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A BULK SERVICE $GI/M/1$ QUEUE WITH SERVICE RATES DEPENDING ON SERVICE BATCH SIZE

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Abstract We consider a bulk service $GI/M/1$ queue with service rates depending on service batch size. If there are n customers waiting at the completion of service, $\min(n, K)$ customers enter service. We show that the queue size and the service batch size at points of arrivals form an embedded Markov chain and the steady-state probabilities of this Markov chain have the matrix geometric form. We describe the rate matrix R of the matrix geometric solution procedure in a readily computable form. We obtain explicit analytic expressions for the steady-state queue length distribution at points of arrivals. Further we obtain the Laplace-Stieltjes transform and the moments of the stationary waiting time distribution of an arbitrary customer.

1 Introduction

Bulk service queueing models are often encountered in applications. For example, transportation processes involving buses, airplanes, trains, ships, elevators and so on, all have a common feature of bulk service.

Several authors studied the bulk service queue with service time distribution depending on service batch size under a general bulk service rule. A general bulk service rule was first introduced by Neuts [2]. Under this rule, let there be n customers waiting at the completion of a service. If $0 \leq n < L$, the server remains idle until the queue length reaches L and then starts serving all L customers. If $L \leq n \leq K$, a group of size n enters service and if $n > K$, a group of size K is served. We may denote this system by $GI/G(L, K)/s$ if the interarrival time distribution is general and independent, the service time distribution is general and the number of servers is s.

For Poisson arrival queueing systems with service time distribution depending on service batch size, Neuts [2] and Neuts [3] studied $M/G(L, K)/1$ queue and derived queue length distribution. Further, for the same system, Neuts [6] studied waiting time distribution. By different approach, Curry and Feldman [1] studied $M/M(L, K)/1$ queue by using the matrix geometric solution procedure by Neuts [4]. They derived the distribution of the number in service as well as in the queue. They also showed that the solution to the matrix geometric equation had a simple structure that led to an easy algorithmic implementation. However, for non-Poisson arrival queue with service time distribution depending on service batch size, there has been no work to our knowledge.

In this paper, we study a $GI/M(1, K)/1$ queue with service rate depending on service batch size. The arrivals to the system occur one at a time, according to a renewal process with interarrival time distribution $F(\cdot)$ of finite mean λ_1' and the Laplace-Stieltjes transform (LST) $A^*(\theta)$. The service times of successive batches are conditionally independent, given that the batch sizes and the cumulative function (CDF) of service times. The CDF of service time S_k of a batch size k $(1 \leq k \leq K)$ obeys the negative exponential distribution with mean $1/\mu_k$. Further assume that $\mu_i \neq \mu_j$ when $i \neq j$. For this queueing system, the paper is organized as follows. In Section 2, we show that the queue size and the service batch size at points of arrivals form an embedded Markov chain. In Section 3, we show that the steady-state probabilities of this Markov chain have the matrix geometric form. We describe the rate matrix R of the matrix geometric solution procedure in a readily computable form. Explicit analytic expressions for the steady-state queue length distribution at points of arrivals are obtained in Section 4. In Section 5, we obtain the LST and the moments of the stationary waiting time distribution of an arbitrary customer.

Throughout this paper, for notational convenience, we denote $\boldsymbol{x}^T$ the transpose of the vector $\boldsymbol{x}, \boldsymbol{e}$ the appropriate dimensional column vector with all elements equal to one and I the appropriate dimensional identity matrix, respectively.

2 Embedded Markov Chain

We consider a $GI/M(1,K)/1$ queue with service rates depending on service batch size at points of arrivals. Suppose that I_r denotes the queue length immediately prior to the rth arrival and J_r the number of customers in service immediately prior to the rth arrival, respectively. Further suppose that τ_r denotes the time between $(r-1)$st and rth arrivals. For convenience, we choose the time origin at an epoch of arrival and set $\tau_0 = 0$. Then, it is easily seen that the sequence $\{(I_r, J_r, \tau_r), r \geq 0\}$ is a Markov renewal sequence on the state space $\{(0,x) : x \geq 0\} \cup \{(i,j,x) : i \geq 0, 1 \leq j \leq K, x \geq 0\}$. Suppose that $\boldsymbol{a} = (a_1, \underbrace{0, \ldots, 0}_{K-1})$ and $A = \text{diag}(a_1, \ldots, a_K)$, where $a_i = A^*(\mu_i)$ $(1 \leq i \leq K)$. Further suppose that B_{ij} $(1 \leq i \leq K, j \geq 0)$ is $K \times K$ matrix with nonzero elements only in the ith column, $\boldsymbol{b}_{ij} = (b_{ij1}, \ldots, b_{ijK})^T$. b_{ijk} $(i = 1, 2, \ldots, K; j = 0, 1, \ldots; k = 1, 2, \ldots, K)$ is the probability that when the customers of batch size k are served immediately prior to an arrival, the service of batch size k finishes, the service of batch size K finishes j times and the customers of batch size i are served immediately prior to next arrival. The details of b_{ijk} will be explained in Section 3. Now, for the case $K = 3$, the transition probability matrix P of the embedded Markov chain $\{(I_r, J_r) : r \geq 0\}$ is given by

$$P = \begin{bmatrix}
c & \boldsymbol{a} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots \\
\boldsymbol{c}_0 & B_{10} & A & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\
\boldsymbol{c}_1 & B_{20} & 0 & A & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\
\boldsymbol{c}_2 & B_{30} & 0 & 0 & A & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\
\boldsymbol{c}_3 & B_{11} & B_{30} & 0 & 0 & A & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\
\boldsymbol{c}_4 & B_{21} & 0 & B_{30} & 0 & 0 & A & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\
\boldsymbol{c}_5 & B_{31} & 0 & 0 & B_{30} & 0 & 0 & A & 0 & 0 & 0 & 0 & 0 & \cdots \\
\boldsymbol{c}_6 & B_{12} & B_{31} & 0 & 0 & B_{30} & 0 & 0 & A & 0 & 0 & 0 & 0 & \cdots \\
\boldsymbol{c}_7 & B_{22} & 0 & B_{31} & 0 & 0 & B_{30} & 0 & 0 & A & 0 & 0 & 0 & \cdots \\
\boldsymbol{c}_8 & B_{32} & 0 & 0 & B_{31} & 0 & 0 & B_{30} & 0 & 0 & A & 0 & 0 & \cdots \\
\boldsymbol{c}_9 & B_{13} & B_{32} & 0 & 0 & B_{31} & 0 & 0 & B_{30} & 0 & 0 & A & 0 & \cdots \\
\boldsymbol{c}_{10} & B_{23} & 0 & B_{32} & 0 & 0 & B_{31} & 0 & 0 & B_{30} & 0 & 0 & A & \cdots \\
\boldsymbol{c}_{11} & B_{33} & 0 & 0 & B_{32} & 0 & 0 & B_{31} & 0 & 0 & B_{30} & 0 & 0 & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots
\end{bmatrix}, \tag{1}$$

where the constant c and K-dimensional column vectors $\boldsymbol{c}_i$ $(i \geq 0)$ are determined to satisfy that each row sum of P is equal to unity.

3 Determination of Rate Matrix

Assume that $\boldsymbol{y} = (p_0, \boldsymbol{x}_0, \boldsymbol{x}_1, \ldots)$ is the steady-state probability vector of P. That is, $\boldsymbol{y}$ is the solution to $\boldsymbol{y}P = \boldsymbol{y}, \boldsymbol{ye} = 1$. Note that p_0 is the steady-state probability corresponding to the state 0 and the vector $\boldsymbol{x}_n$, for $n = 0, 1, \ldots,$ of order K is the steady-state probability vector corresponding to the states $\{(n, 1), \ldots, (n, K)\}$ and is partitioned as $\boldsymbol{x}_n = (x_{n1}, \ldots, x_{nK})$. By exploiting the structure of P, the Markov chain represented by the transition probability matrix P is irreducible. Further we have the following theorem.

Theorem 3.1 The Markov chain represented by the transition matrix P is positive recurrent if and only if the traffic intensity $\rho = \dfrac{1}{\lambda_1' \mu_K K} < 1$.

Proof : The proof will be in Appendix. □

By Neuts' matrix geometric solution procedure [4], when $\rho < 1$, we have

$$\boldsymbol{x}_n = \boldsymbol{x}_0 R^n \qquad \text{for } n = 0, 1, \ldots, \tag{2}$$

where R is the minimal nonnegative solution of

$$R = A + \sum_{n=1}^{\infty} R^{nK} B_{K,n-1}. \tag{3}$$

Then, we can show that the structure of R is a simple form and R can be calculated in a readily computable form.

Theorem 3.2 The rate matrix R is represented by

$$R = \begin{bmatrix} a_1 & & 0 & r_1 \\ & \ddots & & \vdots \\ & & a_{K-1} & r_{K-1} \\ 0 & & & r_K \end{bmatrix}, \tag{4}$$

where

$$a_i = A^*(\mu_i) \qquad \text{for } i = 1, \ldots, K-1 \tag{5}$$

and r_K is the unique real root between 0 and 1 of the equation

$$z = A^*(\mu_K(1 - z^K)). \tag{6}$$

Proof: We assume that $R = \Delta + V_1$, where Δ is some diagonal matrix and V_1 is some matrix with nonzero elements only in the last column. Then, by induction, it follows that $R^n = \Delta^n + V_n$ for all n, where V_n has nonzero elements only in the last column. Neuts [4] showed that the matrix equation (3) is solved by successive substitutions, starting $R = 0$. Therefore, since A is a diagonal matrix and $B_{K,n-1}$ $(n = 1, 2, \ldots)$ have nonzero elements only in the last column, it follows that R has the specified form $\Delta + V_1$. Given this form for R, equation (3) leads immediately to equation (4). By exploiting the elements of the transition matrix P, we have

$$a_i = \int_0^\infty e^{-\mu_i t} dA(t) = A^*(\mu_i) \qquad \text{for } i = 1, \ldots, K-1$$

and

$$b_{K,n-1,K} = \int_0^\infty \frac{(\mu_K t)^n}{n!} e^{-\mu_K t} dA(t) \qquad \text{for } n \geq 1. \tag{7}$$

By considering the (K, K) element of equation (3), we have

$$\begin{aligned} r_K &= a_K + \sum_{n=1}^{\infty} r_K^{nK} b_{K,n-1,K} \\ &= \sum_{n=0}^{\infty} \int_0^{\infty} \frac{(r_K^K \mu_K t)^n}{n!} e^{-\mu_K t} dA(t) \\ &= A^*(\mu_K(1 - r_K^K)). \end{aligned} \tag{8}$$

Finally we prove the existence of the unique real root $z = r_K$ $(0 < r_K < 1)$ for equation (6). We set $f(z) = z - A^*(\mu_K(1 - z^K))$. Then we have

$$f(0) = -A^*(\mu_K) < 0 \quad \text{and} \quad f(1) = 0. \tag{9}$$

Since $f'(z) = 1 + K\mu_K z^{K-1} A^{*(1)}(\mu_K(1 - z^K))$, we have

$$f'(0) = 1 > 0 \quad \text{and} \quad f'(1) = 1 - \lambda_1' \mu_K K = 1 - 1/\rho < 0, \tag{10}$$

where $A^{*(n)}(s) = \frac{d^n}{d\theta^n} A^*(\theta)|_{\theta=s}$. Further it follows that

$$\begin{aligned} f''(z) = K(K-1)\mu_K z^{K-2} A^{*(1)}(\mu_K(1 - z^K)) \\ - K^2 \mu_K^2 z^{2K-2} A^{*(2)}(\mu_K(1 - z^K)) < 0 \qquad \text{for } 0 < z < 1. \end{aligned} \tag{11}$$

From equations (9), (10) and (11), it is shown that the equation (6) has the unique real root between 0 and 1. □

Theorem 3.3 The elements of the last column of R, r_i $(i = 1, \ldots, K-1)$, are given by

$$r_i = \frac{\mu_i A^*(\mu_i)^K [A^*(\mu_i) - r_K][A^*(\mu_K(1 - A^*(\mu_i)^K)) - A^*(\mu_i)]}{[\mu_i - \mu_K + \mu_K A^*(\mu_i)^K][A^*(\mu_i) - r_K + A^*(\mu_K(1 - r_K^K)) - A^*(\mu_K(1 - A^*(\mu_i)^K))]}. \tag{12}$$

Proof: By induction, it can be shown that the elements of the last column of R^n are

$$r_i^{(n)} = r_i \sum_{j=1}^{n} a_i^{n-j} r_K^{j-1} \tag{13}$$

for $i = 1, \ldots, K-1$. The (i, K) element of equation (3) leads immediately to

$$r_i = \sum_{n=1}^{\infty} (a_i^{nK} b_{K,n-1,i} + r_i^{(nK)} b_{K,n-1,K}) \tag{14}$$

for $i = 1, \ldots, K-1$. Since

$$b_{Kni} = \int_0^{\infty} dA(t) \int_0^t \mu_i e^{-\mu_i(t-x)} \frac{(\mu_K x)^n}{n!} e^{-\mu_K x} dx \tag{15}$$

and equation (7) hold, direct calculation shows that

$$\sum_{n=1}^{\infty} a_i^{nK} b_{K,n-1,i} = \frac{\mu_i A^*(\mu_i)^K [A^*(\mu_K(1 - A^*(\mu_i)^K)) - A^*(\mu_i)]}{\mu_i - \mu_K + \mu_K A^*(\mu_i)^K} \tag{16}$$

and

$$\sum_{n=1}^{\infty} r_i^{(nK)} b_{K,n-1,K} = r_i \sum_{n=1}^{\infty} b_{K,n-1,K} \sum_{j=1}^{nK} a_i^{nK-j} r_K^{j-1} = \frac{r_i[A^*(\mu_K(1-A^*(\mu_i)^K)) - A^*(\mu_K(1-r_K^K))]}{A^*(\mu_i) - r_K} \tag{17}$$

equations (14), (16) and (17) lead to equation (12). □

From Theorems 3.2 and 3.3, we can calculate the elements of the rate matrix R by only using the LST of the interarrival time distribution, $A^*(\theta)$, and the service rate μ_i $(i = 1, \ldots, K)$.

4 Stationary Queue Length at Arrivals

In this section, we derive explicit expressions for the steady-state probability vector $\boldsymbol{y}$ of P. By exploiting the special structure of P and using that the steady-state probability vector has a matrix geometric form, p_0 and $\boldsymbol{x}_0$ satisfy the following steady-state equations.

$$x_{01} = p_0 a_1 + \sum_{n=0}^{\infty} \boldsymbol{x}_{nK} \boldsymbol{b}_{1n} = p_0 A^*(\mu_1) + \boldsymbol{x}_0 \sum_{n=0}^{\infty} R^{nK} \boldsymbol{b}_{1n}, \tag{18}$$

$$x_{0i} = \sum_{n=0}^{\infty} \boldsymbol{x}_{nK+i-1} \boldsymbol{b}_{in} = \boldsymbol{x}_0 \sum_{n=0}^{\infty} R^{nK+i-1} \boldsymbol{b}_{in} \qquad (i = 2, \ldots, K). \tag{19}$$

Further, the normalizing equation

$$p_0 + \boldsymbol{x}_0 (I - R)^{-1} \boldsymbol{e} = 1 \tag{20}$$

holds. Since the rate matrix R is completely determined in Section 3 and equations (18), (19) and (20) form a system of simultaneous linear equations with $K+1$ unknowns, we can solve these equations if we can calculate

$$\boldsymbol{d}_i = (d_{i1}, \ldots, d_{iK})^T = \sum_{n=0}^{\infty} R^{nK+i-1} \boldsymbol{b}_{in} \qquad (i = 1, \ldots, K) \tag{21}$$

in closed form.

At first, we calculate $\boldsymbol{d}_K$. For $j = 1, \ldots, K-1$, it follows that

$$\begin{aligned} d_{Kj} &= \sum_{n=0}^{\infty} (a_j^{nK+K-1} b_{Knj} + r_j^{(nK+K-1)} b_{KnK}) \\ &= \sum_{n=0}^{\infty} \left(a_j^{nK+K-1} \int_0^{\infty} dA(t) \int_0^t \mu_j e^{-\mu_j(t-x)} \frac{(\mu_K x)^n}{n!} e^{-\mu_K x} dx \right. \\ &\quad \left. + r_j \sum_{k=1}^{nK+K-1} a_j^{nK+K-1-k} r_K^{k-1} \int_0^{\infty} \frac{(\mu_K t)^{n+1}}{(n+1)!} e^{-\mu_K t} dA(t) \right) \\ &= \frac{\mu_j A^*(\mu_j)^{K-1} [A^*(\mu_K(1-A^*(\mu_j)^K)) - A^*(\mu_j)]}{\mu_j - \mu_K + \mu_K A^*(\mu_j)^K} \\ &\quad + \frac{r_j}{A^*(\mu_j) - r_K} \left[\frac{A^*(\mu_K(1-A^*(\mu_j)^K)) - A^*(\mu_K)}{A^*(\mu_j)} - \frac{A^*(\mu_K(1-r_K^K)) - A^*(\mu_K)}{r_K} \right]. \end{aligned} \tag{22}$$

The Kth element of $\boldsymbol{d}_K, d_{KK}$, is given by

$$\begin{aligned} d_{KK} &= \sum_{n=0}^{\infty} r_K^{nK+K-1} b_{KnK} \\ &= \sum_{n=0}^{\infty} r_K^{nK+K-1} \int_0^\infty \frac{(\mu_K t)^{n+1}}{(n+1)!} e^{-\mu_K t} dA(t) \\ &= \frac{A^*(\mu_K(1-r_K^K)) - A^*(\mu_K)}{r_K}. \end{aligned} \tag{23}$$

For the next step, we calculate $\boldsymbol{d}_i \quad (i = 1, \ldots, K-1)$.

(1) Derivation of d_{iK}

Since it follows that

$$b_{inK} = \int_0^\infty dA(t) \int_0^t e^{-\mu_i(t-x)} \mu_K \frac{(\mu_K x)^n}{n!} e^{-\mu_K x} dx \qquad (n \geq 0), \tag{24}$$

we have

$$\begin{aligned} d_{iK} &= \sum_{n=0}^{\infty} r_K^{nK+i-1} b_{inK} \\ &= \sum_{n=0}^{\infty} r_K^{nK+i-1} \int_0^\infty dA(t) \int_0^t e^{-\mu_i(t-x)} \mu_K \frac{(\mu_K x)^n}{n!} e^{-\mu_K x} dx \\ &= \frac{\mu_K r_K^{i-1}[A^*(\mu_K(1-r_K^K)) - A^*(\mu_i)]}{\mu_i - \mu_K + \mu_K r_K^K}. \end{aligned} \tag{25}$$

(2) Derivation of d_{ii}

Since it follows that

$$b_{i0i} = \int_0^\infty \mu_i t e^{-\mu_i t} dA(t) \tag{26}$$

and

$$\begin{aligned} b_{ini} &= \int_0^\infty dA(t) \int_0^t \mu_K \frac{(\mu_K x)^{n-1}}{(n-1)!} e^{-\mu_K x} dx \int_0^{t-x} \mu_i e^{-\mu_i y} e^{-\mu_i(t-x-y)} dy \\ &= \int_0^\infty \mu_i \mu_K e^{-\mu_i t} dA(t) \int_0^t (t-x) \frac{(\mu_K x)^{n-1}}{(n-1)!} e^{(\mu_i - \mu_K)x} dx \qquad (n \geq 1), \end{aligned} \tag{27}$$

tedious calculations deduce that

$$\begin{aligned} \sum_{n=0}^{\infty} a_i^{nK+i-1} b_{ini} &= \frac{\mu_i(\mu_K - \mu_i) A^*(\mu_i)^{i-1} A^{*(1)}(\mu_i)}{\mu_i - \mu_K + \mu_K A^*(\mu_i)^K} \\ &+ \frac{\mu_i \mu_K A^*(\mu_i)^{K+i-1}\{A^*(\mu_K(1 - A^*(\mu_i)^K)) - A^*(\mu_i)\}}{(\mu_i - \mu_K + \mu_K A^*(\mu_i)^K)^2}. \end{aligned} \tag{28}$$

For $i = 1$, by using equation (24), we have

$$\begin{aligned} \sum_{n=0}^{\infty} r_1^{(nK)} b_{1nK} &= b_{10K} + \sum_{n=1}^{\infty} r_1^{(nK)} b_{1nK} \\ &= \int_0^\infty dA(t) \int_0^t \mu_K e^{-\mu_K x} e^{-\mu_1(t-x)} dx \end{aligned}$$

$$+\sum_{n=1}^{\infty} r_1 \sum_{k=1}^{nK} a_1^{nK-k} r_K^{k-1} \int_0^\infty dA(t) \int_0^t \mu_K \frac{(\mu_K x)^n}{n!} e^{-\mu_K x} e^{-\mu_1 (t-x)} dx \tag{29}$$

$$= \frac{\mu_K[A^*(\mu_K) - A^*(\mu_1)]}{\mu_1 - \mu_K} + \frac{r_1 \mu_K}{A^*(\mu_1) - r_K} \left[\frac{A^*(\mu_K(1 - A^*(\mu_1)^K)) - A^*(\mu_1)}{\mu_1 - \mu_K + \mu_K A^*(\mu_1)^K} - \frac{A^*(\mu_K(1 - r_K^K)) - A^*(\mu_1)}{\mu_1 - \mu_K + \mu_K r_K^K} \right].$$

For $i = 2, \ldots, K-1$, we have

$$\sum_{n=0}^{\infty} r_i^{(nK+i-1)} b_{inK} = \frac{r_i \mu_K}{A^*(\mu_i) - r_K} \left[\frac{A^*(\mu_i)^{i-1}\{A^*(\mu_K(1 - A^*(\mu_i)^K)) - A^*(\mu_i)\}}{\mu_i - \mu_K + \mu_K A^*(\mu_i)^K} - \frac{r_K^{i-1}\{A^*(\mu_K(1 - r_K^K)) - A^*(\mu_i)\}}{\mu_i - \mu_K + \mu_K r_K^K} \right]. \tag{30}$$

Therefore we have

$$d_{ii} = \sum_{n=0}^{\infty} (a_i^{nK+i-1} b_{ini} + r_i^{(nK+i-1)} b_{inK}) \tag{31}$$

by using equations (29) and (30).

(3) Derivation of d_{ij} $\quad (j = 1, \ldots, K-1; i \neq j)$

Since it follows that

$$b_{i0j} = \int_0^\infty dA(t) \int_0^t \mu_j e^{-\mu_j x} e^{-\mu_i (t-x)} dx \tag{32}$$

and

$$b_{inj} = \int_0^\infty dA(t) \int_0^t \mu_K \frac{(\mu_K x)^{n-1}}{(n-1)!} e^{-\mu_i x} dx \int_0^{t-x} \mu_j e^{-\mu_j y} e^{-\mu_i (t-x-y)} dy \quad (n \geq 1), \tag{33}$$

we have

$$\sum_{n=0}^{\infty} a_j^{nK+i-1} b_{inj} = \frac{\mu_j A^*(\mu_j)^{j-1}[A^*(\mu_j) - A^*(\mu_i)]}{\mu_i - \mu_j} + \frac{\mu_j \mu_K A^*(\mu_j)^{K+i-1}}{\mu_j - \mu_i} \left[\frac{A^*(\mu_K(1 - A^*(\mu_j)^K)) - A^*(\mu_j)}{\mu_j - \mu_K + \mu_K A^*(\mu_j)^K} - \frac{A^*(\mu_K(1 - A^*(\mu_j)^K)) - A^*(\mu_i)}{\mu_i - \mu_K + \mu_K A^*(\mu_j)^K} \right]. \tag{34}$$

For $i = 1$, by using equation (24), we have

$$\sum_{n=0}^{\infty} r_j^{(nK)} b_{1nK} = b_{10K} + \sum_{n=1}^{\infty} r_j^{(nK)} b_{1nK}$$

$$= \int_0^\infty dA(t) \int_0^t \mu_K e^{-\mu_K x} e^{-\mu_1 (t-x)} dx + \sum_{n=1}^{\infty} r_j \sum_{k=1}^{nK} a_j^{nK-k} r_K^{k-1} \int_0^\infty dA(t) \int_0^t \mu_K e^{-\mu_K x} e^{-\mu_1 (t-x)} dx \tag{35}$$

$$= \frac{\mu_K[A^*(\mu_K) - A^*(\mu_1)]}{\mu_1 - \mu_K} + \frac{r_1\mu_K}{A^*(\mu_j) - r_K}\left[\frac{A^*(\mu_K(1 - A^*(\mu_j)^K)) - A^*(\mu_1)}{\mu_1 - \mu_K + \mu_K A^*(\mu_j)^K} - \frac{A^*(\mu_K(1 - r_K^K)) - A^*(\mu_1)}{\mu_1 - \mu_K + \mu_K r_K^K}\right].$$

For $i = 2, \dots, K-1$, we have

$$\sum_{n=0}^{\infty} r_j^{(nK+i-1)} b_{inK} = \frac{r_j\mu_K}{A^*(\mu_j) - r_K}\left[\frac{A^*(\mu_j)^{i-1}\{A^*(\mu_K(1 - A^*(\mu_j)^K)) - A^*(\mu_i)\}}{\mu_i - \mu_K + \mu_K A^*(\mu_j)^K} - \frac{r_K^{i-1}\{A^*(\mu_K(1 - r_K^K)) - A^*(\mu_i)\}}{\mu_i - \mu_K + \mu_K r_K^K}\right]. \tag{36}$$

Therefore we have

$$d_{ij} = \sum_{n=0}^{\infty}(a_j^{nK+i-1} b_{inj} + r_j^{(nK+i-1)} b_{inK}) \tag{37}$$

by using equations (34), (35) and (36).

Finally we can calculate $\boldsymbol{d}_i$ $(i = 1, \dots, K)$ by using the LST of the interarrival distribution, $A^*(\theta)$, and its first derivative, $A^{*(1)}(\theta)$, the service rates μ_i $(i = 1, \dots, K)$ and the rate matrix R. We can determine the steady-state probability distribution at points of arrivals by solving a system of equations (18), (19) and (20) and using equation (2).

Further we can obtain the moments of the queue length at points of arrivals. We denote by L and $L(z)$ the steady-state queue length at points of arrivals and its generating function, respectively. By using $\boldsymbol{x}_n$ $(n = 0, 1, \dots)$ and R, $L(z)$ is expressed as

$$\begin{aligned} L(z) &= \sum_{n=0}^{\infty} \boldsymbol{x}_n \boldsymbol{e} z^n \qquad (|z| \le 1) \\ &= \sum_{n=0}^{\infty} \boldsymbol{x}_0 R^n \boldsymbol{e} z^n \\ &= \boldsymbol{x}_0 (I - Rz)^{-1} \boldsymbol{e}. \end{aligned} \tag{38}$$

The mean and variance of L are given by

$$E(L) = L'(1), \qquad V(L) = L''1 + L'(1) - \{L'(1)\}^2, \tag{39}$$

where

$$L'(1) = \left.\frac{dL(z)}{dz}\right|_{z=1} = \boldsymbol{x}_0 R(I - R)^{-2}\boldsymbol{e}, \tag{40}$$

$$L''(1) = \left.\frac{d^2L(z)}{dz^2}\right|_{z=1} = 2\boldsymbol{x}_0 R(I - R)^{-3}\boldsymbol{e}. \tag{41}$$

5 Waiting Time Distribution

In this section, we obtain the LST and the moments of the stationary waiting time distribution of an arbitrary customer.

Theorem 5.1 Let us define

$$\boldsymbol{\mu}(\theta) = \left(\frac{\mu_1}{\theta+\mu_1}, \ldots, \frac{\mu_K}{\theta+\mu_K}\right)^T. \tag{42}$$

Let W and $W^*(\theta)$ denote the steady-state waiting time and its LST, respectively. Then $W^*(\theta)$ is given by

$$W^*(\theta) = p_0 + \boldsymbol{x}_0 \left(I - \frac{\mu_K}{\theta+\mu_K} R^K\right)^{-1} (I - R^K)(I-R)^{-1}\boldsymbol{\mu}(\theta) \tag{43}$$

Proof: If upon arrival a customer finds that the server is idle, then $W = 0$. Further if upon arrival a customer finds that the number of waiting customers is $iK+j$ $(i \geq 0, 0 \leq j \leq K-1)$ and the number in service is k then the conditional waiting time has LST, given by

$$\frac{\mu_k}{\theta+\mu_k}\left(\frac{\mu_K}{\theta+\mu_K}\right)^i. \tag{44}$$

Therefore $W^*(\theta)$ is given by

$$\begin{aligned} W^*(\theta) &= p_0 + \sum_{i=0}^{\infty}\sum_{j=0}^{K-1}\sum_{k=1}^{K} x_{iK+j,k}\frac{\mu_k}{\theta+\mu_k}\left(\frac{\mu_K}{\theta+\mu_K}\right)^i \\ &= p_0 + \sum_{i=0}^{\infty}\sum_{j=0}^{K-1} \boldsymbol{x}_{iK+j}\boldsymbol{\mu}(\theta)\left(\frac{\mu_K}{\theta+\mu_K}\right)^i \\ &= p_0 + \sum_{i=0}^{\infty}\sum_{j=0}^{K-1} \boldsymbol{x}_0 R^{iK+j}\boldsymbol{\mu}(\theta)\left(\frac{\mu_K}{\theta+\mu_K}\right)^i \\ &= p_0 + \sum_{i=0}^{\infty} \boldsymbol{x}_0 R^{iK}(I-R^K)(I-R)^{-1}\boldsymbol{\mu}(\theta)\left(\frac{\mu_K}{\theta+\mu_K}\right)^i \\ &= p_0 + \boldsymbol{x}_0\left(I - \frac{\mu_K}{\theta+\mu_K}R^K\right)^{-1}(I-R^K)(I-R)^{-1}\boldsymbol{\mu}(\theta). \end{aligned} \tag{45}$$

□

Differentiating (43) once and twice and inserting $\theta = 0$, we have the next Corollary.

Corollary 5.2 Let us define

$$\boldsymbol{\mu}_1 = \left(\frac{1}{\mu_1}, \ldots, \frac{1}{\mu_K}\right)^T \quad \text{and} \quad \boldsymbol{\mu}_2 = \left(\frac{1}{\mu_1^2}, \ldots, \frac{1}{\mu_K^2}\right)^T. \tag{46}$$

The first and second moments of W are given by

$$E(W) = \boldsymbol{x}_0\left[\frac{1}{\mu_K}R^K(I-R^K)^{-1}(I-R)^{-1}\boldsymbol{e} + (I-R)^{-1}\boldsymbol{\mu}_1\right], \tag{47}$$

$$\begin{aligned} E(W^2) = 2\boldsymbol{x}_0 &\left[\frac{1}{\mu_K^2}R^{2K}(I-R^K)^{-2}(I-R)^{-1}\boldsymbol{e}\right. \\ &+ \frac{1}{\mu_K^2}R^K(I-R^K)^{-1}(I-R)^{-1}\boldsymbol{e} \\ &\left.+ \frac{1}{\mu_K}R^K(I-R^K)^{-1}(I-R)^{-1}\boldsymbol{\mu}_1 + (I-R)^{-1}\boldsymbol{\mu}_2\right]. \end{aligned} \tag{48}$$

6 Concluding Remarks

In this paper, we studied a bulk service $GI/M/1$ queue with service rates depending on service batch size. In Neuts and Chandramouli [5], we can find a nice application of bulk service queue with service time distribution depending on service batch size. Neuts and Chandramouli [5] studied group testing in quality control tests, with special attention to the trade-off between larger group sizes and time lost due to retesting of groups containing flawed items.

We only treated the case $L = 1$ in this paper. If $L > 1$, it is cumbersome to analyze the steady-state queue length distribution at points of arrivals because the boundary states of the Markov chain represented by the transition matrix P are very complicated. Further, the analysis of the waiting time distribution is very difficult because the waiting time of a tagged customer will be influenced by the customers who will arrive after a tagged customer's arrival.

Thus the treatment of the case $L > 1$ is left to the future study.

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Appendix

The proof of Theorem 3.1

Let $C = A + \sum_{n=0}^{\infty} B_{Kn}$. It is clear that C is stochastic. Since A is a $K \times K$ diagonal matrix and B_{Kn} $(i = 0, 1, \ldots)$ are $K \times K$ matrices with nonzero elements only in the Kth column, C is upper triangular. By adapting the result of Theorem 1.4.1 of Neuts [4] to present case, we see that

$$\sum_{n=0}^{\infty}(n+1)Kb_{KnK} > 1. \tag{49}$$

By using eq. (7), the inequality (49) yields

$$\sum_{n=0}^{\infty}(n+1)K\int_0^{\infty}\frac{(\mu_K t)^{n+1}}{(n+1)!}e^{-\mu_K t}dA(t) = K\mu_K\int_0^{\infty} t dA(t) = K\mu_K\lambda' > 1. \tag{50}$$

This completes the proof. □

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