \mathrm{argmin}_{0 \leq p \leq 1}\left\{\frac{-\dfrac{k_fL + c}{k_rL - c}\cdot\dfrac{m_f}{m_r} + \phi(p)}{-\left(1 + \dfrac{(k_r + k_f)m_f}{k_rL - c}\right) + p}\right\}. \tag{23}$$

The proof is omitted for simplicity.

Next, we interpret the conditions of optimality in Theorem 2.2 geometrically, by using the relationship between the aging and the TTT transform. First, we consider the case where

$G(t)$ is strictly DHR. Since $\phi(p)$ becomes a convex function in $p$, the nonlinear equation (8) is rewritten by

$$q(t_0) = \Big\{k_r + k_f - (k_f L + c)r(t_0)\Big\}m_f + (k_r L - c)\Big\{\overline{G}(t_0) + r(t_0)\int_0^{t_0} \overline{G}(t)dt\Big\}. \qquad (24)$$

Consider the graph $y = \phi(p)$ in the plane $(x, y) = (p, \phi(p))$, where $p \in [0, \infty]$ and $\phi(p) \in [0, 1]$. Define the point $\mathrm{B} = (x_\mathrm{B}, y_\mathrm{B})$, where

$$(x_\mathrm{B}, y_\mathrm{B}) \equiv \left(1 + \frac{(k_r + k_f)m_f}{k_r L - c}, \frac{k_f L + c}{k_r L - c} \cdot \frac{m_f}{m_r}\right). \qquad (25)$$

Since the slope of the equation $y = \phi(p)$ is

$$\frac{d}{dp}\phi(p) = \frac{1}{m_r r(G^{-1}(p))} \qquad (26)$$

(see [2, p. 455]) and the tangent line for the point $\mathrm{M} = (p^*, \phi(p^*))$ is

$$\varphi(p) = \frac{1}{m_r r(G^{-1}(p^*))}(p - p^*) + \phi(p^*), \qquad (27)$$

the condition that the point B is over $y = \varphi(p)$ is $q({t_0}^*) = 0$.

Moreover, define the intersection $\mathrm{Z} = (x_\mathrm{Z}, y_\mathrm{Z})$ of the tangent line for the point $\mathrm{O} = (0, 0)$ in the equation $y = \phi(p)$ and the straight line

$$y = \frac{k_f L + c}{k_r L - c} \cdot \frac{m_f}{m_r}. \qquad (28)$$

Then, we have

$$(x_\mathrm{Z}, y_\mathrm{Z}) \equiv \left(\frac{k_f L + c}{k_r L - c} m_f r(0), \frac{k_f L + c}{k_r L - c} \cdot \frac{m_f}{m_r}\right). \qquad (29)$$

If the coordinate $x = p$ of the point B is greater than that of the point Z, then $q(0) < 0$, otherwise, $q(0) \geq 0$ under the assumption $k_r L < c$. Similarly, the intersection $\mathrm{I} = (x_\mathrm{I}, y_\mathrm{I})$ of the tangent line for the point $\mathrm{U} = (1, 1)$ and the straight line

$$p = 1 + \frac{(k_r + k_f)m_f}{k_r L - c} \qquad (30)$$

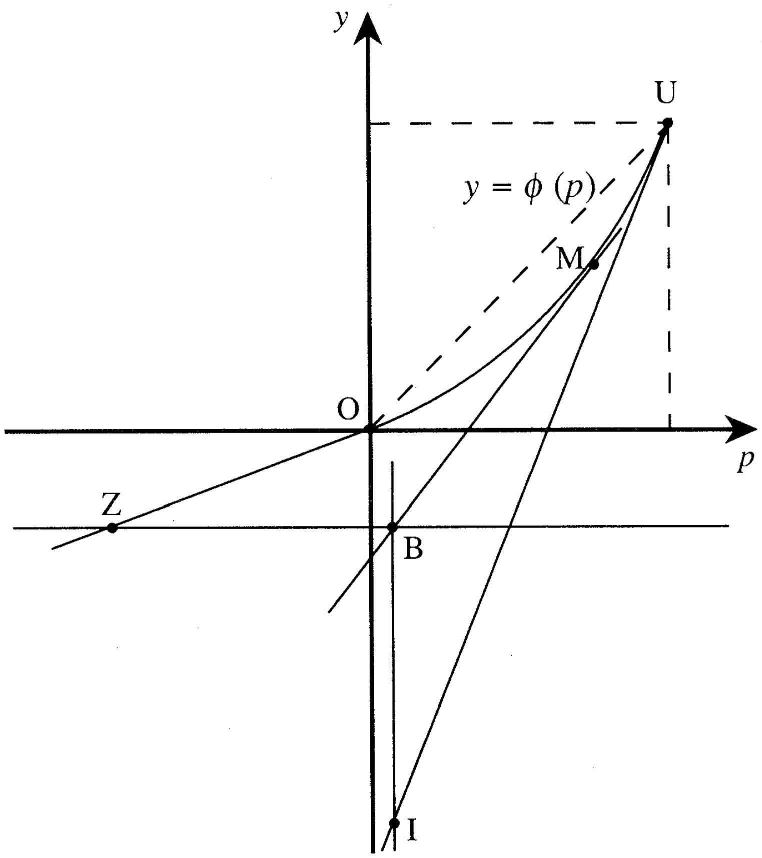
becomes

$$(x_\mathrm{I}, y_\mathrm{I}) \equiv \left(1 + \frac{(k_r + k_f)m_f}{k_r L - c}, 1 + \frac{(k_r + k_f)m_f}{k_r L - c} \cdot \frac{1}{m_r r(\infty)}\right). \qquad (31)$$

If the coordinate $y$ of the point B is greater than that of the point Z, then $q(\infty) > 0$, otherwise, $q(\infty) \leq 0$ under the assumption $k_r L < c$. Thus, the graphical determination of the optimal repair time limit in the strictly DHR case is illustrated in Fig. 2.

$$(k_f L + c)m_f - \Big\{k_r L - c + (k_r + k_f)m_f\Big\}m_r < 0. \qquad (32)$$

Second, consider the case where $G(t)$ is IHR. In this case, $\phi(p)$ becomes a concave function in $p$. Figure 3 shows the schematic illustration of the optimal repair limit policy. If the coordinate $x = p$ of the point B is strictly negative and if the slope of the straight line BO is strictly smaller than that of the line BU, then we have From Eqs. (10) and (11), the inequality above is equivalent to the condition $C(0) < C(\infty)$. Conversely, if $x_\mathrm{B} < 0$ and if the slope of the straight line BO is not small than that of the line BU, then $C(0) \geq C(\infty)$ is satisfied. Similarly, the condition $x_\mathrm{B} \geq 0$ also implies $C(0) > C(\infty)$.




$\mathrm{O}(0,0)$
$\mathrm{U}(1,1)$
$\mathrm{B}\left(1+\frac{(k_r+k_f)m_f}{k_rL-c}, \frac{k_fL+c}{k_rL-c}\cdot\frac{m_f}{m_r}\right)$
$\mathrm{Z}\left(\frac{k_fL+c}{k_rL-c}m_fr(0), \frac{k_fL+c}{k_rL-c}\cdot\frac{m_f}{m_r}\right)$
$\mathrm{I}\left(1+\frac{(k_r+k_f)m_f}{k_rL-c}, 1+\frac{(k_r+k_f)m_f}{k_rL-c}\cdot\frac{1}{m_rr(\infty)}\right)$
$\mathrm{M}(p^*, \phi(p^*))$

Figure 2: Illustration of Graphical Determination of Optimal Repair Time Limit. (Strictly DHR Case)

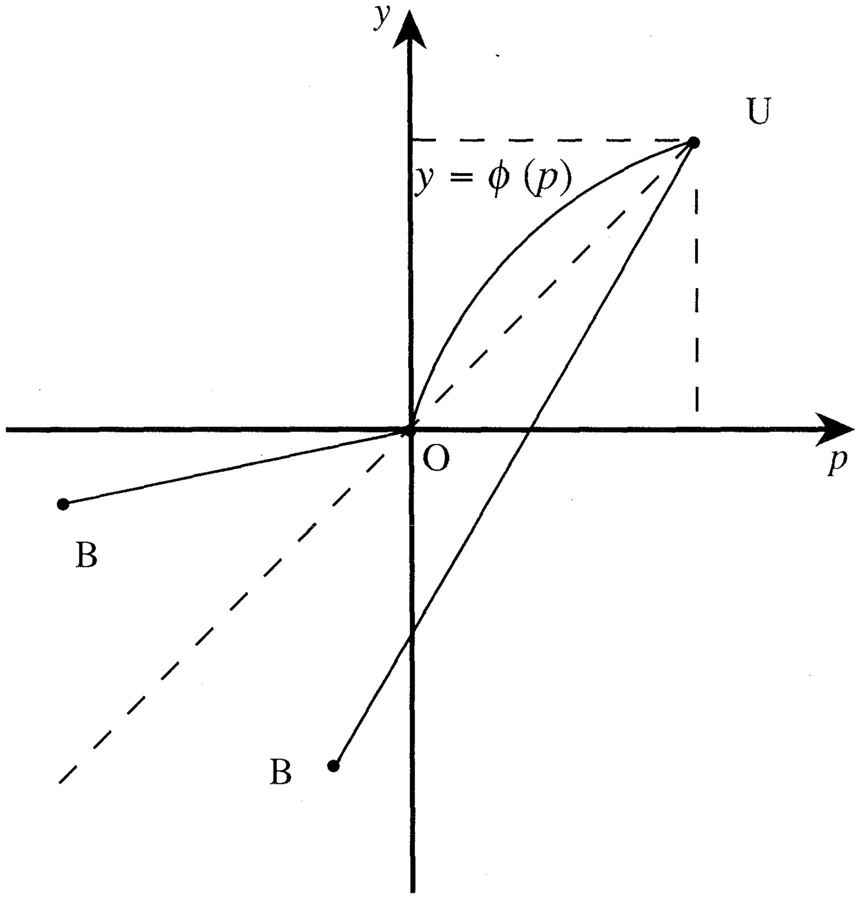
Finally, we summarize the results above as the following theorem.

**Theorem 3.2:** (1) Suppose that the scaled TTT transform $\phi(p)$ is strictly convex.

(i) If $x_{\mathrm{B}} > x_{\mathrm{Z}}$ and $y_{\mathrm{B}} > y_{\mathrm{I}}$, then there exists a unique solution

$$p^* \in \left[\max\left(0, 1+\frac{(k_r+k_f)m_f}{k_rL-c}\right), 1\right] \tag{33}$$

which minimizes the cost function given by Eq. (22). Then, the optimal solution $p^*$ is given by the coordinate $x = p$ in the contact point $M(p, \phi(p))$ for $y = \phi(p)$ from the point B. The corresponding minimum expected cost is given by Eq. (17).




O(0,0)
U(1,1)
$$\mathrm{B}\left(1+\frac{(k_r+k_f)m_f}{k_rL-c},\frac{k_fL+c}{k_rL-c}\cdot\frac{m_f}{m_r}\right)$$

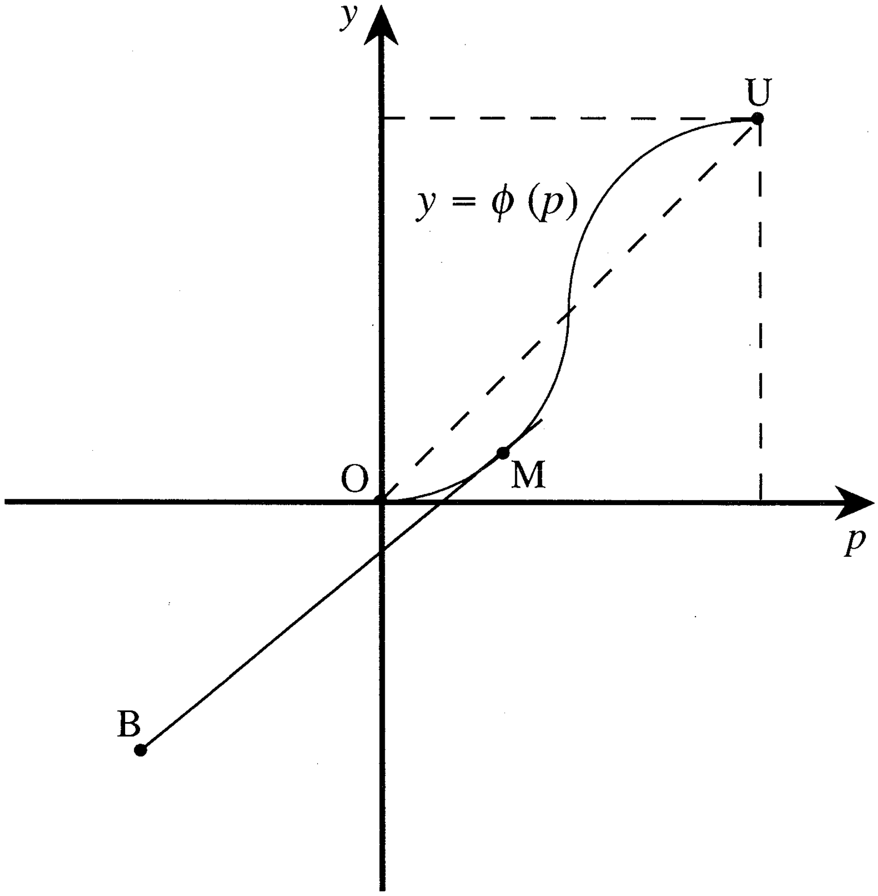
Figure 3: Illustration of Graphical Determination of Optimal Repair Time Limit. (IHR Case)

(ii) If $x_\mathrm{B} \le x_\mathrm{Z}$, then the optimal repair limit policy is $p^* = 0$ ($t_0{}^* = 0$) and the corresponding minimum expected cost is given by Eq. (10).

(iii) If $y_\mathrm{B} \le y_\mathrm{I}$, then the optimal repair limit policy is $p^* = 1$ ($t_0{}^* \to \infty$) and the corresponding minimum expected cost is given by Eq. (11).

(2) Suppose that the scaled TTT transform is concave. If the point B lies completely above the 45° line, then the optimal solution is $p^* = 0$ ($t_0{}^* = 0$), otherwise, $p^* = 1$ ($t_0{}^* \to \infty$).

From the discussion above, we can obtain the optimal repair limit policy graphically when the repair time distribution is strictly DHR or IHR. If the repair time distribution is neither IHR nor DHR and if the algebraic method is adopted, it will be difficult to obtain the optimal repair time limit analytically. In general, the investigation of the aging property from the empirical data is rather troublesome and it is not always possible to determine whether the hypothesis of being IHR or DHR is rejected or not. Thus, we should also consider the case where the repair time distribution belongs to more general classes (see Fig. 4). Then, it is possible to analyze such a case by using the graphical solution procedure proposed here. In the following section, we propose a method to estimate the optimal policy from the empirical data directly.




O(0, 0)
U(1, 1)
$\mathrm{B}\left(1 + \frac{(k_r + k_f)m_f}{k_r L - c}, \frac{k_f L + c}{k_r L - c} \cdot \frac{m_f}{m_r}\right)$
M$(p^*, \phi(p^*))$

Figure 4: Illustration of Graphical Determination of Optimal Repair Time Limit. ($S$-shaped Case)

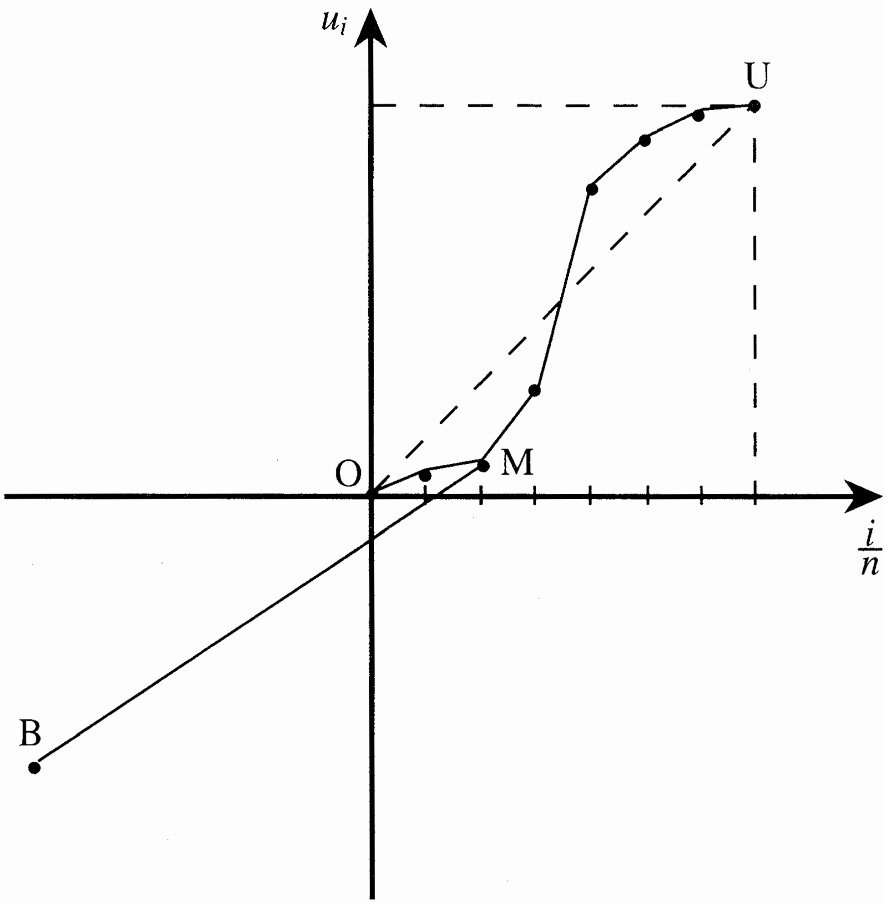
## 4. Optimal Repair Limit Policy by TTT Plot

In this section, we propose a statistical method using the TTT plotting for complete samples to estimate the optimal repair limit policy. Suppose that the optimal repair time limit has to be estimated from an ordered complete sample $0 = x_0 \leq x_1 \leq x_2 \leq \cdots \leq x_n$ of repair times from an absolutely continuous repair time distribution $G(\cdot)$, which is unknown. The TTT statistics based on this sample are defined as

$$T_i \equiv \sum_{j=1}^{i}(n - j + 1)(x_j - x_{j-1}), \qquad i = 1, 2, \cdots, n;\ T_0 = 0. \tag{34}$$

In a similar fashion to Section 3, we define the scaled TTT statistics as

$$u_i \equiv \frac{T_i}{T_n}, \qquad i = 0, 1, 2, \cdots, n. \tag{35}$$




O(0, 0)
U(1, 1)
$\mathrm{B}\left(1+\frac{(k_r+k_f)m_f}{k_rL-c}, \frac{k_fL+c}{k_rL-c}\cdot\frac{m_f}{m_r}\right)$
$\mathrm{M}(\frac{i^*}{n}, u_{i^*})$

Figure 5: Illustration of Graphical Determination of Optimal Repair Time Limit by Using TTT Plot.

By plotting the point $(i/n, u_i)$ $(i = 0, 1, 2, \cdots, n)$ and connecting them by line segments, we obtain a curve called the *scaled TTT plot.* Thus, we can express these plotting positions by using the empirical repair time distribution function $G_n(x)$ defined as (see [2])

$$G_n(x) = \begin{cases} \frac{i}{n} & \text{for} \quad x_i \le x < x_{i+1},\ i = 0, 1, 2, \cdots, n-1, \\ 1 & \text{for} \quad x_n \le x. \end{cases} \tag{36}$$

Since

$$\frac{i}{n} = G_n(x_i), \qquad i = 0, 1, 2, \cdots, n, \tag{37}$$

and from the discussion in Section 3, we estimate the optimal repair limit policy minimizing the total expected cost per unit time in the steady-state. To this end, one calculates each coordinate of the points B, I and Z, and applies the result in the previous section directly. Finally, the estimator $x_{i^*}$ for the optimal repair time limit is obtained from the coordinate $p_{i^*} = i^*/n$, which gives the minimum slope.

We summarize the main result in this paper as follows.

**Theorem 4.1:** Suppose that an ordered sample $0 = x_0 \leq x_1 \leq x_2 \leq \cdots \leq x_n$ of repair time from an absolutely continuous repair time distribution $G(\cdot)$, which is unknown, is given. Then the optimal repait time limit $x_{i^*}$ minimizing the total expected cost per unit time in the steady-state is given by

$$i^* = \mathrm{argmin}_i \Big\{ \frac{u_i - y_{\mathrm{B}}}{i/n - x_{\mathrm{B}}} \Big\}. \tag{38}$$

## 5. Numerical Examples and Concluding Remarks

We have developed a computer software to realize the optimal repair limit policy. In the following discussion, numerical examples using the TTT transform and the TTT plot are presented.

*TTT transform*

Let us consider the case where the repair time distribution is known. Note that one must represent $t_0{}^* \to \infty$ on computer. For convenience, the upper value of $t_0$ is substituted by $t_0 = 125$ in the system. That is, $t_0{}^* = 125$ roughly means $t_0{}^* \to \infty$. In fact, we can check that the total expected cost for $t_0{}^* = 125$ is almost equal to that for $t_0{}^* \to \infty$ and $G(125) \approx 1$, and thus we can conclude that $t_0 = 125$ is a sufficiently large value of repair time limit.

**Example 1-1:** The parameters are $m_f = 0.050$, $L = 0.15$, $c = 5.0$, $k_r = 27$ and $k_f = 10$. The Weibull distribution with $\theta = 2.0$ and $\beta = 0.8$ is assumed as the repair time distribution, where

$$G(t) = 1 - \exp\Big\{ -(\frac{t}{\theta})^{\beta} \Big\}. \tag{39}$$

This case obviously implies that the instantaneous repair rate is monotonically decreasing. Figure 6 displays the output in this example. We obtain $B = (-0.947, -0.151)$ and find that the slope for the point O from the point B is minimum, *i.e.* $p^* = 0$. Thus, we estimate the optimal repair time limit $t_0{}^* = 0$ and the minimum expected cost $C(t_0{}^*) = 32.500$.

**Example 1-2:** The parameters are $m_f = 0.5$, $L = 0.1$, $c = 4.0$, $k_r = 5.0$ and $k_f = 6.5$. The gamma distribution with $\lambda = 1.0$ and $\alpha = 0.8$ is assumed, where

$$G(t) = \int_0^t \lambda \exp(-\lambda s)(\lambda s)^{\alpha - 1} ds. \tag{40}$$

This case corresponds to strictly DHR case. From Fig. 7, we find that the slope for $\mathrm{M} = (0.693, 0.654)$ from $\mathrm{B} = (-0.643, -0.840)$ is minimum. Hence, we have $p^* = 0.693$ and the optimal repair time limit $t_0{}^* = 0.9210$. Then the corresponding expected cost is $C(t_0{}^*) = 7.041$.

**Example 1-3:** The parameters $m_f = 0.050$, $L = 1.5$, $c = 50$, $k_r = 30$ and $k_f = 10$ and the Weibull distribution with $\theta = 2.0$ and $\beta = 0.8$ are assumed. From Fig. 8, we find that the slope for $\mathrm{U} = (1, 1)$ from $\mathrm{B} = (0.600, -0.287)$ is minimum. Hence, we have $p^* = 1$ and $t_0{}^* \to \infty$. This means that it is optimal that no spare should order. The corresponding expected cost is $C(t_0{}^*) = 39.136$.

*TTT plot*

Suppose that the complete sample data of repair time is given. If we can estimate the repair time distribution from the data, then we may test the fit of the distribution to the data,

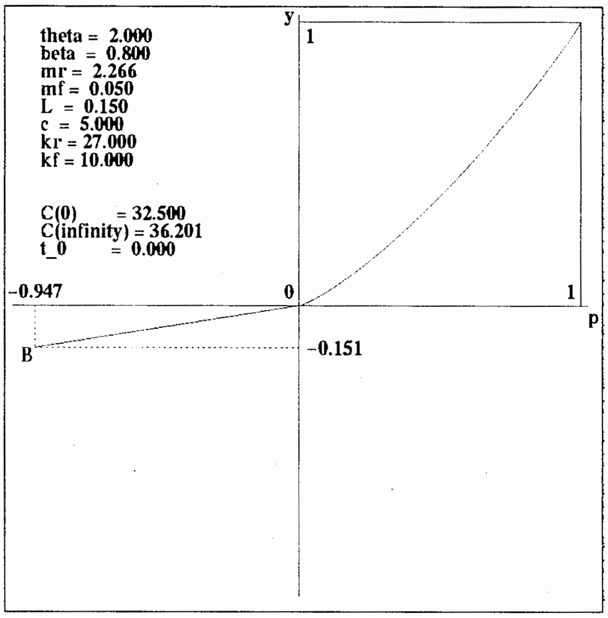



Figure 6: Example 1-1.

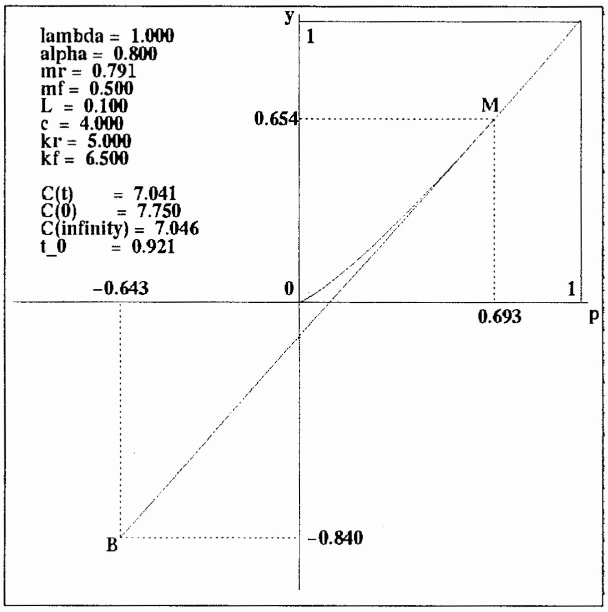



Figure 7: Example 1-2.

and the mean time may be estimated by using probability paper. However, it seems to be common in practical management to calculate the ordinary arithmetic mean

$$m_r \equiv \sum_{i=1}^{n} x_i/n. \tag{41}$$

In most cases, it is essentially difficult to specify the repair time distribution. By substituting the arithmetic mean for the mean repair time, we numerically determine the optimal repair limit policy. Here, three kinds of data sets, which are given in Table 1, are used.

**Example 2-1:** Data Set 1 and the parameters $m_f = 25.292$, $L = 5.724$, $c = 80.215$, $k_r = 3.501$ and $k_f = 1.151$ are assumed. Figure 9 is a plot of the data. Then, the slope

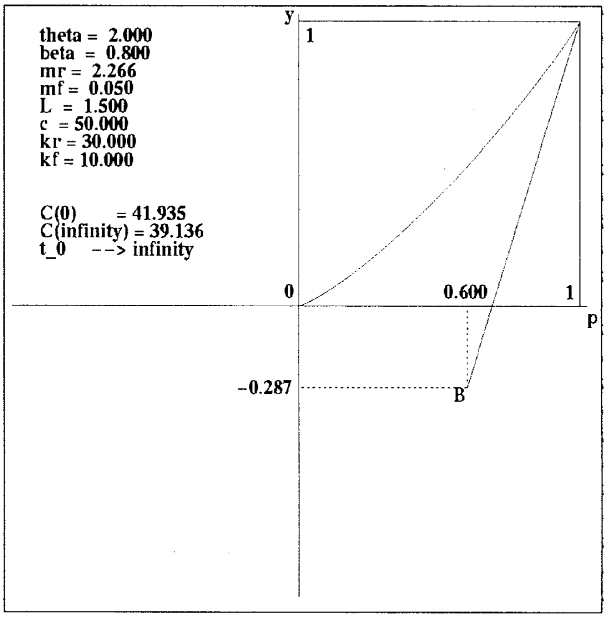



Figure 8: Example 1-3.

Table 1: Sample Data Sets.

Data Set 1.

| $i$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
|---|---|---|---|---|---|---|---|---|---|---|
| $x_i$ | 1.207 | 1.311 | 3.648 | 9.699 | 10.69 | 28.79 | 52.17 | 63.26 | 77.18 | 440.9 |

Data Set 2.

| $i$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $x_i$ | 5 | 24 | 49 | 51 | 63 | 66 | 70 | 71 | 78 | 79 | 80 | 139 | 166 | 198 | 224 |
| $i$ | 16 | 17 | 18 | 19 | 20 | 21 | 22 | 23 | 24 | 25 | 26 | 27 | 28 | 29 | 30 |
| $x_i$ | 225 | 230 | 243 | 260 | 265 | 267 | 291 | 329 | 341 | 343 | 343 | 356 | 358 | 363 | 367 |

Data Set 3.

| $i$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $x_i$ | 21 | 113 | 382 | 517 | 1205 | 1585 | 1930 | 2210 | 2340 | 2870 | 3340 | 4010 | 4850 |

for the point M $= (0.500, 0.116)$ from B $= (-0.955, -0.530)$ is minimum. Then, we have $p_5{}^* = 0.500$ and the optimal repair time limit $t_0{}^* = 10.690$.

**Example 2-2:** Data Set 2 and the parameters $m_f = 25.292$, $L = 5.748$, $c = 80.788$, $k_r = 3.010$ and $k_f = 1.499$ are assumed, respectively. From Fig. 10, the slope for O $= (0, 0)$ from B $= (-0.796, -0.180)$ is minimum. Hence, we have $p_0{}^* = 0$ and the optimal repair time limit $t_0{}^*$ is 0.

**Example 2-3:** Data Set 3 and the parameters $m_f = 46.816$, $L = 15.993$, $c = 278.702$, $k_r = 1.830$ and $k_f = 0.989$ are assumed, respectively. In Fig. 11, the slope for U $= (1, 1)$ from B $= (0.471, -0.950)$ is minimum. Hence, we have $p_{13}{}^* = 1$ and $t_0{}^* \to \infty$.

It should be noted that one can easily carry out the sensitivity analyses of the optimal repair time limit for model parameters, when the solution procedures via the TTT transform and the TTT plot are adopted. Especially, we can observe how the optimal policies are influenced as parameters are changed. This may be an advantage of using the TTT concept.

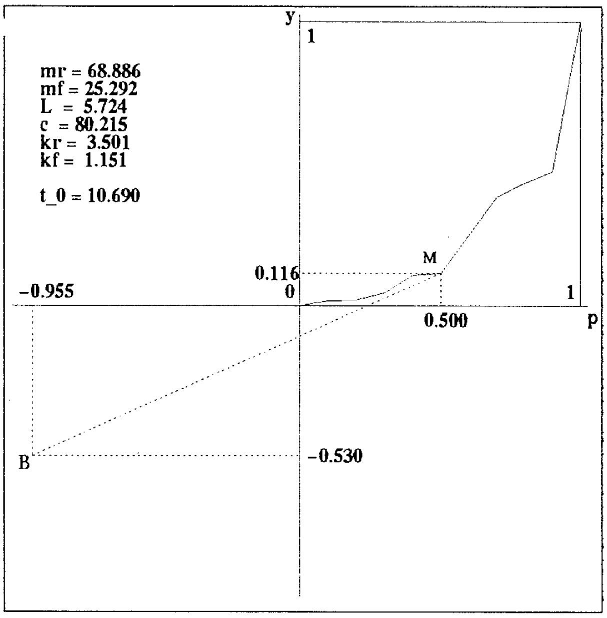



Figure 9: Example 2-1.

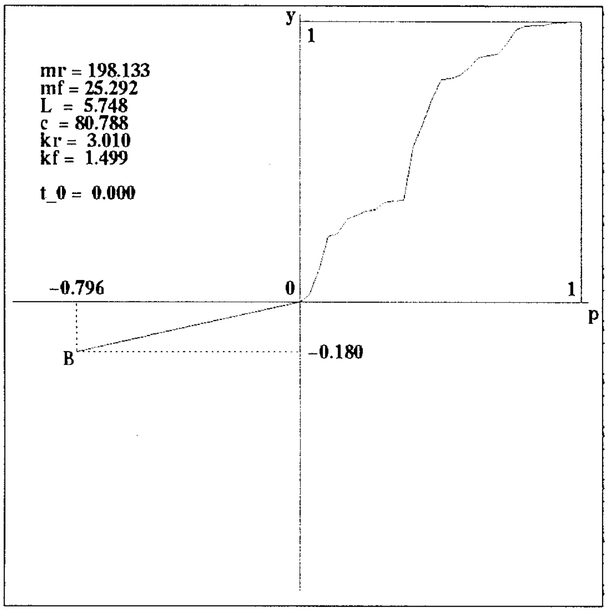



Figure 10: Example 2-2.

Since the software developed is made by using the computer graphics, we can visually catch the optimal repair limit policies. Since practitioners like such visual representations, the software will be useful in planning the maintenance of repairable system.

This paper has addressed the problem to determine optimal repair time limit minimizing the total expected cost per unit time in the steady-state. In particular, we have proposed a graphical method applicable to the repair time limit replacement problem. The procedure provided is nonparametric and can estimate the optimal solution from complete sample data directly, without specifying the repair time distribution. We have focused on only the repair limit replacement with time constraint. Similarly, the problem with cost constraint can be analyzed by using TTT concept, which will be studied in forthcoming papers. Further, it is

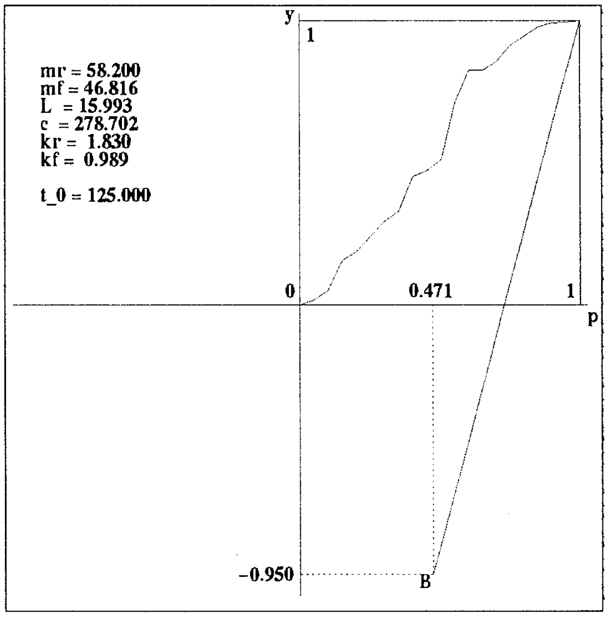



Figure 11: Example 2-3.

noticed that some important problems have been remained in this field. For example, the two-unit redundant systems, the one-unit system with delay should be analyzed by applying the TTT concept.

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