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MAP/G/1 QUEUES UNDER N-POLICY WITH AND WITHOUT VACATIONS

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Abstract This paper considers MAP/G/1 queueing systems under the following two situations: (1) At the end of a busy period, the server is turned off and inspects the queue length every time a customer arrives. When the queue length reaches a pre-specified value N, the server turns on and serves customers continuously until the system becomes empty. (2) At the end of a busy period, the server takes a sequence of vacations. At the end of each vacation, the server inspects the queue length. If the queue length is greater than or equal to a pre-specified value N at this time, the server begins to serve customers continuously until the system becomes empty. For each case, we analyze the stationary queue length and the actual waiting time distributions, and derive the recursive formulas to compute the moments of these distributions. Furthermore, we provide a numerical algorithm to obtain the mass function of the stationary queue length. The numerical examples show that in light traffic, correlation in arrivals leads to a smaller mean waiting time.

1. Introduction

There are many variants in single-server queueing models with server vacations, where a server is unavailable for occasional intervals called vacations. Queueing systems with vacations have been studied for the last two decades and applied to investigate the performance of computer, communication and manufacturing systems. Excellent surveys have been found in [3, 4].

Recently, a single-server queue with multiple vacations and exhaustive services has been analyzed by Lucantoni et al. [9], where customers arrive to the system according to a Markovian arrival process (MAP). MAP includes as special cases the Markov modulated Poisson process (MMPP) and the superposition of phase-type renewal processes. Asmussen and Koole [1] have also shown that MAP is weakly dense in the class of stationary simple point processes. Therefore MAP is a fairly general process and has a capability of representing a wide class of arrival processes.

In this paper, we consider MAP/G/1 vacation models with the following characteristics. Customers arrive to the system according to a MAP with representation (C, D), where C and D are $m \times m$ matrices. Note that m denotes the number of phases in the underlying Markov chain which governs the arrival process. As for the definition of MAP, readers are referred to section 2.1 in [9]. Service times are independent, and identically distributed (i.i.d.) according to a general probability distribution function (PDF) $S(x)$ with finite mean $E[S]$, whose Laplace-Stieltjes transform (LST) is denoted by $S^*(s)$. As for the vacation policy, we consider the following two situations:

1. At the end of a busy period, the server is turned off and inspects the queue length every time a customer arrives. When the queue length reaches a pre-specified value N, the server turns on and serves customers continuously until the system becomes empty.

2. At the end of a busy period, the server takes a sequence of vacations, where vacation times are i.i.d. according to a general PDF $V(x)$ with finite mean $E[V]$. At the end of each vacation, the server inspects the queue length. If the queue length is greater than or equal to a pre-specified value N at this time, the server begins to serve customers continuously until the system becomes empty.

In what follows, Case 1 is referred to as N-policy without vacations and Case 2 as N-policy with vacations. (In [12], Case 1 is referred to as N-policy and Case 2 as vacations with a threshold.) In both cases, there is a possibility that the server remains being idle even when some customers are waiting for their services. Thus, both queues with the above features fall into a category of queues with generalized vacations [2]. Note that when $N = 1$ without vacations, our queueing model is reduced to the ordinary MAP/G/1 queue. Also when $N = 1$ with vacations, our queueing model is reduced to the MAP/G/1 with multiple vacations and the exhaustive service. Thus, the queueing models considered in this paper are regarded as generalizations of those which have been analyzed.

The queueing system under N-policy without vacations has been one of the classical subjects on control of queues (see [5] and references therein). As for the N-policy with vacations, there also have been a number of works. Among them, Hofri [6] and Kella [7] studied the same control policy for the M/G/1 system. Lee and Srinivasan [8] studied the M^X/G/1 system under the N-policy with vacations.

A typical application for N-policy is the quality control problem [7]. A manufacturing plant produces certain items that occasionally are defective. The good items are marketed while the defective ones are kept in storage until they can be reworked to meet specifications. Assume that one of the machines in the plant may be converted as needed from production mode to a repair mode in order to perform this rework. The question is what would be an appropriate cutoff number N such that if the number of defective items is at least N, then the special machine will be converted from the production mode to the repair mode at the next opportunity. After conversion to repair mode, this machine will rework all of the defective items (including new arrivals) exhaustively, and then switch back to the production mode when there are no defective items left.

We can interpret the defective items as the customers and the special machine as the server, where this server is available for serving these customers only when the machine is in the repair mode. The service time is the time required to rework a defective item to meet specifications.

If we count the number of defectives at each time when the defective is produced, we then have a queueing system under N-policy without vacations. On the other hand, if we inspect the number of defectives after a certain period, we have a queueing system under N-policy with vacations.

In [7], authors assumed that defective items occur according to a Bernoulli trial for each machine, and hence, the superposition of the output processes of defective items from the various machines could be regarded as a Poisson process. However, if we consider a few production machines, MAP is suitable for modeling the arrival process.

The queueing models considered in this paper are formulated as Markov chains of M/G/1 type [10]. However, the boundary behavior in our queueing models is complicated, especially in the N-policy with vacations. Thus, the usual approach given in [10] does not seems to be efficient. We provide an alternative approach to compute an essential quantity related to the boundary behavior. Thus, combined with the established methods in [9], [10] and [13], this approach gives a simple and efficient algorithm to compute various quantities of interest.

The remainder of this paper is organized as follows. In section 2, we study the queue length and waiting time distributions for N-policy without vacations. We derive the recursive formulas to compute the queue length distribution, the factorial moments of the queue length distribution and the moments of the actual waiting time distribution. In section 3, we study the queue length and actual waiting time distributions for N-policy with vacations. We derive the recursive formulas to compute the queue length distribution, its factorial moments and the moments of the waiting time distribution. In section 4, we show some numerical examples using the moment formulas of the waiting time for N-policy with and without vacations. In particular, we show that that in light traffic, the correlation in arrivals leads to a smaller mean waiting time. Throughout the paper, we assume that the system is in equilibrium.

2. N-policy without Vacations

In this section, we consider a MAP/G/1 queue under N-policy without vacations in equilibrium. First, we consider the stationary queue length at departures. Then, we consider the stationary queue length distribution at an arbitrary time. We also derive the LST of the actual waiting time distribution for an arriving customer.

2.1. Generating function for queue length at departures

We consider the imbedded Markov chain at departure epochs. Let A_n $(n \geq 0)$ denote an $m \times m$ matrix whose (i,j)th element represents the conditional probability that n customers arrive to the system during a service time of a customer and the underlying Markov chain is in phase j at the end of the service given that the underlying Markov chain is in phase i at the beginning of the service. In the queue under N-policy without vacations, the transition probability matrix P is given by

$$(2.1) \qquad P = \begin{bmatrix} O & O & O & \cdots & O & B_{N-1} & B_N & \cdots \\ A_0 & A_1 & A_2 & \cdots & A_{N-2} & A_{N-1} & A_N & \cdots \\ O & A_0 & A_1 & \cdots & A_{N-3} & A_{N-2} & A_{N-1} & \cdots \\ O & O & A_0 & \cdots & A_{N-4} & A_{N-3} & A_{N-2} & \cdots \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \\ O & O & O & \cdots & A_0 & A_1 & A_2 & \cdots \\ O & O & O & \cdots & O & A_0 & A_1 & \cdots \\ \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots & \end{bmatrix},$$

where B_n $(n \geq N-1)$ denotes an $m \times m$ matrix which is given by

$$B_n = \left[(-C)^{-1}D\right]^N A_{n-N+1}, \qquad n \geq N-1.$$

Note that the factor $(-C)^{-1}D$ represents the phase transition matrix during an interarrival time [9]. As for the computation of A_n, readers are referred to [13]. Let $A(z)$ and $B(z)$ denote matrix generating functions of the A_n and the B_n, respectively:

$$(2.2) \qquad A(z) = \sum_{n=0}^{\infty} A_n z^n, \qquad B(z) = \sum_{n=N-1}^{\infty} B_n z^n.$$

We then have [9]

$$(2.3) \qquad A(z) = \int_0^{\infty} e^{(C+zD)x} dS(x).$$

Furthermore, $B(z)$ is given in terms of $A(z)$:

$$B(z) = \left[(-C)^{-1} Dz\right]^N \frac{A(z)}{z}.$$

Let $\boldsymbol{x}_k$ ($k \geq 0$) denote a $1 \times m$ vector whose ith element represents the stationary joint probability that the number of customers in the system at departures is k and the phase of the arrival process is i. Furthermore, let $\boldsymbol{X}(z) = \sum_{k=0}^{\infty} \boldsymbol{x}_k z^k$, which denotes the generating function of the $\boldsymbol{x}_k$. From (2.1), we have the following equation:

$$\text{(2.4)} \qquad \boldsymbol{X}(z) = \boldsymbol{x}_0 B(z) + [\boldsymbol{X}(z) - \boldsymbol{x}_0] \frac{A(z)}{z},$$

from which we obtain

$$\text{(2.5)} \qquad \boldsymbol{X}(z)\,[zI - A(z)] = \boldsymbol{x}_0 \left\{ \left[(-C)^{-1}D\right]^N z^N - I \right\} A(z).$$

Thus, once we obtain $\boldsymbol{x}_0$, the vector generating function $\boldsymbol{X}(z)$ is completely determined. Before considering $\boldsymbol{x}_0$, we derive some formulas which will be used later.

Let $\boldsymbol{\pi}$ denote a $1 \times m$ vector whose ith element represents the stationary probability of the underlying Markov chain being phase i. Note that $\boldsymbol{\pi}$ satisfies

$$\text{(2.6)} \qquad \boldsymbol{\pi}(C + D) = \boldsymbol{0}, \qquad \boldsymbol{\pi e} = 1,$$

where $\boldsymbol{e}$ denotes an $m \times 1$ vector whose all elements are equal to one. Setting $z = 1$ in (2.5) and adding $\boldsymbol{X}(1)\boldsymbol{e\pi}$ to both sides yield

$$\text{(2.7)} \qquad \boldsymbol{X}(1) = \boldsymbol{\pi} + \boldsymbol{x}_0 \left\{ \left[(-C)^{-1}D\right]^N - I \right\} A(I - A + \boldsymbol{e\pi})^{-1},$$

where $A = A(1)$. We define $\boldsymbol{\beta}$ as $\boldsymbol{\beta} = A'(1)\boldsymbol{e}$. Post-multiplying both sides of (2.7) by $\boldsymbol{\beta}$, we obtain

$$\text{(2.8)} \qquad \boldsymbol{X}(1)\boldsymbol{\beta} = \rho + \boldsymbol{x}_0 \left\{ \left[(-C)^{-1}D\right]^N - I \right\} A(\boldsymbol{e\pi} - C - D)^{-1}D\boldsymbol{e},$$

where ρ denotes the utilization factor which is given by $\boldsymbol{\pi\beta}$. Due to the assumption that the system is in equilibrium, we have $\rho < 1$. In the derivation of (2.8), we use the equality

$$(I - A + \boldsymbol{e\pi})^{-1}\boldsymbol{\beta} = (\boldsymbol{e\pi} - C - D)^{-1}D\boldsymbol{e} + (\rho - \boldsymbol{\pi}D\boldsymbol{e})\boldsymbol{e},$$

which comes from (2.3) and (2.6).

On the other hand, differentiating (2.5) with respect to z, setting $z = 1$ and post-multiplying both sides by $\boldsymbol{e}$ yield

$$\text{(2.9)} \qquad 1 - \boldsymbol{X}(1)\boldsymbol{\beta} = N\boldsymbol{x}_0\boldsymbol{e} + \boldsymbol{x}_0 \left\{ \left[(-C)^{-1}D\right]^N - I \right\} (I - A)(\boldsymbol{e\pi} - C - D)^{-1}D\boldsymbol{e},$$

where we use the equality

$$\boldsymbol{\beta} = (I - A)(\boldsymbol{e\pi} - C - D)^{-1}D\boldsymbol{e} + \rho\boldsymbol{e},$$

which again comes from (2.3) and (2.6). From (2.8) and (2.9), we obtain

$$\begin{aligned}
\text{(2.10)} \qquad 1 - \rho &= N\boldsymbol{x}_0\boldsymbol{e} + \boldsymbol{x}_0 \left\{ \left[(-C)^{-1}D\right]^N - I \right\} (\boldsymbol{e\pi} - C - D)^{-1}D\boldsymbol{e} \\
&= N\boldsymbol{x}_0\boldsymbol{e} + \boldsymbol{x}_0 \sum_{k=0}^{N-1} \left[(-C)^{-1}D\right]^k (-C)^{-1}(C + D)(\boldsymbol{e\pi} - C - D)^{-1}D\boldsymbol{e} \\
&= \lambda\boldsymbol{x}_0 \sum_{k=0}^{N-1} \left[(-C)^{-1}D\right]^k (-C)^{-1}\boldsymbol{e},
\end{aligned}$$

where λ denotes the mean arrival rate which is given by $\boldsymbol{\pi} D \boldsymbol{e}$. Note that $\rho = \lambda E[S]$, which can be verified with (2.3).

Remarks. (2.10) can be rewritten to be

$$\rho = E[S] \Big/ \left\{ E[S] + x_0 e \frac{x_0}{x_0 e} \sum_{k=0}^{N-1} \left[(-C)^{-1} D\right]^k (-C)^{-1} \boldsymbol{e} \right\},$$

where the right hand side is considered as a time fraction of the server being busy between consecutive imbedded points.

2.2. Determination of the vector $\boldsymbol{x}_0$

In this subsection, we obtain a formula to compute $\boldsymbol{x}_0$. We define the level i as the set of states $\{(i, 1), \cdots, (i, m)\}$, $i \geq 0$. We first consider the state transition of the underlying Markov chain during the first passage time from level $i+1$ to level i $(i \geq 0)$. Let G denote an $m \times m$ matrix which represents the state transition matrix of the underlying Markov chain during the first passage time. Then we have [10]

$$(2.11) \quad G = \sum_{\nu=0}^{\infty} A_\nu G^\nu.$$

Note that G is stochastic when $\rho < 1$. Also G satisfies the following equation [9]:

$$G = \int_0^\infty e^{(C+DG)x} dS(x).$$

As for the computation of G, readers are referred to [9] and [13].

Using G, we consider the state transition of the underlying Markov chain during the recurrence time of the level 0. Let K denote an $m \times m$ matrix which represents the state transition matrix of the underlying Markov chain during the recurrence time. Note that K satisfies

$$(2.12) \quad K = \left[(-C)^{-1} D\right]^N G^N.$$

Let $\boldsymbol{\kappa}$ denote the invariant probability vector of K, which satisfies $\boldsymbol{\kappa} K = \boldsymbol{\kappa}$ and $\boldsymbol{\kappa e} = 1$. Once we obtain $\boldsymbol{\kappa}$, we can readily obtain $\boldsymbol{x}_0$. Let $\overline{K}$ denote the mean recurrence time of level zero. By definition, $\boldsymbol{x}_0$ is given in terms of $\boldsymbol{\kappa}$ and $\overline{K}$ [10]

$$(2.13) \quad \boldsymbol{x}_0 = \frac{\boldsymbol{\kappa}}{\overline{K}}.$$

Substituting $\boldsymbol{x}_0$ in (2.13) into (2.10), and solving with respect to $\overline{K}$, we have

$$\overline{K} = \frac{\lambda}{1-\rho} \boldsymbol{\kappa} \sum_{k=0}^{N-1} \left[(-C)^{-1} D\right]^k (-C)^{-1} \boldsymbol{e}.$$

Thus, $\overline{K}$ is given in terms of $\boldsymbol{\kappa}$ and the vector $\boldsymbol{x}_0$ is given by (2.13).

Remarks. In the ordinary M/G/1 paradigm, we first compute the invariant probability vector $\boldsymbol{g}$ of G, and then obtain $\boldsymbol{\kappa}$ and $\overline{K}$ in terms of $\boldsymbol{g}$ [10]. However, in our formulation, we derive the quantities of interest only in terms of $\boldsymbol{\kappa}$ and we don't need to compute $\boldsymbol{g}$.

2.3. Queue length distribution at departure and its moments

In this subsection, we provide the computational algorithm for the queue length distribution $\boldsymbol{x}_k$ $(k \geq 1)$ at departures and its moments. Note that a stable algorithm for the Markov chain of M/G/1 type is provided in [11]. Since (2.1) is of M/G/1 type [10], we follows the algorithm in [11] and obtain the following recursion for $\boldsymbol{x}_k$ $(k \geq 1)$:

$$\boldsymbol{x}_k = \left[\boldsymbol{x}_0 \overline{B}_k + \sum_{j=1}^{k-1} \boldsymbol{x}_j \overline{A}_{k+1-j}\right] \left(I - \overline{A}_1\right)^{-1},$$

where

$$\begin{aligned}
(2.14)\quad \overline{A}_k &= \sum_{n=k}^{\infty} A_n G^{n-k}, \qquad k \geq 1, \\
(2.15)\quad \overline{B}_k &= \sum_{n=N-1}^{\infty} B_n G^{n-k}, \qquad 1 \leq k \leq N-2, \\
(2.16)\quad \overline{B}_k &= \sum_{n=k}^{\infty} B_n G^{n-k}, \qquad k \geq N-1.
\end{aligned}$$

Next we provide a recursive formula to compute the factorial moments of the queue length distribution at departures. We define $\boldsymbol{X}^{(n)}$, $A^{(n)}$ and $B^{(n)}$ as

$$\boldsymbol{X}^{(n)} = \lim_{z \to 1} \frac{d^n}{dz^n} \boldsymbol{X}(z), \qquad A^{(n)} = \lim_{z \to 1} \frac{d^n}{dz^n} A(z), \qquad B^{(n)} = \lim_{z \to 1} \frac{d^n}{dz^n} B(z).$$

We then follow the approach in [10] and obtain the following recursion for the factorial moments of queue length distribution at departures:

$$\boldsymbol{U}^{(n)} = \begin{cases} \boldsymbol{x}_0(B(1) - A(1)), & n = 0, \\ \boldsymbol{x}_0(B^{(1)} + B^{(0)} - A^{(1)}), & n = 1, \\ \sum_{m=0}^{n-2} \binom{n}{m} \boldsymbol{X}^{(m)} A^{(n-m)} + \boldsymbol{x}_0(B^{(n)} + nB^{(n-1)} - A^{(n)}), & n \geq 2, \end{cases}$$

$$\begin{aligned}
\boldsymbol{X}^{(n)}\boldsymbol{e} &= \frac{\boldsymbol{U}^{(n+1)}\boldsymbol{e}}{(n+1)(1-\rho)} + \frac{1}{1-\rho}\{\boldsymbol{U}^{(n)} - n\boldsymbol{X}^{(n-1)}(I - A^{(1)})\} \\
&\quad \cdot [I - A(1) + \boldsymbol{e}\boldsymbol{\pi}]^{-1} A^{(1)}\boldsymbol{e}, \qquad n \geq 1,
\end{aligned}$$

$$\boldsymbol{X}^{(n)} = \begin{cases} \boldsymbol{\pi} + \boldsymbol{U}^{(0)}[I - A(1) + \boldsymbol{e}\boldsymbol{\pi}]^{-1}, & n = 0, \\ \boldsymbol{X}^{(n)}\boldsymbol{e}\boldsymbol{\pi} + \{\boldsymbol{U}^{(n)} - n\boldsymbol{X}^{(n-1)}(I - A^{(1)})\}[I - A(1) + \boldsymbol{e}\boldsymbol{\pi}]^{-1}, & n \geq 1, \end{cases}$$

where

$$\boldsymbol{X}^{(0)} = \boldsymbol{X}(1), \qquad A^{(0)} = A(1), \qquad B^{(0)} = B(1).$$

Namely, computing $\boldsymbol{U}^{(0)}$, $\boldsymbol{X}^{(0)}$, $\boldsymbol{U}^{(1)}$ and then $\boldsymbol{U}^{(k+1)}$, $\boldsymbol{X}^{(k)}\boldsymbol{e}$, $\boldsymbol{X}^{(k)}$ in this order, we obtain the nth factorial moment $\boldsymbol{X}^{(n)}$ of the queue length distribution at departures.

2.4. Queue length distribution at an arbitrary time and its moments

In this subsection, we consider the distribution of the number of customers in the system at an arbitrary time. Let $\boldsymbol{y}_n$ denote a $1 \times m$ vector whose ith element is the stationary joint probability that the number of customers in the system is n and the phase of the arrival process is i at an arbitrary time. Let $\boldsymbol{Y}(z) = \sum_{n=0}^{\infty} \boldsymbol{y}_n z^n$, which denotes the generating function of the $\boldsymbol{y}_n$. $\boldsymbol{Y}(z)$ consists of the idle term and busy one. Let U denote the idle time of the server. Then, we obtain the mean idle time as

$$(2.17) \quad E[U] = \frac{\boldsymbol{x}_0}{\boldsymbol{x}_0 \boldsymbol{e}} \sum_{k=0}^{N-1} \left[(-C)^{-1}D\right]^k (-C)^{-1}\boldsymbol{e}.$$

Using (2.10) and (2.17), we obtain the vector whose ith element represents the conditional probability that the number of customers in the system is k and the phase of the arrival process is i given that the server is idle:

$$(2.18) \quad \frac{1}{E[U]} \frac{\boldsymbol{x}_0}{\boldsymbol{x}_0 \boldsymbol{e}} \left[(-C)^{-1}D\right]^k (-C)^{-1} = \frac{\lambda}{1-\rho} \boldsymbol{x}_0 \left[(-C)^{-1}D\right]^k (-C)^{-1}.$$

Using (2.18), we obtain

$$\begin{aligned}(2.19) \quad \boldsymbol{Y}(z) &= (1-\rho) \sum_{k=0}^{N-1} \frac{\lambda}{1-\rho} \boldsymbol{x}_0 \left[(-C)^{-1}D\right]^k (-C)^{-1} z^k \\ &\quad + \rho \left\{ \boldsymbol{X}(z) - \boldsymbol{x}_0 + \boldsymbol{x}_0 \left[(-C)^{-1}D\right]^N z^N \right\} A^*(z) \\ &= \lambda \boldsymbol{x}_0 \sum_{k=0}^{N-1} \left[(-C)^{-1}D\right]^k (-C)^{-1} z^k \\ &\quad + \rho \left\{ \boldsymbol{X}(z) - \boldsymbol{x}_0 + \boldsymbol{x}_0 \left[(-C)^{-1}D\right]^N z^N \right\} A^*(z),\end{aligned}$$

where $A^*(z)$ is the matrix generating function of the number of arrivals during the forward recurrence time of a service time and given by [9]

$$(2.20) \quad A^*(z) = \frac{1}{E[S]} [A(z) - I] (C + zD)^{-1}.$$

From (2.4) and (2.20), the second term in (2.19) becomes

$$\begin{aligned}(2.21) \quad & \rho \left\{ \boldsymbol{X}(z) - \boldsymbol{x}_0 + \boldsymbol{x}_0 \left[(-C)^{-1}D\right]^N z^N \right\} A^*(z) \\ &= \lambda(z-1)\boldsymbol{X}(z)(C+zD)^{-1} - \lambda \boldsymbol{x}_0 \sum_{k=0}^{N-1} \left[(-C)^{-1}D\right]^k (-C)^{-1} z^k.\end{aligned}$$

Substituting (2.21) into (2.19), we obtain

$$(2.22) \quad \boldsymbol{Y}(z) = \lambda(z-1)\boldsymbol{X}(z)(C+zD)^{-1}.$$

(2.22) shows the relationship between the queue length distribution at departures and at an arbitrary time. Since this relationship holds for any stationary queue with MAP arrivals [14], an independent verification provides a validation for our analysis so far.

Post-multiply both sides of (2.22) by $(C + zD)$ and comparing the coefficients of z^k in both sides, we obtain the following recursion for $\boldsymbol{y}_k$ $(k \geq 0)$ in terms of the $\boldsymbol{x}_k$:

$$\begin{aligned}(2.23) \quad \boldsymbol{y}_0 &= \lambda \boldsymbol{x}_0 (-C)^{-1}, \\ (2.24) \quad \boldsymbol{y}_k &= \boldsymbol{y}_{k-1} D(-C)^{-1} + \lambda(\boldsymbol{x}_k - \boldsymbol{x}_{k-1})(-C)^{-1}, \qquad k \geq 1.\end{aligned}$$

Next we consider the factorial moments of the queue length distribution at an arbitrary time. We define $\boldsymbol{Y}^{(n)}$ as

$$\boldsymbol{Y}^{(n)} = \lim_{z\to 1} \frac{d^n}{dz^n}\boldsymbol{Y}(z).$$

We follow the approach in [9] and obtain the following recursion to compute $\boldsymbol{Y}^{(n)}$ $(n \geq 1)$:

$$\begin{aligned}
\boldsymbol{Y}^{(0)} &= \boldsymbol{\pi}, \\
\boldsymbol{Y}^{(n)}\boldsymbol{e} &= \boldsymbol{X}^{(n)}\boldsymbol{e} - n\left(\boldsymbol{Y}^{(n-1)}D/\lambda - \boldsymbol{X}^{(n-1)}\right)(\boldsymbol{e\pi} - C - D)^{-1}D\boldsymbol{e}, \quad n \geq 1, \\
\boldsymbol{Y}^{(n)} &= \boldsymbol{Y}^{(n)}\boldsymbol{e\pi} + n\left(\boldsymbol{Y}^{(n-1)}D - \lambda\boldsymbol{X}^{(n-1)}\right)(\boldsymbol{e\pi} - C - D)^{-1}, \quad n \geq 1,
\end{aligned}$$

where $\boldsymbol{Y}^{(0)} = \boldsymbol{Y}(1)$.

2.5. LST for actual waiting time and its moments

In this section, we consider the waiting time distribution of an arriving customer. To do so, we first consider the waiting time of a customer which arrives when the server is idle. Let $\boldsymbol{y}_k^+$ denote a $1 \times m$ vector whose ith element represents the joint probability that a customer arrives when the server is idle, finds k waiting customers upon arrival, and the state of the arrival process immediately after the arrival is i. Using (2.18), We then have

$$\boldsymbol{y}_k^+ = (1-\rho)\cdot\frac{\lambda}{1-\rho}\boldsymbol{x}_0\left[(-C)^{-1}D\right]^k(-C)^{-1}D/\lambda = \boldsymbol{x}_0\left[(-C)^{-1}D\right]^{k+1}.$$

Thus, the LST $W_1^*(s)$ of the waiting time distribution when the customer arrives during an idle time of the server is given by

$$\begin{aligned}
W_1^*(s) &= \sum_{k=0}^{N-1}\boldsymbol{y}_k^+[(sI-C)^{-1}D]^{N-k-1}\boldsymbol{e}[S^*(s)]^k \\
&= \boldsymbol{x}_0\sum_{k=0}^{N-1}[(-C)^{-1}D]^{k+1}\left[(sI-C)^{-1}D\right]^{N-k-1}[S^*(s)]^k\boldsymbol{e}.
\end{aligned}$$

Next, we consider the waiting time of a customer which arrives when the server is busy. To do so, we first derive the joint transform for the number of customers and the forward recurrence time of the current service when the server is busy. Note that the server is busy with probability ρ. Given that the server is busy, customers in the system is classified into two types. One includes customers which are in the system when the current service starts. The other includes customers which arrive during the backward recurrence time of the current service. Thus we have the joint transform $Y^*(z,s)$ for the number of customers and the forward recurrence time at an arbitrary point of the current service:

$$Y^*(z,s) = \rho\left\{\boldsymbol{X}(z) - \boldsymbol{x}_0 + z^N\boldsymbol{x}_0[(-C)^{-1}D]^N\right\}A(z,s),$$

where $A(z,s)$ denotes the joint transformed matrix for the number of customers which arrive in the backward recurrence time and the forward recurrence time, and is given by

$$A(z,s) = \int_0^\infty \frac{x\,dS(x)}{E[S]}\int_0^x \frac{dt}{x}e^{(C+zD)t}e^{-s(x-t)} = \frac{A(z)-S^*(s)I}{E[S]}[sI+C+zD]^{-1}.$$

Therefore we obtain the LST $W_2^*(s)$ for the waiting time distribution of a customer which arrives when the server is busy as follows:

$$\begin{aligned} W_2^*(s) &= Y^*(S^*(s),s)D\boldsymbol{e}/\lambda S^*(s) \\ &= \boldsymbol{x}_0\left\{I-[(-C)^{-1}D]^N[S^*(s)]^N\right\}[sI+C+S^*(s)D]^{-1}D\boldsymbol{e}, \end{aligned}$$

where we use the equality

$$\boldsymbol{X}(S^*(s))[S^*(s)I-A(S^*(s))]=\left\{\boldsymbol{x}_0[(-C)^{-1}D]^N[S^*(s)]^N-I\right\}A(S^*(s)),$$

which comes from (2.5).

Let $W^*(s)$ denote the LST for the actual waiting time distribution. By definition, $W^*(s)$ is given by $W_1^*(s)+W_2^*(s)$. Therefore we obtain

$$\begin{aligned} \text{(2.25)}\quad W^*(s) &= \boldsymbol{x}_0\sum_{k=0}^{N-1}[(-C)^{-1}D]^{k+1}\left[(sI-C)^{-1}D\right]^{N-k-1}[S^*(s)]^k\boldsymbol{e} \\ &\quad +\boldsymbol{x}_0\left\{I-[(-C)^{-1}D]^N[S^*(s)]^N\right\}[sI+C+S^*(s)D]^{-1}D\boldsymbol{e}. \end{aligned}$$

We now consider the moments of the actual waiting time. We first define $W^{(n)}$ as

$$W^{(n)}=\lim_{s\to 0}(-1)^n\frac{d^n}{ds^n}W(s),\qquad n\geq 1.$$

To obtain the recursive formula to compute $W^{(n)}$, We rewrite (2.25) as

$$W^*(s)=\boldsymbol{x}_0\sum_{k=0}^{N-1}[(-C)^{-1}D]^{k+1}\,T_k(s)\boldsymbol{e}+\boldsymbol{x}_0\,T(s)\,D\boldsymbol{e},$$

where

$$\begin{aligned} T_k(s) &= \left[(sI-C)^{-1}D\right]^{N-k-1}[S^*(s)]^k, \qquad 0\leq k\leq N-1, \\ \text{(2.26)}\quad T(s) &= \left\{I-[(-C)^{-1}D]^N[S^*(s)]^N\right\}[sI+C+S^*(s)D]^{-1}. \end{aligned}$$

We then have

$$W^{(n)}=\boldsymbol{x}_0\sum_{k=0}^{N-1}[(-C)^{-1}D]^{k+1}T_k^{(n)}\boldsymbol{e}+\boldsymbol{x}_0T^{(n)}D\boldsymbol{e},\qquad n\geq 1,$$

where for $n\geq 1$,

$$T_k^{(n)}=\lim_{s\to 0}(-1)^n\frac{d^n}{ds^n}T_k(s),\qquad T^{(n)}=\lim_{s\to 0}(-1)^n\frac{d^n}{ds^n}T(s).$$

Thus once we have $T_k^{(n)}$ and $T^{(n)}$, $W^{(n)}$ is readily obtained. In what follows, we provide the recursive formula to compute $T_k^{(n)}$ and $T^{(n)}$.

First, we consider $T_k^{(n)}$ $(n\geq 1)$. We define $H_k(s)$ and $S_k(s)$ as

$$H_k(s)=\left[(sI-C)^{-1}D\right]^k,\qquad S_k(s)=[S^*(s)]^k.$$

Then, $T_k(s)=H_{N-k-1}(s)S_k(s)$. Furthermore, we define $H_k^{(n)}$ and $S_k^{(n)}$ as

$$H_k^{(n)}=\lim_{s\to 0}(-1)^n\frac{d^n}{ds^n}H_k(s),\qquad S_k^{(n)}=\lim_{s\to 0}(-1)^n\frac{d^n}{ds^n}S_k(s),$$

and $S^{(n)} = S_1^{(n)}$. Note that $S^{(1)} = E[S]$. Taking the nth derivative of $H_1(s)$, we obtain $H_1^{(n)} = n!\,(-C)^{-(n+1)}D$. Since $H_k(s) = H_1(s)\,H_{k-1}(s)$, we compute the nth derivative $H_k^{(n)}$ using the recursion

$$H_k^{(n)} = \sum_{i=0}^{n} \binom{n}{i} H_1^{(i)}\, H_{k-1}^{(n-i)},$$

where $H_k^{(0)} = [(-C)^{-1}D]^k$. Similarly, we compute the nth derivative $S_k^{(n)}$ using the recursion

$$S_k^{(n)} = \sum_{i=0}^{n} \binom{n}{i} S^{(i)}\, S_{k-1}^{(n-i)},$$

where $S^{(0)} = 1$. Thus we obtain the nth derivative $T_k^{(n)}$ by

$$T_k^{(n)} = \sum_{i=0}^{n} \binom{n}{i} H_{N-k-1}^{(i)}\, S_k^{(n-i)}.$$

Secondly, we consider the nth derivative $T^{(n)}$ of $T(s)$. Using (2.26), it follows

$$T(s)[sI + C + S^*(s)D] = U(s),$$

where

$$U(s) = I - [(-C)^{-1}D]^N[S^*(s)]^N.$$

We define $U^{(n)}$ $(n \geq 1)$ as

$$U^{(0)} = U(0), \qquad U^{(n)} = \lim_{s\to 0}(-1)^n \frac{d^n}{ds^n}U(s), \qquad n \geq 1.$$

Then, we obtain

$$U^{(0)} = I - [(-C)^{-1}D]^N, \qquad U^{(n)} = -[(-C)^{-1}D]^N S_N^{(n)}.$$

According to a similar reasoning in [9], we obtain the following recursion to compute $T^{(n)}$:

$$\begin{aligned}
Z^{(n)} &= -nT^{(n-1)} + \sum_{k=0}^{n-1} \binom{n}{k} T^{(k)} S^{(n-k)} D - U^{(n)}, \qquad n \geq 1,\\
T^{(0)}\boldsymbol{e} &= \frac{1}{1-\rho}\left\{-U^{(1)}\boldsymbol{e} - E[S]\,U^{(0)}(\boldsymbol{e\pi} - C - D)^{-1}D\boldsymbol{e}\right\},\\
T^{(n)}\boldsymbol{e} &= \frac{E[S]}{1-\rho} Z^{(n)}(\boldsymbol{e\pi} - C - D)^{-1}D\boldsymbol{e} + \frac{1}{(n+1)(1-\rho)}\\
&\quad \cdot \left\{\sum_{k=0}^{n-1} \binom{n+1}{k} S^{(n+1-k)} T^{(k)} D\boldsymbol{e} - U^{(n+1)}\boldsymbol{e}\right\}, \qquad n \geq 1,\\
T^{(0)} &= T^{(0)}\boldsymbol{e\pi} - U^{(0)}(\boldsymbol{e\pi} - C - D)^{-1},\\
T^{(n)} &= T^{(n)}\boldsymbol{e\pi} + Z^{(n)}(\boldsymbol{e\pi} - C - D)^{-1}, \qquad n \geq 1,
\end{aligned}$$

where $T^{(0)} = T(0)$. We summarize the procedure to compute $W^{(n)}$.

1. Compute $H_k^{(n)}$ and $S_k^{(n)}$ recursively.

2. Compute $T_k^{(n)}$ using $H_k^{(n)}$ and $S_k^{(n)}$.
3. Compute $U^{(0)}$, $T^{(0)}\boldsymbol{e}$ and $T^{(0)}$ in this order.
4. Compute $U^{(n)}$, $Z^{(n)}$, $T^{(n)}\boldsymbol{e}$ and $T^{(n)}$ recursively.
5. Compute $W^{(n)}$ using $T_k^{(n)}$ and $T^{(n)}$.

Remarks
1. Setting $N = 1$ in (2.25), we obtain

$$W^*(s) = \lambda^{-1}s\boldsymbol{y}_0[sI + C + S^*(s)D]^{-1}D\boldsymbol{e},$$

which is identical to the result in [9].
2. In the case that customers arrive according to a Poisson process with rate λ, $C = -\lambda$ and $D = \lambda$. Substituting these into (2.10) yields $\boldsymbol{x}_0 = (1 - \rho)/N$. Furthermore, (2.25) becomes

$$W^*(s) = \frac{1-\rho}{N}\frac{[\lambda/(s+\lambda)]^N - [S^*(s)]^N}{\lambda/(s+\lambda) - S^*(s)} + \frac{\lambda(1-\rho)\{1-[S^*(s)]^N\}}{N[s - \lambda + \lambda S^*(s)]},$$

which is the LST of the waiting time distribution of M/G/1 under N-policy [12].

3. N-policy with Vacations

In this section, we consider a MAP/G/1 under N-policy with vacations in equilibrium. First, we consider the queue length distribution at departures. Then, we derive the formula of the queue length at an arbitrary time. We also derive the LST of the actual waiting time distribution for an arriving customer.

3.1. Generating function for queue length at departures

We choose the time epochs immediately after the service termination and the vacation termination as imbedded points. Let $\boldsymbol{x}_n^{\rm s}$ ($\boldsymbol{x}_n^{\rm v}$) denote the joint probability vectors whose ith element represents the probability that the imbedded point is the service (vacation) termination, the number of the system is n and the phase of the arrival process is i. We define the following generating functions:

$$\boldsymbol{X}^{\rm s}(z) = \sum_{n=0}^{\infty}\boldsymbol{x}_n^{\rm s}z^n, \qquad \boldsymbol{X}^{\rm v}(z) = \sum_{n=0}^{\infty}\boldsymbol{x}_n^{\rm v}z^n, \qquad \boldsymbol{X}_k^{\rm v}(z) = \sum_{n=0}^{k}\boldsymbol{x}_n^{\rm v}z^n.$$

Let V_n denote an $m \times m$ vector whose (i,j)th element represents the conditional probability that n customers arrive during a vacation and the underlying Markov chain is in state j at the end of the vacation given that the underlying Markov chain being in state i at the beginning of the vacation. Let $V(z) = \sum_{n=0}^{\infty} V_n z^n$, which denotes the matrix generating function of the V_n. We then have [9]

$$V(z) = \int_0^\infty e^{(C+zD)x}\,dV(x).$$

Considering the transition between consecutive imbedded points, we have the following equations:

$$\boldsymbol{X}^{\rm s}(z) = \left[\boldsymbol{X}^{\rm s}(z) - \boldsymbol{x}_0^{\rm s}\right]\frac{A(z)}{z} + \left[\boldsymbol{X}^{\rm v}(z) - \boldsymbol{X}_{N-1}^{\rm v}(z)\right]\frac{A(z)}{z}, \tag{3.1}$$

$$\boldsymbol{X}^{\rm v}(z) = \boldsymbol{x}_0^{\rm s}V(z) + \boldsymbol{X}_{N-1}^{\rm v}(z)V(z), \tag{3.2}$$

(3.3) $\quad [\boldsymbol{X}^{\mathrm{s}}(1)+\boldsymbol{X}^{\mathrm{v}}(1)]\,\boldsymbol{e}=1,$

where $A(z)$ is defined in (2.2).

Note that $\boldsymbol{x}_k^{\mathrm{v}}$ $(0 \le k \le N-1)$ are recursively obtained in terms of $\boldsymbol{x}_0^{\mathrm{s}}$:

$$\boldsymbol{x}_0^{\mathrm{v}} = \boldsymbol{x}_0^{\mathrm{s}} V_0 \left[I - V_0\right]^{-1}, \tag{3.4}$$

$$\boldsymbol{x}_k^{\mathrm{v}} = \left[\boldsymbol{x}_0^{\mathrm{s}} V_k + \sum_{i=0}^{k-1} \boldsymbol{x}_i^{\mathrm{v}} V_{k-i}\right] \left[I - V_0\right]^{-1}, \qquad 1 \le k \le N-1. \tag{3.5}$$

Thus, $\boldsymbol{X}^{\mathrm{v}}(z)$ is given in terms of $\boldsymbol{x}_0^{\mathrm{s}}$ (see (3.2)) and therefore $\boldsymbol{X}^{\mathrm{s}}(z)$ contains only one unknown vector $\boldsymbol{x}_0^{\mathrm{s}}$. Note that the queue length at departures is characterized by $\boldsymbol{X}^{\mathrm{s}}(z)$. Let $\boldsymbol{x}_k$ $(k \ge 0)$ denote a $1 \times m$ vector whose ith element represents the joint probability of k customers in the system and phase i of the underlying Markov chain at departures. Further, let $\boldsymbol{X}(z) = \sum_{k=0}^{\infty} \boldsymbol{x}_k z^k$, which denotes the vector generating function for the $\boldsymbol{x}_k$. By definition, we have $\boldsymbol{X}(z) = \boldsymbol{X}^{\mathrm{s}}(z)/\boldsymbol{X}^{\mathrm{s}}(1)\boldsymbol{e}$. Thus once we obtain $\boldsymbol{x}_0^{\mathrm{s}}$, $\boldsymbol{X}(z)$ is completely determined. Before considering $\boldsymbol{x}_0^{\mathrm{s}}$, we derive some formulas which will be used later. Using (3.1), (3.2) and (3.3), we have the following equation:

$$\left(\boldsymbol{x}_0^{\mathrm{s}} + \boldsymbol{X}_{N-1}^{\mathrm{v}}(1)\right)\boldsymbol{e} = \frac{1-\rho}{1-\rho+\lambda E[V]}. \tag{3.6}$$

The derivation of (3.6) is given in Appendix 1. Using (3.6) and (A.3), we have

$$\boldsymbol{X}^{\mathrm{s}}(1)\boldsymbol{e} = \frac{\lambda E[V]}{1-\rho+\lambda E[V]},$$

and therefore we obtain

$$\boldsymbol{X}(z) = \frac{1-\rho+\lambda E[V]}{\lambda E[V]}\boldsymbol{X}^{\mathrm{s}}(z). \tag{3.7}$$

3.2. Computation of the vector $\boldsymbol{x}_0^{\mathrm{s}}$

In this subsection, we derive a formula to compute $\boldsymbol{x}_0^{\mathrm{s}}$. First, we consider the number of customers at the end of an idle period when the threshold value is equal to n $(1 \le n \le N)$. Let R_k^{n} $(k \ge n)$ denote an $m \times m$ matrix whose (i, j)th element represents the conditional probability that there are k customers in the system and the underlying Markov chain is in state j at the end of an idle period given that the underlying Markov chain being in state i at the beginning of the idle period. Note that the R_k^{n} is computed by the following recursion:

$$R_k^{1} = \left[I - V_0\right]^{-1} V_k, \qquad k \ge 1, \tag{3.8}$$

$$R_k^{\mathrm{n}} = R_k^{\mathrm{n}-1} + R_{n-1}^{\mathrm{n}-1} \cdot R_{k-n+1}^{1}, \qquad 2 \le n \le k. \tag{3.9}$$

For later use, we define the matrix generating function $R^{\mathrm{n}}(z)$ as:

$$R^{\mathrm{n}}(z) = \sum_{k=n}^{\infty} R_k^{\mathrm{n}} z^k.$$

Now we consider the state transition during the recurrence time of the departure instant being in level zero. Let K denote an $m \times m$ matrix which represents the state transition matrix of the underlying Markov chain in the recurrence time. Furthermore, let $\boldsymbol{\kappa}$ denote the invariant probability vector of K. Then $\boldsymbol{x}_0^{\mathrm{s}}$ is given by $\boldsymbol{x}_0^{\mathrm{s}} = \boldsymbol{\kappa}/\overline{K}$, where $\overline{K}$ denotes the mean recurrence time of the departure instant being in level zero.

Note that, with R_n^{N}, K is given by

$$K = \sum_{k=N}^{\infty} R_k^{\mathrm{N}} G^k,$$

where G is defined in (2.11). Thus, $\boldsymbol{\kappa}$ is obtain by solving $\boldsymbol{\kappa}K = \boldsymbol{\kappa}$ and $\boldsymbol{\kappa}\boldsymbol{e} = 1$. We now propose a simple recursive formula to compute $\overline{K}$. Multiplying both sides of (3.4) and (3.5) by $\overline{K}$, we obtain

$$(3.10) \quad \boldsymbol{x}_0^{\mathrm{v}*} = \boldsymbol{\kappa} V_0 \left[I - V_0\right]^{-1},$$

$$(3.11) \quad \boldsymbol{x}_k^{\mathrm{v}*} = \left[\boldsymbol{\kappa} V_k + \sum_{i=0}^{k-1} \boldsymbol{x}_i^{\mathrm{v}*} V_{k-i}\right] \left[I - V_0\right]^{-1}, \qquad 1 \le k \le N-1,$$

where

$$\boldsymbol{x}_k^{\mathrm{v}*} = \overline{K} \boldsymbol{x}_k^{\mathrm{v}}.$$

Also, multiplying both sides of (3.6) by $\overline{K}$, we obtain

$$1 + \sum_{k=0}^{N-1} \boldsymbol{x}_k^{\mathrm{v}*} \boldsymbol{e} = \frac{1-\rho}{1-\rho+\lambda E[V]} \overline{K},$$

from which, it follows that

$$(3.12) \quad \overline{K} = \frac{1-\rho+\lambda E[V]}{1-\rho} \left(1 + \sum_{k=0}^{N-1} \boldsymbol{x}_k^{\mathrm{v}*} \boldsymbol{e}\right).$$

Therefore, $\overline{K}$ is computed as follows. First we compute $\boldsymbol{x}_k^{\mathrm{v}*}$ $(0 \le k \le N-1)$ by (3.10) and (3.11) and then compute $\overline{K}$ by (3.12).

3.3. Queue length distribution at departures and its moments

We first consider the queue length distribution at departures. Observing the system immediately after departures, we have the following transition matrix P:

$$P = \begin{bmatrix} O & O & O & \cdots & O & B_{N-1} & B_N & \cdots \\ A_0 & A_1 & A_2 & \cdots & A_{N-2} & A_{N-1} & A_N & \cdots \\ O & A_0 & A_1 & \cdots & A_{N-3} & A_{N-2} & A_{N-1} & \cdots \\ O & O & A_0 & \cdots & A_{N-4} & A_{N-3} & A_{N-2} & \cdots \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \\ O & O & O & \cdots & A_0 & A_1 & A_2 & \cdots \\ O & O & O & \cdots & O & A_0 & A_1 & \cdots \\ \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots & \end{bmatrix},$$

where

$$B_k = \sum_{n=N}^{k+1} R_n^{\mathrm{N}} A_{k+1-n}, \qquad k \ge N-1.$$

Since the transition matrix P takes the same form as in (2.1), we have the same recursion for $\boldsymbol{x}_k^{\mathrm{s}}$ as in section 2:

$$\boldsymbol{x}_k^{\mathrm{s}} = \left[\boldsymbol{x}_0^{\mathrm{s}} \overline{B}_k + \sum_{j=1}^{k-1} \boldsymbol{x}_j^{\mathrm{s}} \overline{A}_{k+1-j}\right] \left(I - \overline{A}_1\right)^{-1},$$

where $\overline{A}_k$ and $\overline{B}_k$ are given in (2.14), (2.15) and (2.16). Thus, from (3.7), the queue length distribution $\boldsymbol{x}_k$ is computed by

$$\boldsymbol{x}_k = \frac{1-\rho+\lambda E[V]}{\lambda E[V]}\boldsymbol{x}_k^{\mathrm{s}}, \qquad k \geq 0.$$

Since the structure of the transition matrix is exactly the same as in section 2, we can use the same recursive formula in subsection 2.3 to compute the factorial moments for the queue length at departures.

3.4. Queue length distribution at an arbitrary time and its moments

Let $\boldsymbol{Y}(z)$ denote the vector generating function of the number of customers at an arbitrary time. According to a similar reasoning as in subsection 2.4, we obtain

$$\text{(3.13)} \quad \boldsymbol{Y}(z) = (1-\rho)\frac{\boldsymbol{x}_0^{\mathrm{s}}+\boldsymbol{X}_{N-1}^{\mathrm{v}}(z)}{\left(\boldsymbol{x}_0^{\mathrm{s}}+\boldsymbol{X}_{N-1}^{\mathrm{v}}(1)\right)\boldsymbol{e}}V^*(z) + \rho\frac{\boldsymbol{X}^{\mathrm{s}}(z)-\boldsymbol{x}_0^{\mathrm{s}}+\boldsymbol{X}^{\mathrm{v}}(z)-\boldsymbol{X}_{N-1}^{\mathrm{v}}(z)}{1-\left(\boldsymbol{x}_0^{\mathrm{s}}+\boldsymbol{X}_{N-1}^{\mathrm{v}}(1)\right)\boldsymbol{e}}A^*(z),$$

where $A^*(z)$ is given in (2.20) and $V^*(z)$ is the matrix generating function of the number of arrivals during the forward recurrence time of a vacation and given by:

$$V^*(z) = \frac{1}{E[V]}\left[V(z)-I\right]\left[C+zD\right]^{-1}.$$

Substituting $V^*(z)$ and $A^*(z)$ into (3.13) and noting the following equalities

$$\left[\boldsymbol{x}_0^{\mathrm{s}}+\boldsymbol{X}_{N-1}^{\mathrm{v}}(z)\right]\left[V(z)-I\right] = \boldsymbol{x}_0^{\mathrm{s}}\left[R^{\mathrm{N}}(z)-I\right],$$

$$\left[\boldsymbol{X}^{\mathrm{s}}(z)-\boldsymbol{x}_0^{\mathrm{s}}+\boldsymbol{X}^{\mathrm{v}}(z)-\boldsymbol{X}_{N-1}^{\mathrm{v}}(z)\right]\left[A(z)-I\right] = (z-1)\boldsymbol{X}^{\mathrm{s}}(z)-\boldsymbol{x}_0^{\mathrm{s}}\left[R^{\mathrm{N}}(z)-I\right],$$

we rewrite $\boldsymbol{Y}(z)$ as

$$\begin{aligned}\text{(3.14)} \quad \boldsymbol{Y}(z) &= \frac{1-\rho}{E[V]}\frac{\boldsymbol{x}_0^{\mathrm{s}}}{\left(\boldsymbol{x}_0^{\mathrm{s}}+\boldsymbol{X}_{N-1}^{\mathrm{v}}(1)\right)\boldsymbol{e}}\left[R^{\mathrm{N}}(z)-I\right]\left[C+zD\right]^{-1} \\ &\quad + \frac{\rho}{E[S]}\frac{(z-1)\boldsymbol{X}^{\mathrm{s}}(z)-\boldsymbol{x}_0^{\mathrm{s}}\left[R^{\mathrm{N}}(z)-I\right]}{1-\left(\boldsymbol{x}_0^{\mathrm{s}}+\boldsymbol{X}_{N-1}^{\mathrm{v}}(1)\right)\boldsymbol{e}}\left[C+zD\right]^{-1} \\ &= \frac{1-\rho+\lambda E[V]}{E[V]}(z-1)\boldsymbol{X}^{\mathrm{s}}(z)\left[C+zD\right]^{-1} \\ &= \lambda(z-1)\boldsymbol{X}(z)\left[C+zD\right]^{-1}.\end{aligned}$$

In (3.14), $\boldsymbol{X}(z)$ denotes the vector generating function of the queue length at departures (see (3.7)). Since this relationship holds for any stationary queue with MAP arrivals [14], an independent verification provides a validation for our analysis so far.

Since the queue length distributions at departures and at an arbitrary time are related by the common equation, the queue length distribution $\boldsymbol{y}_k$ at an arbitrary time is recursively obtained by (2.23) and (2.24) in terms of the queue length distribution $\boldsymbol{x}_k$ at departures. Furthermore using the recursion in section 2.4, we obtain the factorial moments of the queue length distribution at an arbitrary time.

3.5. Joint PDF of number of arrivals and remaining vacation time

Let $\Omega(i,j,x)$ $(i,j=0,1,\dots)$ denote an $m\times m$ matrix whose (k,l)th element represents the probability that, given the phase being in i at the beginning of the vacation and a customer arrival in the vacation, i customers arrive in the elapsed vacation time, j customers arrive in the remaining vacation time, the remaining vacation time is not greater than x and the phase is j at the end of the vacation. We also define the joint transformed matrix of $\Omega(i,j,x)$ as

$$\Omega^*(z_1,z_2,s)=\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\int_0^{\infty} z_1^i z_2^j e^{-sx} d\Omega(i,j,x).$$

Then, $\Omega^*(z_1,z_2,s)$ becomes

$$\begin{aligned}\Omega^*(z_1,z_2,s) &= \int_0^{\infty}\frac{xdV(x)}{E[V]}\int_0^x \frac{dt}{x}e^{(C+z_1D)t}\cdot\frac{D}{\lambda}\cdot e^{(C+z_2D)(x-t)}e^{-s(x-t)}\\ &= \int_0^{\infty}\frac{dV(x)}{\lambda E[V]}\int_0^x dt\, e^{-\theta t}\sum_{m=0}^{\infty}\frac{t^m}{m!}(\theta I+C+z_1D)^m De^{-\theta(x-t)}\\ &\quad\times\sum_{n=0}^{\infty}\frac{(x-t)^n}{n!}(\theta I+C+z_2D)^n e^{-s(x-t)}\\ &= \int_0^{\infty} e^{-(s+\theta)x}\frac{dV(x)}{\lambda E[V]}\int_0^x e^{st}dt\sum_{i=0}^{\infty}\sum_{n=0}^{i}\frac{t^{i-n}}{(i-n)!}\frac{(x-t)^n}{n!}\\ &\quad\times(\theta I+C+z_1D)^{i-n}D(\theta I+C+z_2D)^n. \end{aligned}\tag{3.15}$$

In order to expand the matrix factor $(\theta I+C+z_1D)^kD(\theta I+C+z_2D)^l$, we introduce matrices $F_{k,l}(m,n)$ $(k,l=0,1,2,\cdots,\ m=0,1,\cdots,k,\ n=0,1,\cdots,l)$ which satisfy

$$\sum_{m=0}^{k}\sum_{n=0}^{l} z_1^m z_2^n F_{k,l}(m,n)=(\theta I+C+z_1D)^kD(\theta I+C+z_2D)^l,$$

where $F_{0,0}(0,0)=D$. Then, matrices $F_{k,l}(m,n)$ satisfies the following recursion

$$F_{k+1,l}(m,n) = \begin{cases}(\theta I+C)F_{k,l}(0,n), & m=0,\\ DF_{k,l}(m-1,n)+(\theta I+C)F_{k,l}(m,n), & 1\le m\le k,\\ DF_{k,l}(k,n), & m=k+1,\end{cases}\tag{3.16}$$

and

$$F_{k,l+1}(m,n) = \begin{cases}F_{k,l}(m,0)(\theta I+C), & n=0,\\ F_{k,l}(m,n-1)D+F_{k,l}(m,n)(\theta I+C), & 1\le n\le l,\\ F_{k,l}(m,l)D, & n=l+1.\end{cases}\tag{3.17}$$

Thus, we obtain

$$\begin{aligned}&\sum_{i=0}^{\infty}\sum_{n=0}^{i}\frac{t^{i-n}}{(i-n)!}\frac{(x-t)^n}{n!}(\theta I+C+z_1D)^{i-n}D(\theta I+C+z_2D)^n\\ &= \sum_{i=0}^{\infty}\sum_{n=0}^{i}\frac{t^{i-n}}{(i-n)!}\frac{(x-t)^n}{n!}\sum_{l=0}^{i-n}\sum_{m=0}^{n} z_1^l z_2^m F_{i-n,n}(l,m)\\ &= \sum_{l=0}^{\infty}\sum_{m=0}^{\infty} z_1^l z_2^m\sum_{n=m}^{\infty}\sum_{i=l}^{\infty}\frac{t^i}{i!}\frac{(x-t)^n}{n!}F_{i,n}(l,m). \end{aligned}\tag{3.18}$$

Substituting (3.18) into (3.15) yields

$$\begin{aligned}\Omega^*(z_1,z_2,s) &= \sum_{l=0}^{\infty}\sum_{m=0}^{\infty} z_1^l z_2^m \\ &\times\left[\sum_{n=m}^{\infty}\sum_{i=l}^{\infty}\int_0^{\infty} e^{-(s+\theta)x}\frac{dV(x)}{\lambda E[V]}\int_0^x e^{st}dt\frac{t^i}{i!}\frac{(x-t)^n}{n!}F_{i,n}(l,m)\right]. \qquad (3.19)\end{aligned}$$

Considering the coefficient matrices of $z_1^i z_2^j$ on both sides of (3.19), we obtain

$$(3.20)\quad \Omega(i,j,s) = \sum_{m=i}^{\infty}\sum_{n=j}^{\infty}\int_0^{\infty} e^{-(s+\theta)x}\frac{dV(x)}{\lambda E[V]}\int_0^x e^{st}dt\frac{t^m}{m!}\frac{(x-t)^n}{n!}F_{m,n}(i,j).$$

3.6. LST for actual waiting time and its moments

In this subsection, we consider the actual waiting time distribution for N-policy with vacations. Let $R_k^{\mathrm{n}}(s)$ denote an $m\times m$ matrix whose (i,j)th element represents the LST for the length of the idle period when the number of customers is k and the phase is j at the end of the idle period given that the phase is i at the beginning of the idle period and the threshold value is n. From the similar reason of (3.8) and (3.9), $R_k^{\mathrm{n}}(s)$ satisfies the following equations

$$\begin{aligned}(3.21)\quad R_k^1(s) &= [I-V_0(s)]^{-1}V_k(s), \qquad k\geq 1,\\ (3.22)\quad R_k^{\mathrm{n}}(s) &= R_k^{\mathrm{n-1}}(s)+R_{n-1}^{\mathrm{n-1}}(s)\cdot R_{k-n+1}^1(s), \qquad 2\leq n\leq k,\end{aligned}$$

where

$$V_k(s) = \int_0^{\infty} e^{-st}P(k,t)dV(t).$$

We also define $R^{\mathrm{n}}(s) = \sum_{k=n}^{\infty} R_k^{\mathrm{n}}(s)$.

First, we consider the waiting time when the tagged customer arrives at the system in a vacation time. We observe the the following two cases:

1. The queue length becomes greater than or equal to N at the end of the vacation time during which the tagged customer arrives.
2. At the end of the vacation, there are k ($< N$) customers in the system. Then, the next service starts after the period according to $R^{\mathrm{N-k}}(s)$.

Thus, the LST $W_1^*(s)$ of the waiting time of a customer when it arrives during a vacation time is given by

$$\begin{aligned}W_1^*(s) &= \frac{1-\rho}{\left(\boldsymbol{x}_0^{\mathrm{s}}+\boldsymbol{X}_{N-1}^{\mathrm{v}}(1)\right)\boldsymbol{e}}\Bigg\{(\boldsymbol{x}_0^{\mathrm{s}}+\boldsymbol{x}_0^{\mathrm{v}})\sum_{i=0}^{\infty}\sum_{j=(N-i-1,0)^+}^{\infty}\Omega(i,j,s)[S^*(s)]^i\\ &+\sum_{k=1}^{N-1}\boldsymbol{x}_k^{\mathrm{v}}\sum_{i=0}^{\infty}\sum_{j=(N-k-i-1,0)^+}^{\infty}\Omega(i,j,s)[S^*(s)]^{k+i}\\ &+(\boldsymbol{x}_0^{\mathrm{s}}+\boldsymbol{x}_0^{\mathrm{v}})\sum_{i=0}^{N-2}\sum_{j=0}^{N-i-2}\Omega(i,j,s)R^{\mathrm{N-i-j-1}}(s)[S^*(s)]^i\\ &+\sum_{k=1}^{N-2}\boldsymbol{x}_k^{\mathrm{v}}\sum_{i=0}^{N-k-2}\sum_{j=0}^{N-k-i-2}\Omega(i,j,s)R^{\mathrm{N-k-i-j-1}}(s)[S^*(s)]^{k+i}\Bigg\}\boldsymbol{e}, \qquad (3.23)\end{aligned}$$

where $(x,y)^+$ indicates the maximum value of x and y.

Next, we consider the waiting time when the server is busy. The joint transform $Y^*(z,s)$ defined in subsection 2.5 becomes

$$Y^*(z,s) = \rho \frac{\boldsymbol{X}^{\mathrm{s}}(z) - \boldsymbol{x}_0^{\mathrm{s}} + \boldsymbol{X}^{\mathrm{v}}(z) - \boldsymbol{X}_{N-1}^{\mathrm{v}}(z)}{1 - \left(\boldsymbol{x}_0^{\mathrm{s}} + \boldsymbol{X}_{N-1}^{\mathrm{v}}(1)\right)\boldsymbol{e}} A(z,s).$$

Then, the LST $W_2^*(s)$ of the waiting time when the server is busy is given by

$$W_2^*(s) = \frac{1}{1 - \left(\boldsymbol{x}_0^{\mathrm{s}} + \boldsymbol{X}_{N-1}^{\mathrm{v}}(1)\right)\boldsymbol{e}} \times [\boldsymbol{x}_0^{\mathrm{s}} + \boldsymbol{X}_{N-1}^{\mathrm{v}}(S^*(s))][I - V(S^*(s))][sI + C + S^*(s)D]^{-1}D\boldsymbol{e}. \tag{3.24}$$

From (3.6), (3.23) and (3.24), the LST of the actual waiting time distribution is obtained as

$$\begin{aligned} W^*(s) &= W_1^*(s) + W_2^*(s) \\ &= \frac{1-\rho+\lambda E[V]}{\lambda E[V]} \Bigg[\lambda E[V] \Bigg\{ (\boldsymbol{x}_0^{\mathrm{s}} + \boldsymbol{x}_0^{\mathrm{v}}) \sum_{i=0}^{\infty} \sum_{j=(N-i-1,0)^+}^{\infty} \Omega(i,j,s)[S^*(s)]^i \\ &\quad + \sum_{k=1}^{N-1} \boldsymbol{x}_k^{\mathrm{v}} \sum_{i=0}^{\infty} \sum_{j=(N-k-i-1,0)^+}^{\infty} \Omega(i,j,s)[S^*(s)]^{k+i} \\ &\quad + (\boldsymbol{x}_0^{\mathrm{s}} + \boldsymbol{x}_0^{\mathrm{v}}) \sum_{i=0}^{N-2} \sum_{j=0}^{N-i-2} \Omega(i,j,s) R^{\mathrm{N-i-j-1}}(s)[S^*(s)]^i \\ &\quad + \sum_{k=1}^{N-2} \boldsymbol{x}_k^{\mathrm{v}} \sum_{i=0}^{N-k-2} \sum_{j=0}^{N-k-i-2} \Omega(i,j,s) R^{\mathrm{N-k-i-j-1}}(s)[S^*(s)]^{k+i} \Bigg\} \\ &\quad + [\boldsymbol{x}_0^{\mathrm{s}} + \boldsymbol{X}_{N-1}^{\mathrm{v}}(S^*(s))][I - V(S^*(s))][sI + C + S^*(s)D]^{-1}D \Bigg] \boldsymbol{e}. \end{aligned} \tag{3.25}$$

For calculating nth moment of the waiting time, we define following notations:

$$T_{ijk}(s) = \Omega(i,j,s)\,[S^*(s)]^{k+i}, \qquad T_{ijk}^{(n)} = \lim_{s\to 0}(-1)^n \frac{d^n}{ds^n} T_{ijk}(s),$$

$$U_{ijk}(s) = \Omega(i,j,s)\,R^{\mathrm{N-k-i-j-1}}(s)\,[S^*(s)]^{k+i}, \qquad U_{ijk}^{(n)} = \lim_{s\to 0}(-1)^n \frac{d^n}{ds^n} U_{ijk}(s),$$

$$\boldsymbol{x}(s) = \boldsymbol{x}_0^{\mathrm{s}} + \boldsymbol{X}_{N-1}^{\mathrm{v}}(S^*(s)), \qquad \boldsymbol{x}^{(n)} = \lim_{s\to 0}(-1)^n \frac{d^n}{ds^n}\boldsymbol{x}(s),$$

$$T(s) = U(s)\,[sI + C + S^*(s)D]^{-1}, \qquad U(s) = I - V(S^*(s)).$$

$T^{(n)}$ and $U^{(n)}$ are defined in subsection 2.5. Then, the nth moment $W^{(n)}$ of the actual waiting time becomes

$$\begin{aligned} W^{(n)} &= \frac{1-\rho+\lambda E[V]}{\lambda E[V]} \Bigg[\lambda E[V] \Bigg\{ (\boldsymbol{x}_0^{\mathrm{s}} + \boldsymbol{x}_0^{\mathrm{v}}) \sum_{i=0}^{\infty} \sum_{j=(N-i-1,0)^+}^{\infty} T_{ij0}^{(n)} \\ &\quad + \sum_{k=1}^{N-1} \boldsymbol{x}_k^{\mathrm{v}} \sum_{i=0}^{\infty} \sum_{j=(N-k-i-1,0)^+}^{\infty} T_{ijk}^{(n)} + (\boldsymbol{x}_0^{\mathrm{s}} + \boldsymbol{x}_0^{\mathrm{v}}) \sum_{i=0}^{N-2} \sum_{j=0}^{N-i-2} U_{ij0}^{(n)} \\ &\quad + \sum_{k=1}^{N-2} \boldsymbol{x}_k^{\mathrm{v}} \sum_{i=0}^{N-k-2} \sum_{j=0}^{N-k-i-2} U_{ijk}^{(n)} \Bigg\} + \sum_{m=0}^{n} \binom{n}{m} \boldsymbol{x}^{(m)} T^{(n-m)} D \Bigg] \boldsymbol{e}. \end{aligned}$$

From definitions of $T_{ijk}(s)$ and $U_{ijk}(s)$, $T_{ijk}^{(n)}$ and $U_{ijk}^{(n)}$ becomes

$$T_{ijk}^{(n)} = \sum_{m=0}^{n} \binom{n}{m} \Omega^{(m)}(i,j)\, S_{k+i}^{(n-m)},$$
$$U_{ijk}^{(n)} = \sum_{m=0}^{n} \binom{n}{m} \left\{ \sum_{l=0}^{m} \binom{m}{l} \Omega^{(l)}(i,j)\, R^{\mathrm{N-k-i-j-1}(m-l)} \right\} S_{k+i}^{(n-m)},$$

where

$$\Omega^{(n)}(i,j) = \lim_{s\to 0}(-1)^n \frac{d^n}{ds^n}\Omega(i,j,s), \qquad R^{\mathrm{m}(n)} = \lim_{s\to 0}(-1)^n \frac{d^n}{ds^n} R^{\mathrm{m}}(s).$$

From (3.20), we obtain

$$\Omega^{(k)}(i,j) = \frac{1}{\lambda E[V]} \sum_{m=i}^{\infty}\sum_{n=j}^{\infty}\sum_{l=0}^{n+k} \frac{F_{m,n}(i,j)}{m!n!} \frac{(-1)^l}{l+m+1} \binom{n+k}{l} \int_0^{\infty} x^{m+n+k+1} e^{-\theta x} dV(x).$$

$R^{\mathrm{m}(n)}$ is expressed as $R^{\mathrm{m}(n)} = \sum_{k=m}^{\infty} R_k^{\mathrm{m}(n)}$. From (3.21) and (3.22), $R_k^{\mathrm{m}(n)}$ can be expressed as

$$R_k^{1(0)} = [I - V_0]^{-1} V_k, \quad R_k^{1(n)} = [I - V_0]^{-1} \left[V_k^{(n)} + \sum_{l=0}^{n-1} \binom{n}{l} V_0^{(n-l)} R_k^{1(l)} \right],$$

$$R_k^{\mathrm{m}(n)} = R_k^{\mathrm{m-1}(n)} + \sum_{l=0}^{n} \binom{n}{l} R_{m-1}^{\mathrm{m-1}(l)} R_{k-m+1}^{1(n-l)}, \qquad 2 \le m \le k.$$

Hence, we can calculate $R_k^{\mathrm{m}(n)}$ recursively.

We can calculate $\boldsymbol{x}^{(n)}$ from the following equations

$$\boldsymbol{x}^{(n)} = \begin{cases} \boldsymbol{x}_0^{\mathrm{s}} + \boldsymbol{X}_{N-1}^{\mathrm{v}}(1), & n = 0, \\ \sum_{k=1}^{N-1} \boldsymbol{x}_k^{\mathrm{v}} S_k^{(n)}, & n > 0. \end{cases}$$

Since we can calculate $T^{(n)}$ according to the same way of the N-policy without vacations, we consider the calculating formula of $U^{(n)}$. According to [13], $V(z)$ can be rewritten as

$$V(z) = \sum_{m=0}^{\infty} \zeta_m [I + \theta^{-1}(C + zD)]^m = \sum_{k=0}^{\infty} z^k \sum_{m=k}^{\infty} \zeta_m F_m(k),$$

where

$$\zeta_m = \int_0^{\infty} \frac{(\theta x)^m}{m!} e^{-\theta x} dV(x),$$

and $F_m(k)$ satisfies following equations

$$F_{m+1}(k) = \begin{cases} (I + \theta^{-1}C)^{m+1}, & k = 0, \\ F_m(k)(I + \theta^{-1}C) + F_m(k-1)(\theta^{-1}D), & 1 \le k \le m, \\ (\theta^{-1}D)^{m+1}, & k = m+1. \end{cases}$$

Then, $U^{(n)}$ can be calculated from following equations

$$U^{(n)} = \begin{cases} I - V, & n = 0, \\ -\sum_{m=0}^{\infty} \zeta_m \sum_{k=0}^{m} F_m(k)\, S_k^{(n)}, & n > 0. \end{cases}$$

We summarize the procedure to compute $W^{(n)}$.

1. Compute $R_k^{N(n)}$ and then $R^{\mathrm{N}(n)}$.
2. Compute $F_{k,l}(i,j)$ and then $\Omega^{(n)}(i,j)$.
3. Compute $T_{ijk}^{(n)}$ and $U_{ijk}^{(n)}$ using $\Omega^{(n)}(i,j)$, $R^{\mathrm{N-k-i-j-1}(n)}$ and $S_{k+i}^{(n)}$.
4. Compute $U^{(n)}$ and then $T^{(n)}$ in the similar manner of section 2.5.
5. Compute $\boldsymbol{x}^{(n)}$.
6. Finally, compute $W^{(n)}$ using $T_{ijk}^{(n)}$, $U_{ijk}^{(n)}$, $\boldsymbol{x}^{(n)}$ and $T^{(n)}$.

4. Numerical examples

In this section, we present some numerical examples of the mean waiting times for N-policy with and without vacations. In our numerical examples, the service time distribution is chosen as an unit distribution with mean $E[S] = 1.0$ and the vacation time distribution as an exponential distribution with mean $E[V] = 1.0$. The arrival process is assumed to be a 2-state MMPP with

$$C = \begin{pmatrix} -r - \frac{2}{11}\rho & r \\ r & -r - \frac{20}{11}\rho \end{pmatrix}, \qquad D = \begin{pmatrix} \frac{2}{11}\rho & 0 \\ 0 & \frac{20}{11}\rho \end{pmatrix}.$$

From this construction, it is easy to see that $\boldsymbol{\pi} = (1/2,\ 1/2)$. Note that the correlation in the arrival process becomes large with the decrease of r. We calculate the mean waiting times with $r = 0.5$ and 1.0.

In computing the matrices G and A under both N-policy with and without vacations, we truncate the infinite sums according to the criteria proposed in [13]. We also truncate the infinite sum for calculating V using the same criterion.

In computing the kth moment $\Omega^{(k)}(i,j)$, we need to truncate the infinite sums of (3.20). The accuracy of $\Omega^{(k)}(i,j)$ depends on how many number of arrays for $F_{k,l}(m,n)$ we can store. Let c denote the index of the set $\{F_{k,l}(m,n) \,:\, 0 \le m \le k,\, 0 \le n \le l\}$, where $c = k + l$. From (3.16) and (3.17), the $c+1$st set of $F_{k,l}(m,n)$ can be calculated using the cth set of $F_{k,l}(m,n)$ (see Fig. 1). Note that we choose a maximum value $c_{\max}$ of c under the constraint of computer resources such as disk space and memory size. In our implementation, we set $c_{\max}$ to be 34. Since $\boldsymbol{\pi}\Omega^*(1,1,0)\boldsymbol{e} = 1$, we can check the accuracy of $\Omega^{(k)}(i,j)$ by summing $\Omega^{(0)}(i,j)$ over all i and j we computed.

We first compare the mean waiting times calculated from moment formulas with those calculated from Little's formula using the mean queue length $\boldsymbol{Y}^{(1)}$. Tables 1 and 2 show the numerical results of N-policy without and with vacations, respectively, where $N = 5$ and $r = 1.0$. In those tables, W_{LST} denotes the mean waiting time calculated by the LST and W_{Little} denotes that by Little's formula.

From Table 1, we observe that W_{LST} gives good agreement with W_{Little}. On the other hand, Table 2 shows that $|W_{LST} - W_{Little}|$ increases as ρ becomes large. This is because the accuracy of $\boldsymbol{\pi}\Omega^*(1,1,0)\boldsymbol{e}$ becomes worse (recall that we fixed $c_{\max}$ to 34). Note, however,

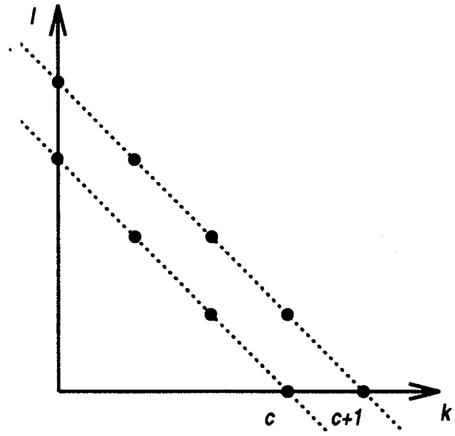

Figure 1: cth and $c+1$st sets of $F_{k,l}(m,n)$

ρ	W_{LST}	W_{Little}	$\lvert W_{LST} - W_{Little} \rvert$
0.10000	19.9516	19.9516	0.00000
0.20000	10.0602	10.0602	0.00000
0.30000	6.86860	6.86860	0.00000
0.40000	5.39333	5.39333	0.00000
0.50000	4.66432	4.66432	0.00000
0.60000	4.40793	4.40793	0.00000
0.70000	4.62049	4.62049	0.00000
0.80000	5.64797	5.64797	0.00000
0.90000	9.53749	9.53749	0.00000

Table 1: Comparison of Mean Waiting Times under N-policy without Vacations.

that W_{LST} agrees with W_{Little} in the order of 10^{-4} as $\rho = 0.9$. Thus it seems that it is sufficient for graphic representations to set $c_{\max} = 34$, except for the region of very high traffic. Therefore we use the result calculated by the LST in the following figures.

Fig. 2 shows the mean waiting times in the case of $N = 5$ and 10 with $r = 1.0$. We observe that the mean waiting time becomes large as the value of N increases, and that the mean waiting time under N-policy with vacations is always larger than that without vacations. We also observe that mean waiting times in all cases diverge to infinity as ρ becomes small. This is because the queue length is hard to reach N when ρ is small.

To investigate the influence of the correlation in arrivals on the mean waiting time, we plot Figs. 3 and 4, which show the mean waiting times with $r = 0.5, 1.0$ and that in Poisson arrivals with the same arrival rate, where $N = 5$. We observe that when ρ is large, the mean waiting time becomes large with the increase of the correlation in arrivals (recall that the correlation in arrivals becomes high with the decrease of r). However, when ρ is small, higher correlation leads to a smaller value of the mean waiting time. Please also see Table 3, which give numerical data of Figs. 3 and 4, respectively.

From these tables, we observe that when ρ is small, $W_{r=0.5} < W_{r=1.0} < W_{Poisson}$, and when ρ is large, $W_{r=0.5} > W_{r=1.0} > W_{Poisson}$. In general, higher correlation in arrival makes

ρ	W_{LST}	W_{Little}	$\lvert W_{LST} - W_{Little}\rvert$	$\boldsymbol{\pi}\Omega^*(1,1,0)\boldsymbol{e}$
0.10000	20.4579	20.4579	0.00000	0.99999
0.20000	10.6128	10.6128	0.00000	0.99999
0.30000	7.47885	7.47885	0.00000	0.99999
0.40000	6.07984	6.07984	0.00000	0.99999
0.50000	5.45828	5.45828	0.00000	0.99999
0.60000	5.36508	5.36508	0.00000	0.99998
0.70000	5.85289	5.85288	0.00001	0.99995
0.80000	7.43537	7.43533	0.00003	0.99991
0.90000	12.9965	12.9965	0.00003	0.99983

Table 2: Comparison of Mean Waiting Times under N-policy with Vacations

	N-policy without vacations			N-policy with vacations		
ρ	$r = 0.5$	$r = 1.0$	$W_{Poisson}$	$r = 0.5$	$r = 1.0$	$W_{Poisson}$
0.01000	199.74357	199.87384	200.00505	200.34455	200.47517	200.60585
0.02000	99.75333	99.88173	100.01020	100.35618	100.48438	100.61180
0.03000	66.42999	66.55646	66.68213	67.03470	67.16042	67.28452
⋮	⋮	⋮	⋮	⋮	⋮	⋮
0.36000	5.85972	5.86487	5.83681	6.79160	6.79476	6.76540
0.37000	5.73609	5.73469	5.69906	6.64972	6.64650	6.60968
0.38000	5.62130	5.61307	5.56961	6.51801	6.50818	6.46367

Table 3: Numerical Data under N-policy with and without Vacations

the mean waiting time larger. However, our numerical results show that it is not the case. Note that, in N-policy, the mean waiting time $E[W]$ consists of two terms; one is the mean waiting time $E[W_1]$ of customers which arrive in the idle period and the other is the mean waiting time $E[W_2]$ of customers which arrive in the busy period. Namely,

$$E[W] = (1 - \rho)E[W_1] + \rho E[W_2]$$

Tables 5 and 6 show $E[W_1]$ and $E[W_2]$ in the same settings as in Tables 3 and 4. We observe that $E[W_1]$ (resp. $E[W_2]$) is a decreasing (resp. an increasing) function of correlation in arrivals for a fixed ρ. We explain this phenomenon. When the correlation in arrivals is high, customers arrive back to back once a customer arrives. Thus after the first customer arrives in the idle period, subsequent customers are likely to arrive in a short interval, so that the mean waiting time of those customers becomes small according to the increase of the correlation in arrivals. On the other hand, the mean waiting time of customers which arrive in the busy period becomes large with the increase of correlation in arrivals, as in a work-conserving queue. In light traffic (i.e., for a small ρ), the former is the dominant factor in the mean waiting time $E[W]$. Thus, correlation in arrivals leads to a smaller mean waiting time in light traffic.

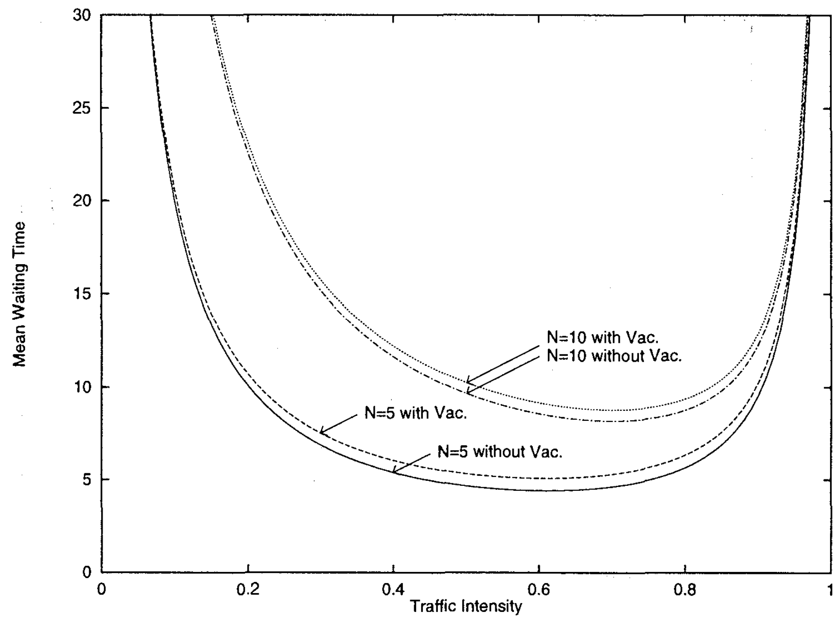

Figure 2: Mean Waiting Times under N-policy with and without Vacations

Appendix 1.
Derivation of Equation (3.6)

In this appendix, we derive (3.6) using (3.1), (3.2) and (3.3). From (3.1), we obtain

$$\text{(A.1)} \quad \boldsymbol{X}^{\mathrm{s}}(z)\,[zI - A(z)] = \boldsymbol{X}^{\mathrm{v}}(z)A(z) - \left[\boldsymbol{x}_0^{\mathrm{s}} + \boldsymbol{X}_{N-1}^{\mathrm{v}}(z)\right] A(z).$$

Using (3.2) and (A.1), we have

$$\boldsymbol{X}^{\mathrm{s}}(z)\,[zI - A(z)] = \left[\boldsymbol{x}_0^{\mathrm{s}} + \boldsymbol{X}_{N-1}^{\mathrm{v}}(z)\right] [V(z) - I]\, A(z).$$

Substituting $z = 1$ and multiplying both sides of (3.2) by $\boldsymbol{e}$, we obtain

$$\text{(A.2)} \quad \boldsymbol{X}^{\mathrm{v}}(1)\boldsymbol{e} = \left(\boldsymbol{x}_0^{\mathrm{s}} + \boldsymbol{X}_{N-1}^{\mathrm{v}}(1)\right) \boldsymbol{e}.$$

ρ	$E[W_1]$			$E[W_2]$		
	$r = 0.5$	$r = 1.0$	$W_{Poisson}$	$r = 0.5$	$r = 1.0$	$W_{Poisson}$
0.01000	201.73042	201.86440	202.00000	3.04533	2.80832	2.50505
0.02000	101.72677	101.86267	102.00000	3.05443	2.81556	2.51020
0.03000	68.38977	68.52759	68.66667	3.06376	2.82296	2.51546
⋮	⋮	⋮	⋮	⋮	⋮	⋮
0.36000	7.14511	7.35454	7.55556	3.57458	3.21655	2.78125
0.37000	6.99066	7.20240	7.40541	3.59992	3.23560	2.79365
0.38000	6.84411	7.05815	7.26316	3.62618	3.25531	2.80645

Table 4: Numerical Data under N-policy without Vacations

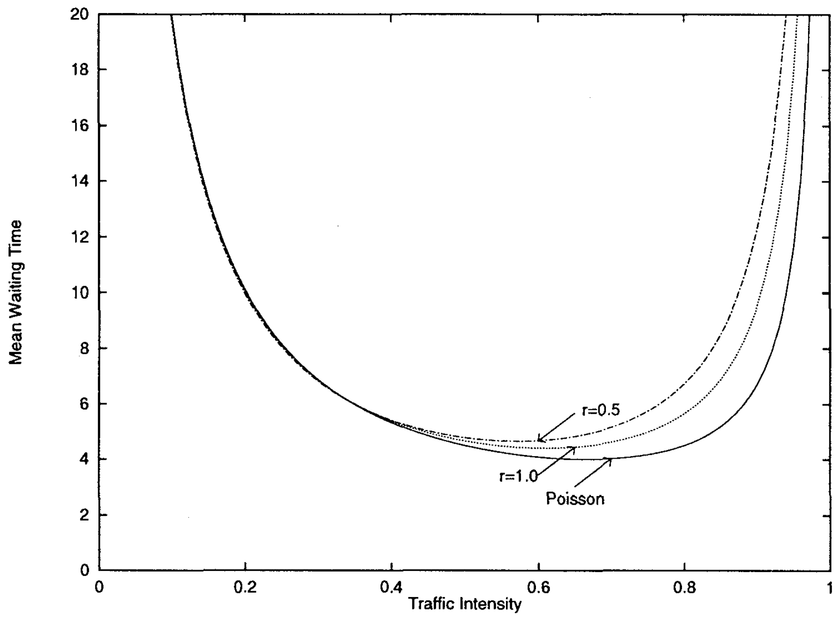

Figure 3: Mean Waiting Time under N-policy without Vacations

It then follows from (3.3) and (A.2) that

$$\text{(A.3)}\quad \boldsymbol{X}^{\mathrm{s}}(1)\boldsymbol{e} = 1 - \left(\boldsymbol{x}_0^{\mathrm{s}} + \boldsymbol{X}_{N-1}^{\mathrm{v}}(1)\right)\boldsymbol{e}.$$

Next, setting $z = 1$ and adding $\boldsymbol{X}^{\mathrm{s}}(1)\boldsymbol{e}\boldsymbol{\pi}$ to both sides of (A.1), we have

$$\text{(A.4)}\quad \boldsymbol{X}^{\mathrm{s}}(1) = \left(1 - \left(\boldsymbol{x}_0^{\mathrm{s}} + \boldsymbol{X}_{N-1}^{\mathrm{v}}(1)\right)\boldsymbol{e}\right)\boldsymbol{\pi} + \left[\boldsymbol{x}_0^{\mathrm{s}} + \boldsymbol{X}_{N-1}^{\mathrm{v}}(1)\right][V - I]A\left(I - A + \boldsymbol{e}\boldsymbol{\pi}\right)^{-1}.$$

Multiplying both sides of (A.4) by $A'(1)\boldsymbol{e}$, we obtain

$$\text{(A.5)}\quad \boldsymbol{X}^{\mathrm{s}}(1)A'(1)\boldsymbol{e} = \rho\left[1 - \left(\boldsymbol{x}_0^{\mathrm{s}} + \boldsymbol{X}_{N-1}^{\mathrm{v}}(1)\right)\boldsymbol{e}\right] + \left[\boldsymbol{x}_0^{\mathrm{s}} + \boldsymbol{X}_{N-1}^{\mathrm{v}}(1)\right][V - I](A - \boldsymbol{e}\boldsymbol{\pi})(\boldsymbol{e}\boldsymbol{\pi} - C - D)^{-1}D\boldsymbol{e}$$

	$E[W_1]$			$E[W_2]$		
ρ	$r = 0.5$	$r = 1.0$	$W_{Poisson}$	$r = 0.5$	$r = 1.0$	$W_{Poisson}$
0.01000	202.34008	202.47375	202.60681	2.78663	2.61574	2.51106
0.02000	102.34701	102.48138	102.61363	2.80528	2.63125	2.52224
0.03000	69.02059	69.15569	69.28712	2.82417	2.64696	2.53354
⋮	⋮	⋮	⋮	⋮	⋮	⋮
0.36000	8.12492	8.29845	8.40809	3.66128	3.32548	3.00692
0.37000	7.98094	8.15603	8.26516	3.69670	3.35345	3.02585
0.38000	7.84483	8.02151	8.13015	3.73306	3.38209	3.04519

Table 5: Numerical Data under N-policy with Vacations

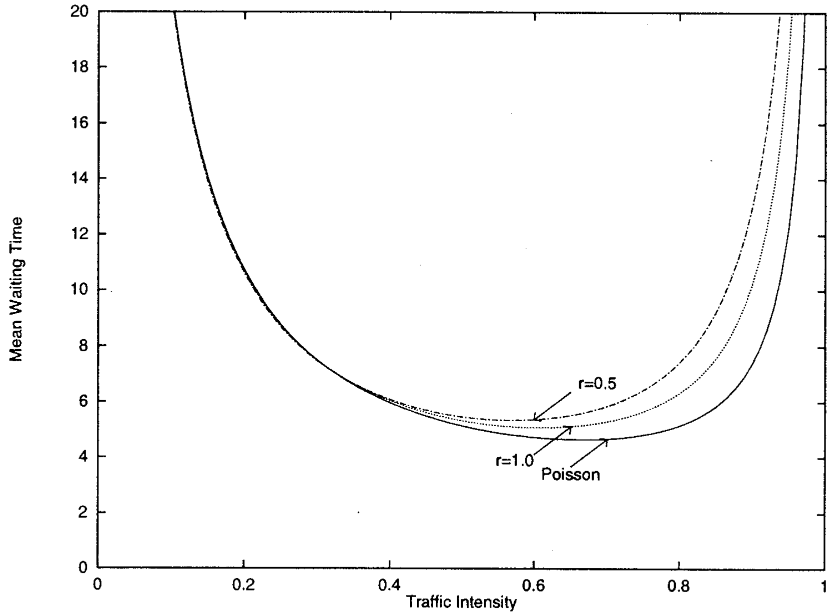

Figure 4: Mean Waiting Time under N-policy with Vacations

$$\begin{aligned} &= \rho\left[1-\left(\boldsymbol{x}_0^{\mathrm{s}}+\boldsymbol{X}_{N-1}^{\mathrm{v}}(1)\right)\boldsymbol{e}\right] \\ &\quad+\left[\boldsymbol{x}_0^{\mathrm{s}}+\boldsymbol{X}_{N-1}^{\mathrm{v}}(1)\right][V-I]A(\boldsymbol{e\pi}-C-D)^{-1}D\boldsymbol{e}, \end{aligned}$$

where we use the equality

$$A'(1)\boldsymbol{e}=\rho\boldsymbol{e}+(I-A)(\boldsymbol{e\pi}-C-D)^{-1}D\boldsymbol{e}.$$

On the other hand, differentiating (A.1) and setting $z=1$ yield

$$\begin{aligned} \boldsymbol{X}^{\mathrm{s}}(1)[I-A'(1)]\boldsymbol{e} &= \lambda E[V]\left(\boldsymbol{x}_0^{\mathrm{s}}+\boldsymbol{X}_{N-1}^{\mathrm{v}}(1)\right)\boldsymbol{e} \\ &\quad+\left[\boldsymbol{x}_0^{\mathrm{s}}+\boldsymbol{X}_{N-1}^{\mathrm{v}}(1)\right](I-V)A(\boldsymbol{e\pi}-C-D)^{-1}D\boldsymbol{e}, \end{aligned} \tag{A.6}$$

where we use the equality

$$V'(1)\boldsymbol{e}=\lambda E[V]\boldsymbol{e}+(I-V)(\boldsymbol{e\pi}-C-D)^{-1}D\boldsymbol{e}.$$

Thus, it follows from from (A.5) and (A.6) that

$$\boldsymbol{X}^{\mathrm{s}}(1)\boldsymbol{e}=\lambda E[V]\left(\boldsymbol{x}_0^{\mathrm{s}}+\boldsymbol{X}_{N-1}^{\mathrm{v}}(1)\right)\boldsymbol{e}+\rho\left[1-\left(\boldsymbol{x}_0^{\mathrm{s}}+\boldsymbol{X}_{N-1}^{\mathrm{v}}(1)\right)\boldsymbol{e}\right]. \tag{A.7}$$

Finally, using (A.3) and (A.7), we obtain

$$\left(\boldsymbol{x}_0^{\mathrm{s}}+\boldsymbol{X}_{N-1}^{\mathrm{v}}(1)\right)\boldsymbol{e}=\frac{1-\rho}{1-\rho+\lambda E[V]}$$

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